Completeness of $C_c^\infty(\mathbb{R})$

Our next goal is to show that $C_c^\infty(\mathbb{R})$ is complete w.r.t. the inductive limit topology that we have defined. However, since $C_c^\infty(\mathbb{R})$ is not metrizable, we need to extend the definition of Cauchy sequences to TVS.

<u>Definition</u>
Let X be a TVS. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is <u>Cauchy</u> on X if given a local base $\mathcal{B}$ for X we have

$\forall V \in \mathcal{B}\ \exists N>0$ s.t.

$$m,n > N \implies x_m - x_n \in V.$$

<u>Exercise</u>: This definition does not depend on which local base for X that we choose, i.e., a sequence is Cauchy w.r.t. one local base if & only if it is Cauchy w.r.t. another local base.

Exercise:
Suppose that X has a translation-invariant metric d. Then

$$d(x_m, x_n) = d(x_m - x_n, 0)$$

Show that the usual definition of Cauchy sequence in a metric space is equivalent to the definition of Cauchy sequence in a TVS.

Remark
Since $C^\infty(K)$ is metrizable, Cauchy sequences in $C^\infty(K)$ can be characterized in terms of the metric, or equivalently in terms of the family of seminorms.
Specifically, a sequence $\{\varphi_k\}_{k\in\mathbb{N}}$ contained in $C^\infty(K)$ is Cauchy if & only if

$$\forall n \geq 0, \quad \{\varphi_k\} \text{ is Cauchy w.r.t. } \rho_{K,n}(f) = \|f^{(n)}\|_\infty$$

Here is a lemma that we needed early but skipped the proof. Symmetric means $U=-U$.

Lemma

Let X be a TVS. If W is an open neighborhood of 0, then $\exists$ symmetric open neighborhood U of 0 such that $U+U \subseteq W$.

Proof:

Since $0+0=0$ and addition is continuous, $\exists$ open neighborhoods V_1, V_2 of 0 such that $V_1+V_2 \subseteq W$.

Set

$$U = V_1 \cap V_2 \cap (-V_1) \cap (-V_2).$$

Then U is symmetric and $U+U \subseteq V_1+V_2 \subseteq W$. ∎

Corollary

We can find a symmetric open neighborhood U such that $U+U+U+U \subseteq W$, etc.

Lemma

Every Cauchy sequence in a TVS X is bounded.

Proof:

Let W be any open neighborhood of 0; without loss of generality we can assume W is balanced. Then $\exists$ balanced open neighborhood V of 0 such that $V + V \subseteq W$. Then $\exists N > 0$ such that $x_m - x_n \in V$ for all $m, n \geq N$. An earlier result shows that $X = \bigcup_{r>0} rV$, so $x_N \in rV$ for some $r > 1$. Then for $n \geq N$ we have

$$x_n \in x_N + V \subseteq rV + rV \subseteq rW.$$

Since $X = \bigcup_{t>1} tW$, we can find some $t > r$ such that $x_1, \ldots, x_N \in tW$, and hence $x_n \in tW$ for all $n \in \mathbb{N}$. Therefore $\{x_n\}$ is a bounded set in X. ∎

Theorem

(a) $\{\varphi_k\}$ is a Cauchy sequence in $C_c^\infty(\mathbb{R})$ if & only if:

i. $\exists$ compact K s.t. $\varphi_k \in C^\infty(K)$ $\forall k$, and

ii. $\{\varphi_k\}$ is Cauchy in $C^\infty(K)$.

(b) $\varphi_k \to 0$ in $C_c^\infty(\mathbb{R})$ if & only if

i. $\exists$ compact K s.t. $\varphi_k \in C^\infty(K)$ $\forall k$, and

ii. $\varphi_k \to 0$ in $C^\infty(K)$, i.e.,

$$\forall n \geq 0, \quad \lim_{k\to\infty} \|\varphi_k^{(n)}\|_\infty = 0.$$

(c) $C_c^\infty(\mathbb{R})$ is complete, i.e., every Cauchy sequence in $C_c^\infty(\mathbb{R})$ converges.

Proof.

(a) By an earlier lemma, a Cauchy sequence $\{\varphi_k\}$ in X must be bounded. A previous theorem then tells us that $\exists$ some compact K such that

$\varphi_k \in C^\infty(K)$ (and $\{\varphi_k\}$ is bounded in $C^\infty(K)$).

