See Rudin, Chapter 6

Notation: $C^\infty(K) = C_c^\infty(K)$

# Distributions

The space of <u>test functions</u> is

$$\mathcal{D}(\mathbb{R}) = C_c^\infty(\mathbb{R}) = \left\{ f:\mathbb{R}\to\mathbb{C} \ : \ \begin{array}{l} f \text{ is infinitely diff} \\ \text{and compactly supported} \end{array} \right\}$$

Set

$$C_c^\infty(K) = \{ f \in C_c^\infty(\mathbb{R}) \ : \ \operatorname{supp}(f) \subseteq K \},$$

$K \subseteq \mathbb{R}$ <u>compact</u>.

Note that, as a set,

$$C_c^\infty(\mathbb{R}) = \bigcup_{K \text{ compact}} C_c^\infty(K).$$

Each space $C_c^\infty(K)$ is a locally convex TVS with respect to the family of seminorms $\{\rho_{n,K}\}_{n\geq 0}$

Note $C_c^\infty(K)$ is metrizable since this is a countable family of seminorms.

where

$$\rho_{n,K}(f) = \| f^{(n)} \|_\infty, \qquad f \in C_c^\infty(K)$$

$K$ <u>fixed</u>.

Since $f \in C_c^\infty(K)$, we have

$$\rho_{n,K}(f) = \| f^{(n)} \chi_K \|_\infty.$$

Sometimes this is a better notation for avoiding confusion.

Even though $C_c^\infty(\mathbb{R}) = \bigcup_K C_c^\infty(K)$, the "correct" topology on $C_c^\infty(\mathbb{R})$ is not induced from the family of seminorms

$$\{\rho_{n,K}\}_{n\geq 0,\ K \text{ compact}}. \quad (*)$$

One reason why this is not the "correct" topology is that $C_c^\infty(\mathbb{R})$ is not complete w.r.t. the topology induced from those seminorms. Indeed, recall the following fact.

Theorem
$C^\infty(\mathbb{R})$ is a metrizable locally convex TVS w.r.t. the family of seminorms in $(*)$, or an equivalent countable family is

$$\{\rho_{M,n}\}_{M,n\geq 0} \text{ where } \rho_{M,n}(f) = \|f^{(n)}\cdot \chi_{[-M,M]}\|_\infty.$$

Moreover, $C^\infty(\mathbb{R})$ is complete w.r.t. this topology and $C_c^\infty(\mathbb{R})$ is dense in $C^\infty(\mathbb{R})$.

Example

Fix $\varphi \in C_c^\infty[0,1]$. Then the series

$$\sum_{k=0}^{\infty}{}' 2^{-k}\varphi(x-k)$$

converges in the topology induced from the seminorms (*), but the limit is not in $C_c^\infty(\mathbb{R})$.

In other words, if we set

$$\psi_N(x) = \sum_{k=0}^{N}{}' 2^{-k}\varphi(x-k)$$

then $\{\psi_N\}$ is contained in $C_c^\infty(\mathbb{R})$, is Cauchy w.r.t. the topology from (*), converges in $C^\infty(\mathbb{R})$ w.r.t. this topology, but does not converge in $C_c^\infty(\mathbb{R})$.

Goal

We will define another "natural" topology on $C_c^\infty(\mathbb{R})$ with respect to which $C_c^\infty(\mathbb{R})$ is complete.

Definition

Let $T_K$ be the topology on $C^\infty(K)$ induced from the family $\{\rho_{n,K}\}_{n\geq 0}$. Note $C^\infty(K)$ is a Fréchet space w.r.t. this topology.

Let

$$\beta = \left\{ V \subseteq C_c^\infty(\mathbb{R}) : \begin{array}{l} V \text{ is convex \& balanced and} \\ V \cap C^\infty(K) \in T_K \ \forall K \end{array} \right\}$$

Let

$$T = \left\{ \bigcup_{i\in I} (\varphi_i + V_i) : I \text{ arbitrary}, \ \varphi_i \in C_c^\infty(\mathbb{R}), V \in \beta \right\}$$

This will be the "correct" topology on $X$.

