Metrizable TVS

Theorem

Every TVS X that has a countable local base is metrizable. In fact, $\exists$ metric d that induces the topology on X and satisfies

(a) open balls centered at 0 are balanced, and
(b) d is translation-invariant.

If X is locally convex, then we can also construct d so that

(c) open balls are convex.

Proof.

Assume X has a countable local base. Then by an earlier theorem it has a balanced countable local base. We claim that we can construct a balanced local base $\{V_n\}_{n\in\mathbb{N}}$ so that

$$V_{n+1} + V_{n+1} + V_{n+1} + V_{n+1} \subseteq V_n, \qquad n \in \mathbb{N}.$$

Let D denote the set of dyadic rationals in $[0,1)$, i.e.,

$$D = \left\{ \frac{k}{2^n} : n \in \mathbb{N},\ k = 0, \ldots, 2^n - 1 \right\}.$$

Given $r = k/2^n \in D$ we can write r uniquely as

$$r = \sum_{m=1}^{n} \frac{c_m}{2^m} \quad \text{where } c_m = 0 \text{ or } 1.$$

Let A_r be the subset

$$A_r = c_1 V_1 + \cdots + c_n V_n,$$

and also set $A_r = X$ for $r \geq 1$. Define

$$f(x) = \inf \{ r : x \in A_r \}, \qquad x \in X.$$

We will show that

$$d(x,y) = f(x-y), \qquad x, y \in X$$

is the metric on X that we are seeking.

<u>Step 1.</u> We show that $A_r + A_s \subseteq A_{r+s}$ for $r, s \in D$.

If $r + s \geq 1$ then $A_{r+s} = X$ and there is nothing to prove. So, assume $r, s, r+s \in D$. Write

$$r = \sum_{m=1}^{n} \frac{a_m}{2^m}, \quad s = \sum_{m=1}^{n} \frac{b_m}{2^m}, \quad r+s = \sum_{m=1}^{n} \frac{c_m}{2^m}$$

where each a_m, b_m, c_m is 0 or 1.

If $a_m + b_m = c_m$ for every m, then one or both of a_m, b_m is zero for every m. In this case

$$\begin{aligned} A_r + A_s &= (a_1V_1 + \cdots + a_nV_n) + (b_1V_1 + \cdots + b_nV_n) \\ &= (a_1V_1 + b_1V_1) + \cdots + (a_nV_n + b_nV_n) \\ &= c_1V_1 + \cdots + c_nV_n \qquad \text{since either } a_mV_m = 0 \\ &\qquad\qquad\qquad\qquad\qquad \text{or } b_mV_m = 0. \end{aligned}$$

The other possibility is $a_m + b_m = c_m$ for $m = 1, \ldots, N-1$ and $a_N + b_N \neq c_N$ for some N.

This can only happen if

$$a_N = b_N = 0 \quad \& \quad c_N = 1.$$

Illustrations

$$r = \frac{1}{2} + 0 + \frac{1}{8} + 0 + \frac{1}{32}$$
$$s = 0 + \frac{1}{4} + 0 + 0 + \frac{1}{32}$$
$$r+s = \underbrace{\frac{1}{2} + \frac{1}{4} + \frac{1}{8}}_{\substack{a_m + b_m = c_m \\ \text{for } m=1,2,3}} + \frac{1}{16} + 0$$

$N = 4$

$$r = 0 + 0 + \frac{1}{8} + \frac{1}{16}$$
$$s = 0 + \frac{1}{4} + 0 + \frac{1}{16}$$
$$r+s = \frac{1}{2} + 0 + 0 + 0$$

$N = 1$

Consequently,

$$A_r = a_1V_1 + \cdots + a_{N-1}V_{N-1} + 0V_N + a_{N+1}V_{N+1} + \cdots + a_nV_n$$
$$\subseteq a_1V_1 + \cdots + a_{N-1}V_{N-1} + V_{N+1} + \cdots + V_{n-1} + V_n$$
$$\subseteq a_1V_1 + \cdots + a_{N-1}V_{N-1} + V_{N+1} + \cdots + V_{n-1} + V_{n-1}$$
$$\subseteq a_1V_1 + \cdots + a_{N-1}V_{N-1} + V_{N+1} + \cdots + V_{n-2}$$
$$\subseteq$$
$$\vdots$$
$$\subseteq a_1V_1 + \cdots + a_{N-1}V_{N-1} + V_{N+1} + V_{N+1}$$

