# Operators on TVS

**Exercise**

Let $L: X \to Y$ be linear, where $X, Y$ are TVS.

(a) $A \subseteq X$ is balanced $\Rightarrow$ $L(A) \subseteq Y$ is balanced

(b) $B \subseteq Y$ is balanced $\Rightarrow$ $L^{-1}(B) \subseteq X$ is balanced

**Theorem**

Let $X$ be a TVS and $\mu: X \to \mathbb{F}$ a linear functional (other than the zero functional). Then TFAE.

(a) $\mu$ is continuous.

(b) $\ker(\mu) = \mu^{-1}\{0\}$ is closed

(c) $\ker(\mu)$ is not dense

(d) $\mu$ is bounded on some open neighborhood of $0$, i.e., $\exists$ open bounded neighborhood $U$ of $0$ such that $\mu(U)$ is a bounded subset of $\mathbb{F}$.

**Proof:**

Exercises: (a) $\Rightarrow$ (b) & (b) $\Rightarrow$ (c).

(c) $\Rightarrow$ (d). Assume $\ker(\mu)$ is not dense in $X$.

Then $\overline{\ker(\mu)}$ is a proper closed subset of $X$,

so $\mathcal{U} = X \setminus \overline{\ker(\mu)}$ is open & nonempty.

Fix any $x \in \mathcal{U}$. Then $\mathcal{U} - x$ is an open neighborhood of $0$, so contains an balanced open neighborhood $V$ of $0$.

Now, $\mu(V)$ is balanced since $V$ is balanced and $\mu$ is linear. Since $\mu(V) \subseteq \mathbb{C}$, by an earlier exercise if $\mu(V)$ is unbounded then $\mu(V) = \mathbb{C}$.

But then $-\langle x, \mu \rangle \in \mu(V)$, so $\exists\, y \in V$ such that $\langle y, \mu \rangle = -\langle x, \mu \rangle$. Hence $y + x \in \ker(\mu)$.

But $y \in V \subseteq \mathcal{U} - x$ so $x + y \in \mathcal{U}$. This is a contradiction since $\mathcal{U} = X \setminus \overline{\ker(\mu)}$.

Hence $\mu(V)$ must be bounded.

(d) $\Rightarrow$ (a). Suppose $\mu$ is bounded on some open neighborhood $\mathcal{U}$ of $0$. Then $\mu(\mathcal{U})$ is a bounded subset of the normed space $\mathbb{F}$, so

$$M = \sup_{x \in \mathcal{U}} |\langle x, \mu \rangle| < \infty.$$

Fix $r > 0$. Then $W = \frac{r}{M+\varepsilon}\mathcal{U}$ is an open neighborhood of $0$, and if $x \in W$ then

$$|\langle x, \mu \rangle| \le M \cdot \frac{r}{M+\varepsilon} < r.$$

Thus

$$\mu^{-1}(B_r(0)) \supseteq W,$$

so $\mu$ is continuous at $0$. Since $\mu$ is linear, it is therefore continuous at every point. $\blacksquare$