MATH 7338 HOMEWORK #4 Due date: December 3, 2009

Work FOUR of the following problems and hand in your solutions. You may work together with other people in the class, but you must each write up your solutions independently. A subset of these will be selected for grading. Write LEGIBLY on the FRONT side of the page only, and STAPLE your pages together.

Definition 1. If μ is a continuous linear functional on $C_c^\infty(\mathbb{R})$ and there exists a single integer $N \geq 0$ such that for each compact $K \subseteq \mathbb{R}$ there is a constant $C_K > 0$ such that

$$|\langle f, \mu\rangle| \;\leq\; C_K \sum_{n=0}^{N} \|f^{(n)}\|_\infty, \qquad f \in C^\infty(K), \tag{1}$$

then we say that μ has *finite order*. In this case, the *order of* μ is the smallest integer $N \geq 0$ such that we can find $C_K > 0$ so that equation (1) holds for every compact $K \subseteq \mathbb{R}$. If no such N exists, then the order of μ is ∞.

Definition 2. If $\mu \in \mathcal{D}'(\mathbb{R})$ and $\theta \in C^\infty(\mathbb{R})$, then $\theta\mu\colon C_c^\infty(\mathbb{R}) \to \mathbb{C}$ is given by

$$\bigl\langle f, \theta\mu \bigr\rangle \;=\; \bigl\langle f\bar{\theta}, \mu \bigr\rangle, \qquad f \in C_c^\infty(\mathbb{R}). \tag{2}$$

Definition 3. The *distributional derivative*, or simply *derivative* for short, of a distribution $\mu \in \mathcal{D}'(\mathbb{R})$ is the functional $D\mu\colon C_c^\infty(\mathbb{R}) \to \mathbb{C}$ given by

$$\langle f, D\mu\rangle \;=\; -\langle f', \mu\rangle, \qquad f \in C_c^\infty(\mathbb{R}). \tag{3}$$

1. (a) Given $f \in C_c^\infty(\mathbb{R})$, show that $\frac{f - T_a f}{a} \to f'$ in $C_c^\infty(\mathbb{R})$ as $a \to 0$

(b) Given $\mu \in \mathcal{D}'(\mathbb{R})$, show that $\frac{\mu - T_a\mu}{a} \xrightarrow{\text{w*}} D\mu$ as $a \to 0$.

2. (a) Define $\delta_a^{(j)} \in \mathcal{D}'(\mathbb{R})$ by $\langle f, \delta_a^{(j)}\rangle = (-1)^j f^{(j)}(a)$. Show that $\delta_a^{(j)}$ has order j.

(b) Define $\mu = \sum_{n\in\mathbb{Z}} \delta_n$ by

$$\langle f, \mu\rangle \;=\; \sum_{n\in\mathbb{Z}} \langle f, \delta_n\rangle \;=\; \sum_{n\in\mathbb{Z}} f(n), \qquad f \in C_c^\infty(\mathbb{R}).$$

Show that $\mu \in \mathcal{D}'(\mathbb{R})$ and μ has order 0, but the constants C_K in equation (1) cannot be chosen to be independent of K.

(c) Define $\delta_a^{(j)} \in \mathcal{D}'(\mathbb{R})$ by $\langle f, \delta_a^{(j)}\rangle = (-1)^j f^{(j)}(a)$. Define $\nu = \sum_{n\in\mathbb{N}} \delta_n^{(n)}$ by

$$\langle f, \nu\rangle \;=\; \sum_{n=1}^{\infty} \langle f, \delta_n^{(n)}\rangle \;=\; \sum_{n=1}^{\infty} (-1)^n f^{(n)}(n), \qquad f \in C_c^\infty(\mathbb{R}).$$

Show that $\nu \in \mathcal{D}'(\mathbb{R})$ and ν has infinite order.

3. Show that $\delta_n \xrightarrow{\text{w}^*} 0$ in $\mathcal{D}'(\mathbb{R})$ and in $\mathcal{S}'(\mathbb{R})$ as $|n| \to \infty$. Is this also true in $\mathcal{E}'(\mathbb{R})$?

4. (a) Show that if $\mu \in \mathcal{D}'(\mathbb{R})$ and $\theta \in C^\infty(\mathbb{R})$, then $\theta\mu \in \mathcal{D}'(\mathbb{R})$.

(b) Given $j \geq 0$, show that $x^j \delta^{(j)} = (-1)^j\, j!\, \delta$, where $x^j \delta^{(j)}$ denotes the product of the function x^j with the distribution $\delta^{(j)}$.

5. (a) Fix $k \in C_c^\infty(\mathbb{R})$ with $\int k = 1$. Show that every $f \in C_c^\infty(\mathbb{R})$ can be written uniquely as $f = c_f k + g_f$ where $c_f \in \mathbb{C}$ and $g_f = h'$ for some $h \in C_c^\infty(\mathbb{R})$.

(b) Use part (a) to show that if $\mu \in \mathcal{D}'(\mathbb{R})$ and $D\mu = 0$, then $\mu = c$ where c is a constant function (this means that μ and c act in the same way, i.e., $\langle f, \mu \rangle = \langle f, c \rangle = c \int f$ for $f \in C_c^\infty(\mathbb{R})$).