<u>Corollary</u>

Assume $\mathcal{B}$ is a convex balanced local base for a TVS. Then the topology on X induced from the family of seminorms $\{\mu_V : V \in \mathcal{B}\}$ is equivalent to the original topology on X.

<u>Proof:</u>

Let T be the original topology on X and T_μ the topology induced from the family of Minkowski functionals. If $V \in \mathcal{B}$ then by a previous theorem $V = B_1^{\mu_V}(0) = \{\mu_V < 1\} = \mu_V^{-1}(-1,1)$, so V is continuous w.r.t. the seminorm topology T_μ. Since $\mathcal{B}$ is a local base for T, we conclude $T \subseteq T_\mu$.

Conversely, μ_V is continuous w.r.t. T by an earlier theorem. Hence every open strip w.r.t. is T-open. As these strips generate the T_μ topology, we have $T_\mu \subseteq T$. $\blacksquare$

generate the $\mathcal{T}_\mu$ topology, we have $\mathcal{T}_\mu \subseteq \mathcal{T}$. ∎

Theorem

Given a TVS X,

X is normable $\Leftrightarrow$ $\exists$ convex bounded neighborhood of 0.

Proof:

$\Rightarrow$ If X is normable then the unit open ball is a convex open bounded neighborhood of 0.

$\Leftarrow$ In this case, $\exists$ convex balanced bounded open neighborhood V of 0. Define

$$\|x\| = \mu_V(x).$$

This is at least a seminorm on X. By a previous theorem, $\{rV\}_{r>0}$ is a local base for the topology on X. Since X is Hausdorff, if $x \neq 0$ then $x \notin rV$ for some $r>0$. Hence $\|x\| = \mu_V(x) \geq r > 0$. Therefore $\|\cdot\|$ is a norm.

The final issue is whether $\|\cdot\|$ induces the topology on X. We have

$$\{x \in X : \|x\| < r\} = rV$$

open in norm topology

open in topology of X

Hence the two topologies coincide. $\blacksquare$

Example

The topology on $C^\infty(\mathbb{R})$ is induced from the countable family of seminorms $\{\rho_{mn}\}_{m,n\geq 0}$ where

$$\rho_{mn}(f) = \|f^{(n)}\chi_{[-m,m]}\|_\infty, \qquad f\in C^\infty(\mathbb{R}).$$

Exercise: $\exists f\in C^\infty(\mathbb{R})$ such that

$$\sup_{n\geq 0}|f^{(n)}(0)| = \infty.$$

The theorem therefore implies that $\{f\}$ is an unbounded set in the B case.

Exercise: Construct $f\in C^\infty$ so that

$$\rho_{1,n}(f) < \delta \text{ for } n=1,\dots,N \quad \& \quad \rho_{1,N+1}(f) > R.$$

Then $B = \bigcap_{n=1}^{N} B_1^{1,n}(0)$ is a local base element but not all seminorms are bounded on B, so B is not a bounded set. The radius 1 can be replaced any radius & $n=1$ by any m, so no local base element is bounded. Hence 0 does not have a convex bounded open neighborhood, so $C^\infty(\mathbb{R})$ is not normable. On the other hand, it is metrizable since its topology is induced from a countable family of seminorms.

<u>Exercise</u>: Show $\mathcal{S}(\mathbb{R})$ is metrizable but not normable. The topology on $\mathcal{S}(\mathbb{R})$ is induced from the seminorms

$$\rho_{mn}(f) = \| x^m f^{(n)}(x) \|_\infty \qquad m, n \geq 0.$$