Since the topology on $C^\infty(K)$ coincides with the subspace topology inherited from $C_c^\infty(\mathbb{R})$, it follow that $\{\varphi_k\}$ is Cauchy in $C^\infty(K)$.

(b) This is similar to (a), there must exist some compact K such that $\{\varphi_k\}$ is contained in $C^\infty(K)$ and converges to 0 in $C^\infty(K)$.

(c) If $\{\varphi_k\}$ is Cauchy in $C_c^\infty(\mathbb{R})$ then it is Cauchy in some $C^\infty(K)$ and therefore must converge since $C^\infty(K)$ is complete. As convergence in $C^\infty(K)$ implies convergence in $C_c^\infty(\mathbb{R})$, it follows that $C_c^\infty(\mathbb{R})$ is complete. $\blacksquare$

Example

Differentiation $Df = f'$ is a continuous mapping of $C_c^\infty(\mathbb{R})$ into itself.

Proof:

Suppose $\varphi_k \to 0$ in $C_c^\infty(\mathbb{R})$. Then $\exists$ compact K such that $\varphi_k \to 0$ in $C^\infty(K)$. Hence $\forall n \geq 0$,

$$\|(D\varphi_k)^{(n)}\|_\infty = \|\varphi_k^{(n+1)}\|_\infty \to 0 \quad \text{as } k \to \infty,$$

so $D\varphi_k \to 0$ in $C^\infty(K)$ and therefore in $C_c^\infty(\mathbb{R})$.

Alternatively, we can use "continuity = boundedness."

We have

$$\forall \varphi \in C^\infty(K), \quad \|(D\varphi)^{(n)}\|_\infty = \|\varphi^{(n+1)}\|_\infty \quad \forall n \geq 0,$$

which says that D is bounded w.r.t. every seminorm on $C^\infty(K)$, and hence is continuous on $C^\infty(K)$.

4.2.4 $C_c^\infty(\mathbb{R})$ and the Space of Distributions

Now we come to the space of distributions, which is the dual space of $C_c^\infty(\mathbb{R})$. A first guess might be that the topology on $C_c^\infty(\mathbb{R})$ should be induced from the same family of seminorms $\{\rho_{K,n} : n \geq 0, \text{ compact } K \subseteq \mathbb{R}\}$ that induces the topology on $C^\infty(\mathbb{R})$. These seminorms are defined by

$$\rho_{K,n}(f) \;=\; \|f^{(n)}\,\chi_K\|_\infty, \qquad f \in C_c^\infty(\mathbb{R}).$$

This family of seminorms does define a topology on $C_c^\infty(\mathbb{R})$, but it is not the "correct" topology. In particular, $C_c^\infty(\mathbb{R})$ is not complete under this topology (see Problem E.4).

To motivate the "correct" topology on $C_c^\infty(\mathbb{R})$, let us first consider the space $C^\infty(K)$ introduced in equation (4.3), which consists of those infinitely differentiable functions that are supported within a particular compact set $K \subseteq \mathbb{R}$. With K fixed, the countable family of seminorms

$$\{\rho_{K,n}\}_{n\geq 0}$$

induces a metrizable topology on K, and $C^\infty(K)$ is complete with respect to this topology. In this topology, a sequence $\{f_k\}_{k\in\mathbb{N}}$ in $C^\infty(K)$ converges to f in $C^\infty(K)$ if and only if $\lim_{k\to\infty} \rho_{K,n}(f - f_k) = 0$ for each $n \geq 0$. Since each function f_k, f is supported within K,

$$f_k \to f \text{ in } C^\infty(K) \quad\iff\quad \forall\, n \geq 0, \quad \lim_{k\to\infty} \big\|f^{(n)} - f_k^{(n)}\big\|_\infty \;=\; 0.$$

Now consider that, as a set, $C_c^\infty(\mathbb{R})$ is the union of the Fréchet spaces $C^\infty(K)$ over all compact K:

$$C_c^\infty(\mathbb{R}) \;=\; \bigcup \big\{C^\infty(K) : K \subseteq \mathbb{R},\, K \text{ compact}\big\} \tag{4.7}$$

(and, if we like, we need only consider the countably many compact sets $K_m = [-m, m]$ instead of all compact K). Now, to form the topology on $C_c^\infty(\mathbb{R})$, we do not simply throw together all the seminorms from each space $C^\infty(K)$. Instead, we create a topology that is reflective of the union in equation (4.7). This topology is called the *inductive limit* of the topologies on the spaces $C^\infty(K)$, and we will refer to it as the *inductive limit topology on* $C_c^\infty(\mathbb{R})$. The specific definition of open subsets with respect to this topology is discussed in Appendix E, but much more important to us is the definition of the corresponding convergence criterion. This is that $f_k \to f$ in the inductive limit topology on $C_c^\infty(\mathbb{R})$ if and only if there exists a single compact set K such that $f_k \to f$ in $C^\infty(K)$. We formalize this as the following definition.