Lemma

If $U \in T$ & $\varphi \in U$ then $\exists\, V \in \beta$ such that

$$\varphi \in \varphi + V \subseteq U.$$

Proof:

By definition of $T$, $\exists\, \psi \in C_c^\infty(\mathbb{R})$ & $W \in \beta$ such that

$$\varphi \in \psi + W \subseteq U.$$

Choose $K$ large enough that it contains the supports of both $\varphi$ & $\psi$, so $\varphi, K \in C^\infty(K)$. Since $\varphi - \psi \in W \cap C^\infty(K)$, which is open in $C^\infty(K)$, we have $\varphi - \psi \in (1-\delta)W$ for some $\delta > 0$. Since $W$ is convex, we therefore have

$$\varphi - \psi + \delta W \subseteq (1-\delta)W + \delta W = W,$$

so

$$\varphi + \delta W \subseteq \psi + W \subseteq U.$$

Hence $V = \delta W$ is the set we seek. ∎

**Theorem**

(a) $\mathcal{T}$ is a topology on $C_c^\infty(\mathbb{R})$.

(b) $\beta$ is a local base for $\mathcal{T}$.

(c) [redacted] Each set

$$A_r^n = \{ f \in C_c^\infty(\mathbb{R}) \ : \ \| f^{(n)} \|_\infty < r \}$$

belongs to $\beta$.

(d) $C_c^\infty(\mathbb{R})$ is a locally convex TVS w.r.t. $\mathcal{T}$.

**Proof:**

(a) By construction, $\emptyset, C_c^\infty(\mathbb{R}) \in \mathcal{T}$ and $\mathcal{T}$ is closed under unions, so it remains to show that $\mathcal{T}$ is closed under finite intersections.

Suppose $U_1, U_2 \in \mathcal{T}$ & $\varphi \in U_1 \cap U_2$. By the lemma, $\exists\, V_1, V_2 \in \beta$ such that

$$\varphi + V_1 \subseteq U_1 \quad \& \quad \varphi + V_2 \subseteq U_2.$$

Then $V = V_1 \cap V_2 \in \beta$, and

$$\varphi + V \subseteq \varphi + V_1 \subseteq U_1 \quad \& \quad \varphi + V \subseteq \varphi + V_2 \subseteq U_2$$

so $\varphi + V \subseteq U_1 \cap U_2$. Consequently $U_1 \cap U_2 \in \mathcal{T}$.

(b) Suppose $U$ is any open neighborhood of $0$.

Then by the lemma $\exists\, V \in \beta$ such that

$0 \in 0 + V \subseteq U$. Hence $\beta$ is a local base

(c) Write

$$A_r^n = \{ f \in C_c^\infty(\mathbb{R}) \;:\; \|f^{(n)}\|_\infty < r \}$$

$$= \bigcup_{K \text{ compact}} \{ f \in C^\infty(K) \;:\; \rho_{n,K}(f) < r \}$$

$$= \bigcup_{K \text{ compact}} \underbrace{B_r^{n,K}(0)}_{\text{open ball in } C^\infty(K)}$$

It follows that $A_r^n \cap C^\infty(K)$ is open in $C^\infty(K)$

for each compact $K$. As $A_r^n$ is balanced &

convex, it belongs to $\beta$.

(d). Fix $\varphi \neq 0$, and set $r = \|\varphi\|_\infty$. Then

$$A_r^\circ = \{ f \in C_c^\infty(\mathbb{R}) : \|f\|_\infty < r \}$$

$\varphi + A_r^\circ$ is open & $0 \notin \varphi + A_r^\circ$. Hence the complement of $\{0\}$ is open, so $\{0\}$ is closed.

By construction, the topology $\mathcal{T}$ is translation-invariant, so $\{\varphi\}$ is closed for every $\varphi \in C_c^\infty(\mathbb{R})$.

To show that vector addition is continuous, choose any $\varphi, \psi \in C_c^\infty(\mathbb{R})$. If $V \in \beta$ then since $V$ is convex we have

$$\left(\varphi + \tfrac{1}{2}V\right) + \left(\psi + \tfrac{1}{2}V\right) = \varphi + \psi + V.$$

Hence

$$(\varphi, \psi) \in (\varphi + \tfrac{1}{2}V) \times (\psi + \tfrac{1}{2}V) \subseteq +^{-1}(\varphi + \psi + V)$$

Since $\beta$ is a local base, this shows that the inverse image under $+$ of any open neighborhood of $\varphi + \psi$ is an open neighborhood of $(\varphi, \psi)$. Therefore $+$ is continuous and a similar argument using both balancedness & convexity shows that $\cdot$ is continuous.

