# The Minkowski Functional

<u>Definition</u>

A subset $A$ of a TVS $X$ is <u>absorbing</u> if

$$\forall x \in X \quad \exists \delta_x > 0 \quad \text{s.t.}$$

$$0 < t < \delta_x \implies tx \in A.$$

<u>Remark</u>

If $A$ is absorbing then by considering the point $x=0$ in the preceding definition we see that we must have $0 \in A$.

<u>Definition</u>

The <u>Minkowski functional</u> or <u>gauge</u> of a set $A$ is

$$\mu_A(x) = \inf \{ t>0 : x \in tA \}.$$

Thus $\mu_A(x)$ measures how much we must expand $A$ in order to contain $x$. For an arbitrary set $A$ we can have $\mu_A(x) = \infty$ for some $x$ (i.e., $x \notin tA$ for any $t>0$), but if $A$ is absorbing then $\mu_A(x) < \infty$ for all $x \in X$, and thus $\mu_A$ is a (nonlinear) functional on $X$.

We illustrate the Minkowski function by evaluating it on an open strip determined by a seminorm $\rho$.

Theorem

If $\rho$ is a seminorm on a vector space $X$, then the "unit strip"

$$B = B_1^{\rho}(0) = \{x \in X : \rho(x) < 1\}$$

is convex, balanced, & absorbing, and its Minkowski functional is $\mu_B = \rho$.

Proof:

The fact that $B$ is convex follows from the Triangle Inequality for $\rho$.

If $x \in B$ & $|c| \leq 1$ then we have

$$\rho(cx) = |c|\,\rho(x) \leq \rho(x) < 1$$

so $cx \in B$. Hence $B$ is balanced.

Suppose that $x \in X$ and $t > \rho(x)$. Then

$\rho(t^{-1}x) = t^{-1}\rho(x) < 1$ so $t^{-1}x \in B$ and hence $x \in tB$.

Therefore $B$ is absorbing, and this also shows that

$\mu_B(x) \le t$. Consequently $\mu_B(x) \le \rho(x)$.

Conversely, if $0 < t \le \rho(x)$ then $\rho(t^{-1}x) \ge 1$ so $t^{-1}x \notin B$, and therefore $\mu_B(x) \ge \rho(x)$. ∎

**Theorem**

Let $A$ be a convex, absorbing subset of a vector space $X$. Then:

(a) $0 \le \mu_A(x) < \infty$

(b) $\mu_A(x+y) \le \mu_A(x) + \mu_A(y)$

(c) $\mu_A(cx) = c\,\mu(x)$ for all $c \ge 0$

(d) If $A$ is balanced then $\mu_A$ is a seminorm on $X$.

(e) If we set $B = \{\mu_A < 1\}$ and $C = \{\mu_A \le 1\}$

then $B \subseteq A \subseteq C$ and $\mu_B = \mu_A = \mu_C$.

**Proof.**

(a) Follows from the definition of absorbing.

(b) Fix $x, y \in X$, and any $t > \mu_A(x)$, $s > \mu_A(y)$.

Then $x \in tA$ & $y \in sA$, so $\frac{x}{t}, \frac{y}{s} \in A$. As $A$ is convex, the line segment between these points is contained in $A$. In particular,

$$\frac{x+y}{s+t} = \frac{t}{s+t} \cdot \frac{x}{t} + \frac{s}{s+t} \cdot \frac{y}{s} \in A$$

so $\mu_A(x+y) \leq s+t$. Taking the inf over all such $s,t$, we obtain $\mu_A(x+y) \leq \mu_A(x) + \mu_A(y)$.

(c) Follows from scaling the definition of the Minkowski functional.

(d) Suppose $A$ is balanced, $x \in X$, & $c \in \mathbb{F}$. Write $c = \alpha|c|$ where $\alpha \in \mathbb{C}$, $|\alpha| = 1$. Then

$$\begin{aligned}
\mu_A(cx) &= \mu_A(|c|\,\alpha x) \\
&= |c|\,\mu_A(\alpha x) \\
&= |c| \inf\{t>0 : \alpha x \in tA\} \\
&= |c| \inf\{t>0 : x \in tA\} \quad \text{by balancing} \\
&= |c|\,\mu_A(x).
\end{aligned}$$

Hence $\mu_A$ is a seminorm.