Definition 4.12 (Convergence in $C_c^\infty(\mathbb{R})$). Given f_k, $f \in C_c^\infty(\mathbb{R})$, we say that f_k *converges to* f *in* $C_c^\infty(\mathbb{R})$, denoted $f_k \to f$ *in* $C_c^\infty(\mathbb{R})$, if

(a) there exists a single compact set $K \subseteq \mathbb{R}$ such that $\operatorname{supp}(f_k) \subseteq K$ for each $k \in \mathbb{N}$, and

(b) $f_k \to f$ in $C^\infty(K)$, i.e.,

$$\forall n \geq 0, \quad \lim_{k\to\infty} \|f^{(n)} - f_k^{(n)}\|_\infty = 0. \tag{4.8}$$

Since convergence in $C_c^\infty(\mathbb{R})$ means convergence in some particular $C^\infty(K)$, which is a metric space, we need only consider convergence of ordinary sequences rather than needing to consider nets.

A sequence $\{f_k\}_{k\in\mathbb{N}}$ is Cauchy in the inductive limit topology on $C_c^\infty(\mathbb{R})$ if there exists a single compact $K \subseteq \mathbb{R}$ such that $f_k \in K$ for each k and $\{f_k\}_{k\in\mathbb{N}}$ is Cauchy in $C^\infty(K)$. Since $C^\infty(K)$ is complete, it follows that there exists an $f \in C^\infty(K)$ such that $f_k \to f$ in $C^\infty(K)$. By definition, we then have $f_k \to f$ in $C_c^\infty(\mathbb{R})$, and thus $C_c^\infty(\mathbb{R})$ is complete. However, the inductive limit topology on $C_c^\infty(\mathbb{R})$ is not metrizable, and $C_c^\infty(\mathbb{R})$ is not a Fréchet space.

A functional μ on $C_c^\infty(\mathbb{R})$ is continuous if and only if $f_k \to f$ in $C_c^\infty(\mathbb{R})$ implies $\langle f_k, \mu\rangle \to \langle f, \mu\rangle$ as $k \to \infty$. This is equivalent to requiring that for each compact $K \subseteq \mathbb{R}$ the restriction of μ to $C^\infty(K)$ be continuous. Since the topology on any particular $C^\infty(K)$ is given by a countable family of seminorms, for each individual compact K we have a "continuity equals boundedness" theorem. However, the constants involved can be different for each compact K. Combining these facts together gives us the following result.

Theorem 4.13. *If $\mu\colon C_c^\infty(\mathbb{R}) \to \mathbb{C}$ is a linear functional on $C_c^\infty(\mathbb{R})$, then the following statements are equivalent.*

(a) *μ is continuous.*

(b) *If $f_k \to 0$ in $C_c^\infty(\mathbb{R})$, then $\langle f_k, \mu\rangle \to 0$ as $k \to \infty$.*

(c) *$\mu|_{C^\infty(K)}$ is continuous for each compact $K \subseteq \mathbb{R}$. That is, if $K \subseteq \mathbb{R}$ is compact and $f_k \in C^\infty(K)$ satisfy $f_k \to 0$ in $C^\infty(K)$, then $\lim_{k\to\infty} \langle f_k, \mu\rangle = 0$.*

(d) *If $f_k \in C_c^\infty(\mathbb{R})$ and there exists a compact $K \subseteq \mathbb{R}$ such that $\operatorname{supp}(f_k) \subseteq K$ for each k and*

$$\forall n \geq 0, \quad \lim_{k\to\infty} \|f_k^{(n)}\|_\infty = 0,$$

then $\lim_{k\to\infty} \langle f_k, \mu\rangle = 0$.

