D.8 The Riesz Representation Theorem for Positive Linear Functionals on $C_c(\mathbb{R})$

In this section we will discuss one version of the Riesz Representation Theorem, which shows the equivalence between Radon measures and positive linear functionals on $C_c(\mathbb{R})$. This version is only concerned with positive measures and positive functionals, while in Section D.10 we will see a Riesz Representation Theorem for complex Radon measures and bounded functionals.

In this section we deal both with measures and functionals. Typically, we will let μ denote a functional and ν a measure.

Notation D.71. In keeping with the notations introduced in Appendix C (see Notation C.36), we write $\langle f, \mu\rangle$ to denote the action of a linear functional $\mu\colon C_c(\mathbb{R}) \to \mathbb{C}$ on a vector $f \in C_c(\mathbb{R})$. Further, $\langle f, \mu\rangle$ is a sesquilinear form, linear in f but antilinear in μ.

Each Radon measure ν on $\mathbb{R}$ induces an associated linear functional μ on $C_c(\mathbb{R})$ by the formula

$$\langle f, \mu\rangle = \int f\,d\nu, \qquad f \in C_c(\mathbb{R}). \tag{D.7}$$

Note that, by definition, ν is a positive measure here. In order to ensure that $\langle\cdot,\cdot\rangle$ is a sesquilinear form, whenever we extend our consideration to complex measures we will need to replace $d\nu$ by $d\bar{\nu}$ in equation (D.7).

This example immediately raises several questions, which we will address in this section. First, is the functional μ defined in equation (D.7) continuous on $C_c(\mathbb{R})$? Of course, continuity is not even defined until we specify the topology on $C_c(\mathbb{R})$, and, as it turns out, there is more than one "natural" choice.

Second, once we specify the topology on $C_c(\mathbb{R})$, does every continuous linear functional on $C_c(\mathbb{R})$ have the form given in equation (D.7)? In other words, can we characterize the dual space of $C_c(\mathbb{R})$? This question also requires some refinement, since we have specified that Radon measures are positive measures, whereas if we let ν be a complex measure then we can still define a functional μ by equation (D.7).

To address these questions, we next discuss two particular topologies on $C_c(\mathbb{R})$.

D.8.1 Topologies on $C_c(\mathbb{R})$

Since we wish to study the continuity of linear functionals on $C_c(\mathbb{R})$, we must specify a topology or a convergence criterion on $C_c(\mathbb{R})$. There are two natural choices.

(a) *The uniform (or L^∞-norm) topology.* $C_c(\mathbb{R})$ is a normed space with respect to the topology induced by the uniform, or L^∞, norm. A linear functional

μ on $C_c(\mathbb{R})$ is continuous with respect to the uniform topology if and only if it is bounded with respect to the L^∞ norm. That is, μ is continuous if and only if there exists a constant $C > 0$ such that

$$|\langle f, \mu\rangle| \le C\,\|f\|_\infty, \quad \text{all } f \in C_c(\mathbb{R}).$$

Since $C_c(\mathbb{R})$ is a dense subspace of the Banach space $C_0(\mathbb{R})$, such a μ has a unique extension to a bounded linear functional on all of $C_0(\mathbb{R})$, which we also refer to as μ (see Exercise C.26).

(b) *The inductive limit topology.* For each compact $K \subseteq \mathbb{R}$, define

$$C(K) \;=\; \bigl\{f \in C_c(\mathbb{R}) : \operatorname{supp}(f) \subseteq K\bigr\}.$$

Each $C(K)$ is a Banach space with respect to the L^∞-norm. Further, as a set,

$$C_c(\mathbb{R}) \;=\; \bigcup \bigl\{C(K) : K \subseteq \mathbb{R},\ K \text{ compact}\bigr\}.$$