(e) If $x \in B$ then $\mu_A(x) < 1$ so $x \in tA$ for some $t < 1$. Hence $t^{-1}x \in A$. But $t^{-1} > 1$, so $x$ is on the line segment joining $0$ & $t^{-1}x$. As $A$ is convex, it follows that $x \in A$.

If $x \in A$ then by definition we have $\mu_A(x) \leq 1$ so $x \in C$.

Thus $B \subseteq A \subseteq C$, which implies $\mu_B \leq \mu_A \leq \mu_C$. Fix any $x \in X$, and choose scalars $\mu_C(x) < s < t$. Then $\mu_C(\frac{x}{s}) < 1$ so $\frac{x}{s} \in C$. By definition, this implies $\mu_A(\frac{x}{s}) \leq 1$. But then $\mu_A(\frac{x}{t}) < \mu_A(\frac{x}{s}) \leq 1$, so $\frac{x}{t} \in B$ by definition of $B$. Hence $\mu_B(\frac{x}{t}) \leq 1$, so $\mu_B(x) \leq t$. As $t > \mu_C(x)$ is arbitrary, we conclude $\mu_B(x) \leq \mu_C(x)$. ∎

Theorem

Let $\mathcal{B}$ be a convex balanced local base in a TVS.

(In particular, $\mu_V$ is a seminorm on $X$ for each $V \in \mathcal{B}$.)

Then:

(a) $V = B_1^{\mu_V}(0) = \{x \in X : \mu_V(x) < 1\}$

(b) $\{\mu_V\}_{V \in \mathcal{B}}$ is a separating family of continuous seminorms on $X$.

Note: Separating is the Hausdorff property:
$\mu_V(x) = 0 \ \forall V \in \mathcal{B} \iff x = 0$.

Proof:

(a) If $x \in V$ then since $V$ is open & scalar multiplication is continuous, we have $tx \in V$ for all scalars $t$ with $|t|$ small enough. In particular, $tx \in V$ for some $t > 1$, so $x \in t^{-1}V$ with $t^{-1} < 1$. By definition of $\mu_V$ we therefore have $\mu_V(x) \le t^{-1} < 1$.

Conversely, if $x \notin V$ then, since $V$ is balanced,

$x \in V$ can only happen if $t \geq 1$. Hence $\mu_V(x) \geq 1$.

(b) We know each $\mu_V$ is a seminorm. Note that $\mu_V : X \to [0, \infty)$. If $r > 0$ then

$$\mu_V^{-1}[0,r) = \{x \in X : \mu_V(x) < r\} = rV,$$

which is open.

Consider an open interval $(t-r, t+r) \subseteq [0, \infty)$. We have

$$U := \mu_V^{-1}(t-r, t+r) = \{x \in X : t-r < \mu_V(x) < t+r\}.$$

Fix any particular $x$ in this set. Suppose $y \in cV + x$ (which is an open set). Then

$$\begin{aligned} |\mu_V(x) - \mu_V(y)| &\leq \mu_V(x-y) && \text{Reverse triangle} \\ &< c && \text{since } y - x \in cV \end{aligned}$$

Hence if we choose $c$ small enough then we will have $y \in U$. Thus $cV + x$, which is an open neighborhood of $x$, is contained in $U$. We can do this for each $x \in U$, so $U$ is open. Consequently $\mu_V$ is continuous.

If $\mu_V(x) = 0$ for every $V$ then we have $x \in V$ for every $V \in \mathcal{B}$. Since $\mathcal{B}$ is a local base for the topology on $X$, which is Hausdorff, this ~~...~~ implies that $x = 0$ (why?). Hence $\{\mu_V\}_{V \in \mathcal{B}}$ is separating. $\blacksquare$

<u>Theorem</u>

Suppose $\{\rho_\alpha\}_{\alpha \in J}$ is a family of separating seminorms on a vector space $X$. For $n \in \mathbb{N}$ define

$$V_\alpha(n) = \{x \in X : \rho_\alpha(x) < \tfrac{1}{n}\}. = B^\alpha_{1/n}(0)$$

Let $\mathcal{B}$ be the collection of all finite intersections of the sets $V_\alpha(n)$. Then

(a) $\mathcal{B}$ is a convex balanced local base for a topology $\mathcal{T}$ on $X$,

(b) $X$ is locally convex TVS w.r.t. $\mathcal{T}$,

(c) $\rho_\alpha$ is continuous $\forall \alpha \in J$,

(d) $E \subseteq X$ is bounded $\iff \sup_{x \in E} \rho_\alpha(x) < \infty \quad \forall \alpha \in J$.