(e) *For each compact $K \subseteq \mathbb{R}$, there exists a constant $C_K > 0$ and an integer $N_K \geq 0$ such that*

$$|\langle f, \mu\rangle| \leq C_K \sum_{n=0}^{N_K} \|f^{(n)}\|_\infty, \qquad f \in C^\infty(K). \tag{4.9}$$

If a single integer N can be used for all compact K (with possibly different C_K), then we say that μ has finite order.

Definition 4.14. If μ is a continuous linear functional on $C_c^\infty(\mathbb{R})$ and there exists a single integer $N \geq 0$ such that for each compact $K \subseteq \mathbb{R}$ there is a constant $C_K > 0$ such that

$$|\langle f, \mu\rangle| \;\le\; C_K \sum_{n=0}^{N} \|f^{(n)}\|_\infty, \qquad f \in C^\infty(K), \tag{4.10}$$

then we say that μ has *finite order*. In this case, the *order of* μ is the smallest integer $N \ge 0$ such that we can find $C_K > 0$ so that equation (4.10) holds for every compact $K \subseteq \mathbb{R}$. If no such N exists, then the order of μ is ∞.

The space of distributions is the set of continuous linear functionals on $C_c^\infty(\mathbb{R})$. Since $C_c^\infty(\mathbb{R})$ is often denoted by $\mathcal{D}(\mathbb{R})$, its dual space is usually denoted by $\mathcal{D}'(\mathbb{R})$.

Definition 4.15 (Distributions). The *space of distributions* $\mathcal{D}'(\mathbb{R})$ is the dual space of $\mathcal{D}(\mathbb{R}) = C_c^\infty(\mathbb{R})$:

$$\mathcal{D}'(\mathbb{R}) \;=\; C_c^\infty(\mathbb{R})^* \;=\; \bigl\{\mu\colon C_c^\infty(\mathbb{R}) \to \mathbb{C} \,:\, \mu \text{ is linear and continuous}\bigr\}.$$

Here are some examples of distributions with finite or infinite order.

Exercise 4.16. (a) Show that $\delta^{(j)} \in \mathcal{D}'(\mathbb{R})$ and that $\delta^{(j)}$ has order j.

(b) Define $\delta_a \in \mathcal{D}'(\mathbb{R})$ by $\langle f, \delta_a\rangle = f(a)$. Define $\mu = \sum_{n\in\mathbb{Z}} \delta_n$ by

$$\langle f, \mu\rangle \;=\; \sum_{n\in\mathbb{Z}} \langle f, \delta_n\rangle \;=\; \sum_{n\in\mathbb{Z}} f(n), \qquad f \in C_c^\infty(\mathbb{R}).$$

Show that $\mu \in \mathcal{D}'(\mathbb{R})$ and μ has order 0, but the constants C_K in equation (4.10) cannot be chosen to be independent of K.

(c) Define $\delta_a^{(j)} \in \mathcal{D}'(\mathbb{R})$ by $\langle f, \delta_a^{(j)}\rangle = (-1)^j f^{(j)}(a)$. Define $\nu = \sum_{n\in\mathbb{N}} \delta_n^{(n)}$ by

$$\langle f, \nu\rangle \;=\; \sum_{n=1}^{\infty} \langle f, \delta_n^{(n)}\rangle \;=\; \sum_{n=1}^{\infty} (-1)^n f^{(n)}(n), \qquad f \in C_c^\infty(\mathbb{R}).$$

Show that $\nu \in \mathcal{D}'(\mathbb{R})$ and that ν has infinite order.

With some poetic license, we can imagine that the distribution $\mu = \sum_{n\in\mathbb{Z}} \delta_n$ defined in Exercise 4.16(b) "looks like" the illustration in Figure 4.1. This picture inspires many names for μ, including the "delta train," the "Dirac comb," and the "Shah distribution," the last because the Cyrillic letter Sha or Shah is written Ш.

We will see in Section 4.7.4 that the delta train equals its own Fourier transform. Moreover, the fact that $\bigl(\sum \delta_n\bigr)^\wedge = \sum \delta_n$ is a distributional version of the Poisson Summation Formula.

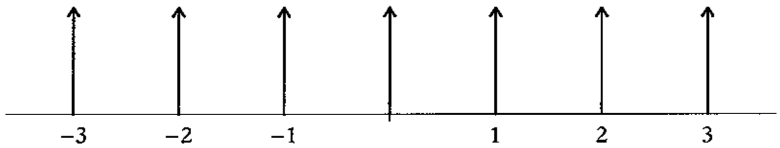

Fig. 4.1. "Graph" of the delta train.