We can define a topology on $C_c(\mathbb{R})$ by declaring that a function whose domain is $C_c(\mathbb{R})$ is continuous if for each compact K its restriction to $C(K)$ is continuous with respect to the L^∞-norm on $C(K)$. In particular, a linear functional $\mu\colon C_c(\mathbb{R}) \to \mathbb{C}$ is continuous with respect to this topology if and only if for each compact K the restriction $\mu|_{C(K)}\colon C(K) \to \mathbb{C}$ is continuous. Since $C(K)$ is a normed space, this happens if and only if each $\mu|_{C(K)}$ is bounded with respect to the norm on $C(K)$, which means that for each compact K there exists a constant $C_K > 0$ such that

$$|\langle f, \mu\rangle| \;\le\; C_K\,\|f\|_\infty, \qquad \text{all } f \in C(K). \tag{D.8}$$

However, unlike boundedness with respect to the uniform topology, where there is a *single* constant C that determines the boundedness, the constants C_K in equation (D.8) can depend on the compact set K. In technical language, this topology corresponds to the *inductive limit* of the topologies $(C(K), \|\cdot\|_\infty)$ with K compact, and hence we will refer to it as the *inductive limit topology* on $C_c(\mathbb{R})$. We refer to [Con90] for details on the inductive limit of topologies. This type of topology is also discussed in Section E.5.

The following definition of the convergence criterion on $C_c(\mathbb{R})$ corresponds to convergence with respect to each of these two topologies.

Definition D.72. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $C_c(\mathbb{R})$.

(a) We say that f_n *converges to* f *uniformly*, or *in* L^∞*-norm*, if $\|f - f_n\|_\infty \to 0$. In this case, we write $f_n \to f$ *uniformly.*

(b) We say that f_n *converges to* f *in* $C_c(\mathbb{R})$ if there exists a compact set K such that $\operatorname{supp}(f_n) \subseteq K$ for all n, and $\|f - f_n\|_\infty \to 0$. In this case, we write $f_n \to f$ *in* $C_c(\mathbb{R})$.

In particular,

$$f_n \to f \text{ in } C_c(\mathbb{R}) \implies f_n \to f \text{ uniformly}. \tag{D.9}$$

However, the converse implication does not hold in general, so these are two distinct topologies on $C_c(\mathbb{R})$. Equation (D.9) implies that the uniform topology on $C_c(\mathbb{R})$ is weaker than the inductive limit topology.

In this section we are focusing on Radon measures (which by definition are positive but possibly unbounded) and corresponding positive linear functionals on $C_c(\mathbb{R})$. For these results it is the inductive limit topology on $C_c(\mathbb{R})$ that will be important. In contrast, in Section D.10 we will consider complex Radon measures (which are necessarily bounded) and corresponding linear functionals on $C_c(\mathbb{R})$, and there it will be the L^∞-topology on $C_c(\mathbb{R})$ that will be important.

D.8.2 Positive Linear Functionals on $C_c(\mathbb{R})$

The next exercise shows that every Radon measure induces a linear functional on $C_c(\mathbb{R})$ that is continuous with respect to the inductive limit topology on $C_c(\mathbb{R})$. Further, this functional is positive in the following sense.

Definition D.73. A functional $\mu\colon C_c(\mathbb{R}) \to \mathbb{C}$ is *positive* if $\langle f, \mu\rangle \geq 0$ for all $f \in C_c(\mathbb{R})$ with $f \geq 0$.

Exercise D.74. Let ν be a Radon measure on $\mathbb{R}$. Define $\mu\colon C_c(\mathbb{R}) \to \mathbb{C}$ by

$$\langle f, \mu\rangle = \int f\,d\nu, \quad f \in C_c(\mathbb{R}).$$

(a) Show that μ is a positive linear functional on $C_c(\mathbb{R})$.

(b) Show that $\mu|_{C(K)}\colon C(K) \to \mathbb{R}$ is continuous for every compact $K \subseteq \mathbb{R}$, i.e.,

$$\begin{gathered} \forall \text{ compact } K \subseteq \mathbb{R}, \quad \exists\, C_K > 0 \text{ such that} \\ f \in C(K) \implies |\langle f, \mu\rangle| \leq C_K \, \|f\|_\infty. \end{gathered} \tag{D.10}$$

Thus, those positive linear functionals on $C_c(\mathbb{R})$ that are induced from Radon measures are continuous with respect to the inductive limit topology on $C_c(\mathbb{R})$. Next we will show directly *every* positive linear functional on $C_c(\mathbb{R})$ is continuous with respect to the inductive limit topology on $C_c(\mathbb{R})$.