<u>Proof:</u>

(a) Let $\mathcal{B}_X$ be the collection of all translates of elements in $\mathcal{B}$, and let $\mathcal{T}$ be the set of all possible unions of elements of $\mathcal{B}_X$. Then $\mathcal{T}$ is a topology, $\mathcal{B}_X$ is a base for $\mathcal{T}$, and $\mathcal{B}$ is a local base for $\mathcal{T}$. Further, each element of $\mathcal{B}$ is convex & balanced since it is an open strip determined by a seminorm.

(b) Since each element of $\mathcal{B}$ is convex, the locally convex requirement is satisfied. We need to show that $X$ is a TVS w.r.t. $\mathcal{T}$.

Suppose $x \neq 0$. Then since our family of seminorms is separating, we have $p_\alpha(x) > 0$ for some $\alpha$. Hence $x \notin V_\alpha(n)$ for some $n$. As $V_\alpha(n)$ is an open neighborhood of $0$, it follows that $x$ is not in the closure of $\{0\}$. This is true for all $x \neq 0$, so the closure of $\{0\}$ is $\{0\}$. That is, $\{0\}$ is closed. By construction the topology $\mathcal{T}$ is translation-invariant, so $\{x\}$ is closed $\forall x \in X$

(this gives us Rudin's Hausdorffness requirement for a TVS).

Let $U$ be an open neighborhood of $0$. Then $U$ contains a base element from $\mathcal{B}$, say

$$\bigcap_{k=1}^{N} V_{\alpha_k}(n_k) \subseteq U.$$

Set

$$V = \bigcap_{k=1}^{N} V_{\alpha_k}(2n_k).$$

Then given $x, y \in V$ we have $\rho_{\alpha_k}(x) < \frac{1}{2n_k}$ and $\rho_{\alpha_k}(y) < \frac{1}{2n_k}$, so $\rho_{\alpha_k}(x+y) < \frac{1}{n_k}$. Hence $x+y \in U$, so $V+V \subseteq U$. Therefore

$$(0,0) \in V \times V \subseteq +^{-1}(U)$$

This shows that vector addition is continuous.

A similar argument shows scalar multiplication is continuous (see Rudin, p. 28).

(c) Since $V_\alpha(n) = B^\alpha_{1/n}(0)$ is open, $\rho_\alpha$ is continuous at 0, and the Reverse Triangle inequality can be used to show $\rho_\alpha$ is continuous at each $x$.

(d) Suppose $E \subseteq X$ is bounded. Since $V_\alpha(1)$ is an open neighborhood of 0, we must have $E \subseteq t V_\alpha(1)$ for some $t>0$. But then for any $x \in E$ we have $\frac{x}{t} \in V_\alpha(1)$, so $\rho_\alpha(\frac{x}{t}) < 1$ and therefore $\rho_\alpha(x) < t$. Thus $\sup_{x \in E} \rho_\alpha(x) < \infty$.

Conversely, suppose each seminorm $\rho_\alpha$ is bounded on $E$. Let $U$ be any open neighborhood of 0. Then as before, $\exists \bigcap_{k=1}^{N} V_{\alpha_k}(n_k) \subseteq U$. Let $M_k = \sup_{x \in E} \rho_{\alpha_k}(x)$. If $t > n_k M_k$ for $k = 1, \ldots, N$ then for any $x \in E$ we have

$$\rho_{\alpha_k}\left(\frac{x}{t}\right) < \frac{M_k}{t} < \frac{1}{n_k},$$

so $\frac{x}{t} \in \mathcal{U}$. Thus $E \subseteq t\mathcal{U}$, so $E$ B

bounded. ∎