4.2.5 Inclusions

As sets, we have

$$C_c^\infty(\mathbb{R}) \subseteq \mathcal{S}(\mathbb{R}) \subseteq C^\infty(\mathbb{R}).$$

Hence every functional on $C^\infty(\mathbb{R})$ is also a functional on $\mathcal{S}(\mathbb{R})$, and likewise every functional on $\mathcal{S}(\mathbb{R})$ is also a functional on $C_c^\infty(\mathbb{R})$. If there is an appropriate relation between the topologies of these spaces, then there will be a corresponding relation between the continuity of functionals on these spaces.

Exercise 4.17. Prove the following statements.

(a) If $f_k \to f$ in $C_c^\infty(\mathbb{R})$ then $f_k \to f$ in $\mathcal{S}(\mathbb{R})$.

(b) If $f_k \to f$ in $\mathcal{S}(\mathbb{R})$ then $f_k \to f$ in $C^\infty(\mathbb{R})$.

(c) If $\mu \in \mathcal{S}'(\mathbb{R})$ then $\mu|_{C_c^\infty(\mathbb{R})} \in \mathcal{D}'(\mathbb{R})$.

(d) If $\mu \in \mathcal{E}'(\mathbb{R})$ then $\mu|_{\mathcal{S}(\mathbb{R})} \in \mathcal{S}'(\mathbb{R})$.

We show next that each $\mu \in \mathcal{S}'(\mathbb{R})$ is *uniquely* determined by its restriction to $C_c^\infty(\mathbb{R})$. Hence it is safe to identify $\mu \in \mathcal{S}'(\mathbb{R})$ with $\mu|_{C_c^\infty(\mathbb{R})} \in \mathcal{D}'(\mathbb{R})$, and in this sense every $\mu \in \mathcal{S}'(\mathbb{R})$ also belongs to $\mathcal{D}'(\mathbb{R})$. We can likewise identify functionals in $\mathcal{E}'(\mathbb{R})$ with their restriction to $\mathcal{S}(\mathbb{R})$, so in this sense of identification we have the inclusions

$$\mathcal{E}'(\mathbb{R}) \subseteq \mathcal{S}'(\mathbb{R}) \subseteq \mathcal{D}'(\mathbb{R}). \tag{4.11}$$

Problems 4.6 and 4.7 show that these inclusions are proper.

Lemma 4.18. *$C_c^\infty(\mathbb{R})$ is dense in $\mathcal{S}(\mathbb{R})$, i.e., if $f \in \mathcal{S}(\mathbb{R})$ then there exist $f_k \in C_c^\infty(\mathbb{R})$ such that $f_k \to f$ in $\mathcal{S}(\mathbb{R})$. Consequently, $\mu \mapsto \mu|_{C_c^\infty(\mathbb{R})}$ is an injective map of $\mathcal{S}'(\mathbb{R})$ into $\mathcal{D}'(\mathbb{R})$.*

Proof. By the C^∞ Urysohn Lemma (Theorem 1.60), we can find a function $\theta \in C_c^\infty(\mathbb{R})$ such that $0 \le \theta \le 1$ and $\theta = 1$ on $[-1,1]$. For each $k \in \mathbb{N}$ define $\theta_k(x) = \theta(x/k)$. Note that $\theta_k^{(j)}(x) = k^{-j}\,\theta^{(j)}(x/k)$.

Choose any $f \in \mathcal{S}(\mathbb{R})$. Then we have $f\theta_k \in C_c^\infty(\mathbb{R})$, and we will show that $f\theta_k \to f$ in $\mathcal{S}(\mathbb{R})$. Choose m, $n \ge 0$. By the product rule for derivatives,

$$\begin{aligned}
&\|x^m f^{(n)}(x) - x^m (f\theta_k)^{(n)}(x)\|_\infty \\
&\quad = \left\| x^m f^{(n)}(x) - x^m \sum_{j=0}^{n} \binom{n}{j} f^{(j)}(x)\,\theta_k^{(n-j)}(x) \right\|_\infty
\end{aligned}$$

$$
\begin{aligned}
&\le \|x^m f^{(n)}(x) - x^m f^{(n)}(x)\theta_k(x)\|_\infty + \sum_{j=0}^{n-1} \binom{n}{j} \|x^m f^{(j)}(x)\, \theta_k^{(n-j)}(x)\|_\infty \\
&\le \sup_{|x|\ge k} |x^m f^{(n)}(x)| + \sum_{j=0}^{n-1} \binom{n}{j} \|x^m f^{(j)}(x)\|_\infty \frac{\|\theta^{(n-j)}\|_\infty}{k^{n-j}} \\
&\to 0 \quad \text{as } k \to \infty.
\end{aligned}
$$

Since this is valid for every m, $n \ge 0$, we conclude that $f\theta_k \to f$ in $\mathcal{S}(\mathbb{R})$.