Theorem D.75. *If $\mu\colon C_c(\mathbb{R}) \to \mathbb{C}$ is a positive linear functional on $C_c(\mathbb{R})$, then μ is continuous on $C_c(\mathbb{R})$ with respect to the inductive limit topology. Specifically, $\mu|_{C(K)}\colon C(K) \to \mathbb{C}$ is continuous for each compact $K \subseteq \mathbb{R}$.*

Proof. Given a compact set K, by Urysohn's Lemma (Theorem 1.60 or Theorem A.109), we can find $\theta_K \in C_c(\mathbb{R})$ such that $\theta_K \geq 0$ and $\theta_K = 1$ on K.

Suppose first that $f \in C(K)$ is real-valued. Then

$$|f(x)| \;=\; |f(x)|\,\theta_K(x) \;\le\; \|f\|_\infty\,\theta_K(x)$$

for all $x \in \mathbb{R}$. Therefore $\|f\|_\infty\,\theta_K \pm f \ge 0$, and so

$$0 \;\le\; \big\langle \|f\|_\infty\,\theta_K \pm f,\, \mu \big\rangle \;=\; \|f\|_\infty\,\langle \theta_K, \mu\rangle \pm \langle f, \mu\rangle.$$

Consequently,

$$|\langle f, \mu\rangle| \;\le\; \langle \theta_K, \mu\rangle\,\|f\|_\infty.$$

Second, given an arbitrary $f \in C(K)$, we have

$$|\langle f, \mu\rangle| \;\le\; |\langle \mathrm{Re}(f), \mu\rangle| + |\langle \mathrm{Im}(f), \mu\rangle| \;\le\; 2\,\langle \theta_K, \mu\rangle\,\|f\|_\infty.$$

Hence the result follows with $C_K = 2\,\langle \theta_K, \mu\rangle$. □

~~Although we will not prove it, the Riesz Representation Theorem completes the characterization of positive linear functionals on $C_c(\mathbb{R})$: Every positive linear functional on $C_c(\mathbb{R})$ is induced from a Radon measure.~~

Theorem D.76 (Riesz Representation Theorem I). *If $\mu\colon C_c(\mathbb{R}) \to \mathbb{C}$ is a positive linear functional, then there exists a unique positive Radon measure ν on $\mathbb{R}$ such that*

$$\langle f, \mu\rangle \;=\; \int f\,d\nu, \quad f \in C_c(\mathbb{R}).$$

Moreover, if $U \subseteq \mathbb{R}$ is open, then

$$\nu(U) \;=\; \sup\big\{\langle f, \mu\rangle : f \in C_c(\mathbb{R}), 0 \le f \le 1, \mathrm{supp}(f) \subseteq U\big\},$$

and if $K \subseteq \mathbb{R}$ is compact then

$$\nu(K) \;=\; \inf\big\{\langle f, \mu\rangle : f \in C_c(\mathbb{R}), f \ge \chi_K\big\}.$$

Thus, Radon measures and positive linear functionals on $C_c(\mathbb{R})$ are equivalent. Therefore, we often use the same symbol to represent a Radon measure ν and the positive functional $f \mapsto \langle f, \nu\rangle = \int f\,d\nu$ that it induces.

Additional Problems

D.25. This problem will show that the locally finite positive measures on $\mathbb{N}$ (which by Problem D.24 are precisely the Radon measures on $\mathbb{N}$) are in 1-1 correspondence with the positive linear functionals on c_{00}.

(a) Give the convergence criterion corresponding to the inductive limit topology on c_{00}.

(b) Show that if ν is a positive locally finite measure on $\mathbb{N}$, then $\langle f, \nu\rangle = \sum f(k)\,\nu\{k\}$ defines a positive linear functional on c_{00} that is continuous with respect to the inductive limit topology on c_{00}.