Now we can show how the topology $\mathcal{T}$ for $C_c^\infty(\mathbb{R})$ relates to the topologies $\mathcal{T}_K$ for $C^\infty(K)$.

Theorem

(a) If $V$ is an open subset of $C_c^\infty(\mathbb{R})$, then $V \cap C^\infty(K)$ is open in $C^\infty(K)$ for each compact $K$.

(b) A convex balanced set $V \subseteq C_c^\infty(\mathbb{R})$ is open in $C_c^\infty(\mathbb{R})$ if & only if $V \in \beta$.

(c) The topology $\mathcal{T}_K$ on $C^\infty(K)$ coincides with the topology on $C_c^\infty(\mathbb{R})$ restricted to $C^\infty(K)$

Proof:

(a) Suppose $V$ is open in $C_c^\infty(\mathbb{R})$. Fix $\varphi \in V \cap C^\infty(K)$.

By the lemma, $\exists\, W \in \beta$ such that $\varphi \in \varphi + W \subseteq V$.

Hence
$$\varphi \in \varphi + \big(\underbrace{W \cap C^\infty(K)}_{\text{open in } C^\infty(K) \text{ by def. of } \beta}\big) \subseteq V \cap C^\infty(K)$$

Therefore $V \cap C^\infty(K)$ is open in $C^\infty(K)$.

(b) $\Rightarrow$. Suppose $V$ is convex, balanced, & open. Then by part (a) we have $V \cap C^\infty(K)$ open in $C^\infty(K)$ for every compact $K$, so $V \in \beta$ by definition of $\beta$.

$\Leftarrow$ Every element of $\beta$ is open by definition of $T$.

(c) $\Rightarrow$ If $V \subseteq C_c^\infty(\mathbb{R})$ is open then $V \cap C^\infty(K)$ is open in $C^\infty(K)$ by part (a).

$\Leftarrow$ Suppose that $W \subseteq C^\infty(K)$ is open in $C^\infty(K)$.

Fix $\varphi \in W$. Then $\exists$ base element s.t.

$$\bigcap_{n=0}^{N} B_r^{n,K}(\varphi) \subseteq W.$$

By a previous theorem, $A_r^n \in \beta$, so

$$A_\varphi = \bigcap_{n=0}^{N} A_r^n = \{ f \in C_c^\infty(\mathbb{R}) : \|f^{(n)}\|_\infty < r, \; n = 0, \dots, N \}$$

belongs to $\beta$. Here $N$ & $r$ depend implicitly on $\varphi$

and we have

$$(\varphi + A_\varphi) \cap C^\infty(K) = \bigcap_{n=0}^{N} B_r^{n,K}(\varphi) \subseteq W.$$

Set

$$V = \bigcup_{\varphi \in W} (A_\varphi + \varphi)$$

Then $V$ is open and $V \cap C^\infty(K) = W$.

Hence $W$ is open in the topolog on $C_c^\infty(\mathbb{R})$ restricted to $C^\infty(K)$. ∎

Now we derive some properties of this topology on $C_c^\infty(\mathbb{R})$.

Theorem

$E \subseteq C_c^\infty(\mathbb{R})$ is bounded if & only if

$E \subseteq C^\infty(K)$ for some compact $K$ and

$$\sup_{\varphi \in E} \| \varphi^{(n)} \chi_K \|_\infty < \infty \quad \forall n \geq 0.$$

Proof:

$\Rightarrow$ Suppose that a subset $E$ of $C_c^\infty(K)$ is not contained in any $C^\infty(K)$. Then we can find functions $\varphi_k \in E$ and points $x_k \nearrow \infty$ such that

$$\varphi_k(x_k) \neq 0 \quad \forall k \in \mathbb{N}.$$

Set

$$W = \left\{ \varphi \in C_c^\infty(\mathbb{R}) : |\varphi(x_k)| < \frac{|\varphi_k(x_k)|}{k} \quad \forall k \in \mathbb{N} \right\}$$

Fix any particular compact set $K$. Then $K$ contains finitely many $x_k$, say $x_1, \ldots, x_N$.