Finally, to show injectivity, suppose that $\mu \in \mathcal{S}'(\mathbb{R})$ and $\mu|_{C_c^\infty(\mathbb{R})} = 0$. Then given $f \in \mathcal{S}(\mathbb{R})$ we can find $f_k \in C_c^\infty(\mathbb{R})$ that converge to f in $\mathcal{S}(\mathbb{R})$, so it follows from the continuity of μ that $0 = \langle f_k, \mu\rangle \to \langle f, \mu\rangle$. Therefore $\mu = 0$. □

While it seems natural that $C_c^\infty(\mathbb{R})$ should be dense in $\mathcal{S}(\mathbb{R})$, the next exercise feels much more surprising at first glance: $C_c^\infty(\mathbb{R})$ is dense in $C^\infty(\mathbb{R})$, even though elements of $C_c^\infty(\mathbb{R})$ are compactly supported while functions in $C^\infty(\mathbb{R})$ can be unbounded and growing rapidly at infinity! This goes against our intuition based on our experience with L^p-norm topologies, but is in fact natural once we consider the seminorms that define the topologies on $C_c^\infty(\mathbb{R})$ and $C^\infty(\mathbb{R})$.

Exercise 4.19. Show that $C_c^\infty(\mathbb{R})$, and hence $\mathcal{S}(\mathbb{R})$, is dense in $C^\infty(\mathbb{R})$, and that $\mu \mapsto \mu|_{\mathcal{S}(\mathbb{R})}$ is an injective map of $\mathcal{E}'(\mathbb{R})$ into $\mathcal{S}'(\mathbb{R})$.

In particular, we see again that $C_c^\infty(\mathbb{R})$ is not complete with respect to the $C^\infty(\mathbb{R})$ topology, whereas it is complete with respect to the inductive limit topology.

4.2.6 Convergence of Distributions*

The topology on $\mathcal{D}'(\mathbb{R})$, $\mathcal{S}'(\mathbb{R})$, and $\mathcal{E}'(\mathbb{R})$ is the weak* topology. The weak* topology on the dual of a normed space is discussed in detail in Section E.6. Although our spaces of distributions are not normed vector spaces but rather are duals of metric vector spaces or an inductive limit of such spaces, the differences are small. Choosing $\mathcal{D}'(\mathbb{R})$ as an example, by definition, a sequence $\{\mu_k\}_{k\in\mathbb{N}}$ of distributions converges weak* to $\mu \in \mathcal{D}'(\mathbb{R})$ if

$$\forall\, f \in C_c^\infty(\mathbb{R}), \quad \lim_{k\to\infty} \langle f, \mu_k\rangle = \langle f, \mu\rangle.$$

In many circumstances, this is all we need to know about convergence of distributions. However, the weak* topology is not metrizable, so to properly characterize weak* continuity of operators on $\mathcal{D}'(\mathbb{R})$ we must use convergence of nets instead of sequences (see Lemma A.53). A net in $\mathcal{D}'(\mathbb{R})$ is a sequence $\{\mu_i\}_{i\in I}$ that is indexed by a *directed set* I. Weak* convergence with respect to the directed set I is denoted by $\mu_i \xrightarrow{\text{w}^*} \mu$, and by definition it means that for each $f \in C_c^\infty(\mathbb{R})$, the scalars $\langle f, \mu_i \rangle$ converge to $\langle f, \mu \rangle$ with respect to I. The exact meaning of a directed set and convergence with respect to a directed set is given in Appendix A (see Definition A.42).

When dealing with nets, convergence is still related to continuity in the usual way. For example, if $T\colon \mathcal{D}'(\mathbb{R}) \to \mathcal{D}'(\mathbb{R})$ is given, then T is continuous if and only if given any net $\{\mu_i\}_{i\in I}$ in $\mathcal{D}'(\mathbb{R})$ and any $\mu \in \mathcal{D}'(\mathbb{R})$ we have that $T(\mu_i) \xrightarrow{\mathrm{w}^*} T(\mu)$ whenever $\mu_i \xrightarrow{\mathrm{w}^*} \mu$ (see Lemma A.53).