(c) Show that if μ is a positive linear functional on c_{00} then there exists a unique sequence of nonnegative scalars $w = (w_k)_{k\in\mathbb{N}}$ such that $\langle f, \mu\rangle = \sum f(k)\,w_k$ for $f \in c_{00}$. Show there is a unique locally finite positive measure ν on $\mathbb{N}$ such that $w_k = \nu\{k\}$ for every k.

Although it seems obvious, is it true that μ is zero on $\mathbb{R}\backslash\text{supp}(\mu)$? By definition, we have

$$\mathbb{R}\backslash\text{supp}(\mu) = \bigcup\{U \subseteq \mathbb{R} : U \text{ is open and } \mu \text{ is zero on } U\}.$$

Hence, if $f \in C_c^\infty(\mathbb{R})$ and $\text{supp}(f) \subseteq \mathbb{R}\backslash\text{supp}(\mu)$, then, since $\text{supp}(f)$ is compact, there exist finitely many open sets $U_1, \dots, U_N$ that cover $\text{supp}(f)$ and are such that μ is zero on each U_k. If f was supported within a single U_k, then we would have $\langle f, \mu\rangle = 0$. However, there is no reason this has to happen, so we need the following lemma.

Lemma 4.50. *Let $U_1, \dots, U_N$ be open subsets of $\mathbb{R}$, and let $K \subseteq U_1 \cup \cdots \cup U_N$ be compact. Then there exist $\theta_k \in C_c^\infty(\mathbb{R})$ with $\text{supp}(\theta_k) \subseteq U_k$ such that $\sum_{k=1}^N \theta_k = 1$ on K.*

Proof. Exercise: For each $k = 1, \dots, N$, construct an open set V_k with compact closure that satisfies $\overline{V_k} \subseteq U_k$, and such that $K \subseteq V_1 \cup \cdots \cup V_N$.

By the C^∞ Urysohn Lemma (Theorem 1.60), for each $k = 1, \dots, N$ we can find a function $\varphi_k \in C_c^\infty(\mathbb{R})$ such that $0 \le \varphi_k \le 1$, $\varphi_k = 1$ on $\overline{V_k}$, and $\text{supp}(\varphi_k) \subseteq U_k$. We can also find a function $\psi \in C_c^\infty(\mathbb{R})$ such that $0 \le \psi \le 1$, $\psi = 1$ on K, and $\text{supp}(\psi) \subseteq V_1 \cup \cdots \cup V_N$. The function

$$\alpha(x) = \bigl(1 - \psi(x)\bigr) + \sum_{k=1}^{N} \varphi_k(x)$$

is then infinity differentiable and everywhere nonzero. Since $\psi = 1$ on K, it follows that the functions

$$\theta_k(x) = \frac{\varphi_k(x)}{\alpha(x)}, \qquad k = 1, \dots, N$$

satisfy $\sum_{k=1}^N \theta_k = 1$ on K. □

Exercise 4.51. Show that if $\mu \in \mathcal{D}'(\mathbb{R})$, then μ is zero on $\mathbb{R}\backslash\text{supp}(\mu)$.

As a consequence of Exercise 4.51, $\text{supp}(\mu)$ is the smallest closed set such that μ is zero on $\mathbb{R}\backslash\text{supp}(\mu)$. In particular, it follows that $\text{supp}(\mu) = \emptyset$ if and only if $\mu = 0$.

Here are some of the basic properties of the support of a distribution.

Exercise 4.52. Prove the following statements.

(a) $\text{supp}(\delta^{(j)}) = \{0\}$ for each $j \ge 0$.

(b) If $g \in L^1_{\text{loc}}(\mathbb{R})$ and μ_g is the distribution that is determined by g, then $\text{supp}(\mu_g) = \text{supp}(g)$ (see Notation 1.20 for the meaning of the support of a function).