Combined with a similar calculation for A_s, we obtain

$$A_r + A_s \subseteq a_1V_1 + \cdots + a_{N-1}V_{N-1} + V_{N+1} + V_{N+1}$$
$$+\, b_1V_1 + \cdots + b_{N-1}V_{N-1} + V_{N+1} + V_{N+1}$$

$$\subseteq (a_1V_1 + b_1V_1) + \cdots + (a_{N-1}V_{N-1} + b_{N-1}V_{N-1}) + V_N$$
$$= c_1V_1 + \cdots + c_{N-1}V_{N-1} + c_NV_N$$
$$= A_{r+s}.$$

<u>Step 2.</u> If $r < s$ then since $0 \in A_{s-r}$ we have

$$A_r \subseteq A_r + A_{s-r} \subseteq A_{r+(s-r)} = A_s.$$

Thus $\{A_r\}_{r\in D}$ is linearly ordered by inclusion.

<u>Step 3</u> We claim that f satisfies the triangle inequality, i.e., $f(x+y) \le f(x) + f(y)$ for $x, y \in X$.

Since $f \le 1$ on X, there is nothing to prove if $f(x) + f(y) \ge 1$. Hence we may assume that $f(x) + f(y) < 1$.

Fix $\varepsilon > 0$, and choose any $r, s \in D$ such that

$$f(x) < r < f(x) + \frac{\varepsilon}{2} \quad \& \quad f(y) < s < f(y) + \frac{\varepsilon}{2}.$$

Then $x \in A_r$ and $y \in A_s$ by definition of f and the linear ordering of $\{A_r\}_{r \in D}$. Step 1 therefore implies that $x + y \in A_r + A_s \subseteq A_{r+s}$, so

$$f(x+y) \leq r + s < f(x) + f(y) + \varepsilon.$$

Since ε is arbitrary, $f(x+y) \leq f(x) + f(y)$.

<u>Step 4</u> We will show that $d(x,y) = f(x-y)$ is a translation-invariant metric on X.

By construction, $d(x,y) = f(x-y) \geq 0$ for all x, y.

If $f(x) = 0$ then $x \in A_r$ for every $r \in D$.

In particular, $x \in V_n$ for every n. Since $\{V_n\}_{n \in \mathbb{N}}$ is a local base, this implies that x belongs to every open neighborhood of 0. As X is Hausdorff, this implies that $x=0$. Therefore $d(x,y)=0$ implies $f(x-y)=0$, which implies $x=y$.

Since each V_n is balanced, so is each set A_r. Consequently $f(x)=f(-x)$, so $d(x,y)=d(y,x)$.

The triangle inequality for d follows from the one for f:

$$d(x,y) = f(x-y) \leq f(x-z)+f(z-y) = d(x,z)+d(z,y).$$

Therefore d is a metric. It is translation-invariant because

$$d(x+z,y+z) = f\big((x+z)-(y+z)\big) = f(x-y) = d(x,y).$$

<u>Step 5.</u> We show that d induces the topology on X.

The open balls centered at 0 determined by d are

$$B_\delta(0) = \{x\in X : f(x)<\delta\} = \bigcup_{r<\delta} A_r.$$

Each A_r is open so $B_\delta(0)$ is open in the topology on X. Further, if $\delta < 2^{-n}$ then

$$B_\delta(0) = \bigcup_{r<\delta} A_r \subseteq A_{2^{-n}} = V_n.$$

Hence $\{B_\delta(0)\}_{\delta>0}$ is a local base for the topology on X. Since it is also a local base for the topology induced by d, these two topologies coincide.

Step 6. The open balls $B_\delta(0) = \bigcup_{r<\delta} A_r$ are balanced because each A_r is balanced.

Step 7. If X is locally convex, then we can make the sets V_n convex, & it follows that $B_\delta(0)$ is convex as well. ∎