As an illustration, we show that the embedding of $\mathcal{S}'(\mathbb{R})$ into $\mathcal{D}'(\mathbb{R})$ given in Lemma 4.18 is continuous. That embedding is the map $T\colon \mathcal{S}'(\mathbb{R}) \to \mathcal{D}'(\mathbb{R})$ given by $T(\mu) = \mu|_{C_c^\infty(\mathbb{R})}$. To show continuity, suppose that $\{\mu_i\}_{i\in I}$ is any net in $\mathcal{S}'(\mathbb{R})$ such that $\mu_i \xrightarrow{\mathrm{w}^*} 0$ (it suffices to consider convergence to 0 since T is linear). By definition, $\mu_i \xrightarrow{\mathrm{w}^*} 0$ means that

$$\forall\, f \in \mathcal{S}(\mathbb{R}), \quad \langle f, \mu_i \rangle \;\to\; 0 \quad \text{with respect to } I.$$

Since $C_c^\infty(\mathbb{R}) \subseteq \mathcal{S}(\mathbb{R})$, we therefore have

$$\forall\, f \in C_c^\infty(\mathbb{R}), \quad \langle f, T(\mu_i) \rangle \;=\; \bigl\langle f,\, \mu_i|_{C_c^\infty(\mathbb{R})} \bigr\rangle \;=\; \langle f, \mu_i \rangle \;\to\; 0,$$

where again the convergence is with respect to the directed set I. Thus $T(\mu_i) \xrightarrow{\mathrm{w}^*} 0$, so T is continuous.

Although nets are more general than sequences indexed by $\mathbb{N}$, they behave similarly in many respects, and we refer to Section A.7 for detailed discussion. Henceforth we shall ignore the technical need to use nets to describe weak* convergence and will only consider convergence of sequences.

Additional Problems

4.1. Prove the following "approximate" version of equation (4.2): If $\{k_\lambda\}_{\lambda>0}$ is an approximate identity and $f \in C_b(\mathbb{R})$, then $\lim_{\lambda\to\infty} \int f(x)\, k_\lambda(x)\, dx = f(0)$.

4.2. Show that differentiation is a continuous operation on $C_c^\infty(\mathbb{R})$, i.e., if $f_k \to f$ in $C_c^\infty(\mathbb{R})$ then $f_k' \to f'$ in $C_c^\infty(\mathbb{R})$. Conclude that if $\mu \in \mathcal{D}'(\mathbb{R})$ and $f_k \to f$ in $C_c^\infty(\mathbb{R})$, then $\langle f_k', \mu \rangle \to \langle f', \mu \rangle$. Prove analogous statements for $\mathcal{S}(\mathbb{R})$ and $C^\infty(\mathbb{R})$.

4.3. Prove that translation $T_a f(x) = f(x-a)$, modulation $M_\eta f(x) = e^{2\pi i \eta x} f(x)$, and dilation $D_\lambda f(x) = \lambda f(\lambda x)$ are each continuous operations on $C_c^\infty(\mathbb{R})$, $\mathcal{S}(\mathbb{R})$, and $C^\infty(\mathbb{R})$.

4.4. Given $f \in C_c^\infty(\mathbb{R})$, show that $T_a f \to f$ in $C_c^\infty(\mathbb{R})$ as $a \to 0$, $M_\eta f \to f$ in $C_c^\infty(\mathbb{R})$ as $\eta \to 0$, and $D_\lambda f \to f$ in $C_c^\infty(\mathbb{R})$ as $\lambda \to 1$. Also prove analogous statements for $\mathcal{S}(\mathbb{R})$ and $C^\infty(\mathbb{R})$.

4.5. Given $f \in C_c^\infty(\mathbb{R})$, show that $\frac{f - T_a f}{a} \to f'$ in $C_c^\infty(\mathbb{R})$ as $a \to 0$, and prove analogous statements for $\mathcal{S}(\mathbb{R})$ and $C^\infty(\mathbb{R})$.

4.6. Let $\nu = \sum_{n\in\mathbb{N}} \delta_n^{(n)}$ be the distribution in $\mathcal{D}'(\mathbb{R})$ defined in Exercise 4.16(c). Show that $\nu \notin \mathcal{S}'(\mathbb{R})$, i.e., there is no way to extend ν to a continuous linear functional on $\mathcal{S}(\mathbb{R})$.