Set $r_k = |\varphi_k(x_k)| / k$. Then

$$W \cap C^\infty(K) = \bigcap_{k=1}^{N} \{ \varphi \in C^\infty(K) : |\varphi(x_k)| < r_k \}$$

We must show that $W \cap C^\infty(K)$ is open in $C^\infty(K)$.

Fix any $\varphi \in W \cap C^\infty(K)$. Then

$$\varphi \in \bigcap_{k=1}^{N} \{ \varphi \in C^\infty(K) : \|\varphi\|_\infty < r_k \}$$

$$= \bigcap_{k=1}^{N} B_{r_k}^{0,K}(0)$$

$$\subseteq W \cap C^\infty(K)$$

Since $B_{r_k}^{0,K}(0)$ is an open strip in $C^\infty(K)$, we conclude that $W \cap C^\infty(K)$ is open in $K$. As it is also convex & balanced, we have $W \in \beta$.

In particular, $W$ is an open neighborhood of $0$ in $C_c^\infty(\mathbb{R})$. Yet $\varphi_k \notin kW$, so $E \not\subseteq kW$ for any $k$. Therefore $E$ is unbounded.

Hence, if $E \subseteq C_c^\infty(K)$ is bounded then $E \subseteq C^\infty(K)$ for some compact $K$. Since the topology on $C^\infty(K)$ coincides with the subspace topology inherited from $C_c^\infty(\mathbb{R})$, it follows that $K$ is a bounded subset of $C^\infty(K)$. Since the topology on $C^\infty(K)$ is induced from the family of seminorms $\{\rho_{n,K}\}_{n\geq 0}$, we know that in order to be bounded we must have

$$\sup_{\varphi\in E} \|\varphi^{(n)}\|_\infty < \varepsilon \quad \text{for each } n\geq 0. \quad (*)$$

$\Leftarrow$ If $E \subseteq C^\infty(K)$ and $(*)$ holds then $E$ is bounded in $C^\infty(K)$ & hence is bounded in $C_c^\infty(\mathbb{R})$ by the subspace topology relation. ∎

Definition

A subset $S$ of $C(K)$ is equicontinuous if

$$\forall \varepsilon > 0 \quad \forall x_0 \in K \quad \exists \delta > 0 \text{ s.t.}$$

$$f \in S, \ |x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon.$$

Arzela - Ascoli Theorem

If $K$ is compact in $\mathbb{R}$ and $S \subseteq C(K)$ then

$$S \text{ is totally bounded} \iff S \text{ is bounded \& equicontinuous.}$$

Consequently, in the case $\bar{S}$ is compact,

any every ~~seq~~ sequence $\{f_k\}$ contained in $S$

has a uniformly convergent subsequence.

Definition

We say that a topological vector space has the Heine-Borel property if every closed & bounded subset is compact.

Fact

A locally bounded TVS (one where $\exists$ a bounded neighborhood of $0$) has the Heine-Borel property if & only if it is finite-dimensional.

Theorem

(a) $C^\infty(K)$ has the Heine-Borel property for each compact $K$.

(b) $C_c^\infty(\mathbb{R})$ has the Heine-Borel property.

Proof:

(a) Suppose that $E \subseteq C^\infty(K)$ is closed & bounded. Then

$$M_n = \sup_{\varphi \in E} \|\varphi^{(n)}\|_\infty < \infty \qquad \forall n \geq 0.$$

By the Mean-Value Theorem, if $f$ is differentiable and $f'$ is bounded then $f$ is Lipschitz with

Constant $\|f'\|_\infty$ :

$$|f(x-y)| \leq \|f'\|_\infty \; |x-y|.$$

Consequently all functions in $E$ have a common Lipschitz constant & therefore $E$ is equicontinuous.

Similarly $E^{(n)} = \{\varphi^{(n)} : \varphi \in E\}$ is equicontinuous for every $n$. Applying Arzela-Ascoli & a Cantor diagonalization argument, every sequence $\{f_k\}_{k\in\mathbb{N}}$ in $E$ has a subsequence that converges in $C^\infty(K)$.

Since $C^\infty(K)$ is a complete metric space & $E$ is closed, it follows that $E$ is compact. $\blacksquare$

Corollary

$C^\infty(K)$ & $C_c^\infty(\Omega)$ are not locally bounded, and hence are not normable. (exception: if $K$ has no interior then $C^\infty(K) = \{0\}$ ).