(c) If $\mu \in \mathcal{D}'(\mathbb{R})$ then $\text{supp}(\widetilde{\mu}) = -\text{supp}(\mu)$ and $\text{supp}(T_a\mu) = \text{supp}(\mu) + a$ for all $a \in \mathbb{R}$.

Road Map to the Proof of RRT I.

0. Define ν on open sets U by

$$\nu(U) = \sup\{\langle f, \mu\rangle : f \in C_c(\mathbb{R}),\ 0 \le f \le 1,\ \operatorname{supp}(f) \subseteq U\}$$

Define an outer measure ν^* on all $E \subseteq \mathbb{R}$ by

$$\nu^*(E) = \inf\left\{\sum_{k=1}^{\infty} \nu(U_k) : U_k \text{ open},\ E \subseteq \bigcup U_k\right\}$$

Facts:

ν^* is monotonic

ν^* is countably subadditive

$\Sigma_1 = \{E \subseteq \mathbb{R} : E \text{ is } \nu^*\text{-measurable}\}$ is a σ-algebra

$\nu^*|_{\Sigma_1}$ is a positive measure.

1. Show ν is countably $\overset{\text{sub}}{\text{additive}}$ on open sets.

2. Show

$$\nu^*(E) = \inf\{\nu(U) : \text{open } U \supseteq E\}$$

for $E \subseteq \mathbb{R}$.

3. Show

ν is monotonic on open sets

$\nu^*(U) = \nu(U)$ for open U

4. Show every open U is ν^*-measurable.

5. Conclude $\mathcal{B}_\sigma \subseteq \Sigma_r$ and $\nu^*|_{\mathcal{B}_\sigma}$ is a positive Borel measure that agrees with ν on the open sets. Extend ν to a Borel measure by setting $\nu = \nu^*|_{\mathcal{B}_\sigma}$. Note ν is outer regular & satisfies equation (A).

6. Show equation (B) is satisfied.

7. Show ν is a Radon measure.

8. Show $\langle f, \mu \rangle = \int f \, d\nu$ for $f \in C_c(\mathbb{R})$.

9. Uniqueness.

Riesz Representation Theorem I

If $\mu : C_c(\mathbb{R}) \to \mathbb{C}$ is a positive linear functional, then there exists a unique positive Radon measure ν on $\mathbb{R}$ such that

$$\langle f, \mu \rangle = \int f \, d\nu, \qquad f \in C_c(\mathbb{R}).$$

Moreover, if $U \subseteq \mathbb{R}$ is open, then

$$\nu(U) = \sup \{ \langle f, \mu \rangle : f \in C_c(\mathbb{R}),\ 0 \le f \le 1,\ \mathrm{supp}(f) \subseteq U \}, \quad (A)$$

and if $K \subseteq \mathbb{R}$ is compact then

$$\nu(K) = \inf \{ \langle f, \mu \rangle : f \in C_c(\mathbb{R}),\ f \ge \chi_K \}. \quad (B)$$

Proof:

We begin with some motivation. What we know is the values $\langle f, \mu \rangle$ for $f \in C_c(\mathbb{R})$, and we want to find a measure ν so that $\langle f, \mu \rangle = \int f \, d\mu$.

How should we define $\nu(E)$? What we would like to do is to define

$$\nu(E) = \int \chi_E \, d\nu = \langle \chi_E, d\mu \rangle,$$

but this makes no sense since $\chi_E \notin C_c(\mathbb{R})$.

Instead, let us define ν on the open sets by declaring

$$\nu(U) = \sup \{ \langle f, \mu \rangle : f \in C_c(\mathbb{R}),\ 0 \le f \le 1,\ \operatorname{supp}(f) \subseteq U \}$$

for U open (note that this is simply equation (A)).

This only defines ν on the open sets, but there is a standard procedure for defining a measure once the measures of some collection of "elementary sets" has been set. It is a two-step process. First we define an outer measure ν^* on the subsets of $\mathbb{R}$ by setting

$$\nu^*(E) = \inf \left\{ \sum_{k=1}^{\infty} \nu(U_k) : U_k \text{ open},\ E \subseteq \bigcup U_k \right\}$$

for $E \subseteq \mathbb{R}$. This outer measure ν^* is not a measure in general. However, the collection

$$\Sigma_1' = \{E \subseteq \mathbb{R} : E \text{ is } \nu^*\text{-measurable}\}$$

is a σ-algebra, and $\nu^*|_{\Sigma_1'}$ is a measure

(we'll define ν^*-measurable sets when we need them).