4.7. (a) Define $\mu_1 \colon \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ by $\langle f, \mu_1\rangle = \langle f, 1\rangle = \int f(x)\,dx$ for $f \in \mathcal{S}(\mathbb{R})$. Show that $\mu_1 \in \mathcal{S}'(\mathbb{R})$ but μ_1 cannot be extended to belong to $\mathcal{E}'(\mathbb{R})$.

(b) Let $\mu = \sum_{n\in\mathbb{Z}} \delta_n$ be the delta train distribution defined in Exercise 4.16(b). Show that $\mu \in \mathcal{S}'(\mathbb{R})$ but μ cannot be extended to belong to $\mathcal{E}'(\mathbb{R})$.

4.8. Given $g \in C^\infty(\mathbb{R})$, show that $f \mapsto fg$ is a continuous map of $C_c^\infty(\mathbb{R})$ into itself, and is also a continuous map of $C^\infty(\mathbb{R})$ into itself (see Problem 4.14 for an analogous result on the Schwartz space).

4.9. (a) Show that if $g \in L^1(\mathbb{R})$ is compactly supported, then $f \mapsto f * g$ is a continuous map of $C_c^\infty(\mathbb{R})$ into itself. and is also a continuous map of $C^\infty(\mathbb{R})$ into itself.

(b) Show that if $x^m g(x) \in L^1(\mathbb{R})$ for every $m \geq 0$, then $f \mapsto f * g$ is a continuous map of $\mathcal{S}(\mathbb{R})$ into itself.

4.10. Let $k \in L^1(\mathbb{R})$ be compactly supported with $\int k = 1$, and set $k_\lambda(x) = \lambda k(\lambda x)$. Show that if $f \in C_c^\infty(\mathbb{R})$ then $f * k_\lambda \to f$ in $C_c^\infty(\mathbb{R})$ as $\lambda \to \infty$, and prove analogous statements for $\mathcal{S}(\mathbb{R})$ and $C^\infty(\mathbb{R})$.

4.11. Show that $\delta_n \xrightarrow{\text{w*}} 0$ in $\mathcal{D}'(\mathbb{R})$ and in $\mathcal{S}'(\mathbb{R})$ as $|n| \to \infty$. Is this also true in $\mathcal{E}'(\mathbb{R})$?

4.12. This problem will show that there exist linear functionals on $C_c^\infty(\mathbb{R})$ that are not continuous (compare Problem C.8). Fix any nonzero $f \in C_c^\infty(\mathbb{R})$, and set $f_n = \frac{1}{n} T_{1/n} f$. By Problem 1.36, the collection $\{f_n\}_{n\in\mathbb{N}}$ is finitely linearly independent. By Zorn's Lemma, there exists a Hamel basis for $C_c^\infty(\mathbb{R})$ that contains $\{f_n\}_{n\in\mathbb{N}}$, say $\{g_\alpha\}_{\alpha\in I}$ (compare Theorem G.3). Define $\langle g_\alpha, \mu\rangle = 1$ if $g_\alpha = f_n$ for some $n \in \mathbb{N}$; otherwise set $\langle g_\alpha, \mu\rangle = 0$. Extend μ linearly to $C_c^\infty(\mathbb{R})$, and show that the resulting functional is not continuous on $C_c^\infty(\mathbb{R})$.

4.3 Functions as Distributions

Given $1 \leq p < \infty$, each function $g \in L^{p'}(\mathbb{R})$ induces a continuous linear functional μ_g on $L^p(\mathbb{R})$ by the formula

$$\langle f, \mu_g\rangle = \langle f, g\rangle = \int f(x)\,\overline{g(x)}\,dx, \qquad f \in L^p(\mathbb{R}),$$

and the mapping $g \mapsto \mu_g$ is an antilinear isometry of $L^{p'}(\mathbb{R})$ into $L^p(\mathbb{R})^*$ that is surjective if $1 < p < \infty$. In the same way, functions can often induce continuous linear functionals on $C_c^\infty(\mathbb{R})$, $\mathcal{S}(\mathbb{R})$, or $C^\infty(\mathbb{R})$, and hence those functions can be naturally identified with distributions. For this reason, distributions are sometimes referred to as "generalized functions."