The issues are:

a. Is $\mathcal{B}_\sigma \subseteq \Sigma_1'$? (so we get a Borel measure)

b. Does $\nu(U) = \nu^*|_{\Sigma_1'}(U)$ for open U,

i.e., does $\nu^*|_{\Sigma_1'}$ extend ν?

c. Is $\langle f, \mu \rangle = \int f \, d\nu$ for $f \in C_c(\mathbb{R})$?

d. Is $\nu(K) = \inf \{\langle f, \mu \rangle : f \in C_c(\mathbb{R}),\ f \geq \chi_K\}$

for K compact?

e. Is ν a Radon measure?

f. Is ν unique?

We proceed through steps to verify that each required property holds.

Step 1. We will show that ν is countably ~~[illegible]~~ subadditive on open sets.

Suppose that $U_k \subseteq \mathbb{R}$ is open for $k \in \mathbb{N}$, and set $U = \bigcup U_k$. Let $f \in C_c(\mathbb{R})$ be any function such that $\text{supp}(f) \subseteq U$ and $0 \leq f \leq 1$. Then $\{U_k\}_{k \in \mathbb{N}}$ is an open cover of the compact set $K = \text{supp}(f)$, so $\exists$ some N s.t.

$$K \subseteq \bigcup_{k=1}^{N} U_k.$$

By a preceding lemma, we can find $\theta_k \in C_c(\mathbb{R})$ with $\text{supp}(\theta_k) \subseteq U_k$ and $0 \leq \theta_k \leq 1$ such that $\sum_{k=1}^{N} \theta_k = 1$ on K. Each function $f\theta_k$ belongs to $C_c(\mathbb{R})$ and satisfies $\text{supp}(f\theta_k) \subseteq U_k$ and $0 \leq f\theta_k \leq 1$. Further, $f = \sum_{k=1}^{N} f\theta_k$, so by ~~[illegible]~~ applying the definition of ν we have

$$0 \le \langle f, \mu \rangle = \sum_{k=1}^{N} \langle f\theta_k, \mu \rangle$$

$$\le \sum_{k=1}^{N} \nu(U_k)$$

$$\le \sum_{k=1}^{\infty} \nu(U_k).$$

Taking the supremum over all such functions f we obtain

$$\nu(U) \le \sum_{k=1}^{\infty} \nu(U_k).$$

<u>Step 2</u> We will show that the outer measure ν^* satisfies

$$\nu^*(E) = \inf\{\nu(U) : \text{open } U \supseteq E\}, \qquad E \subseteq \mathbb{R}.$$

To see this, let $\rho(E) = \inf\{\nu(U) : \text{open } U \supseteq E\}$.

More precisely, take $U_1 = U$ and $U_k = \emptyset$ for $k > 1$.

Since $U \supseteq E$ is a covering of E by one open set, we have by definition of ν^* that $\nu^*(E) \le \rho(E)$.

For the opposite inequality, let U_k be any open sets such that $E \subseteq \bigcup U_k$, and let $U = \bigcup U_k$.

By Step 1,

$$\rho(E) \le \nu(U) \le \sum_{k=1}^{\infty} \nu(U_k).$$

Taking the supremum over all countable open covers of E, it follows that $\rho(E) \le \nu^*(E)$.

<u>Step 3.</u> Exercises. Show that:

a. U, V open & $U \subseteq V \Rightarrow \nu(U) \le \nu(V)$

b. $\nu^*(U) = \nu(U)$ for all open $U \subseteq \mathbb{R}$.

<u>Step 4</u> We will show that every open set U is ν^*-measurable.

To do this, we must show that

$$\nu^*(E) = \nu^*(E \cap U) + \nu^*(E \setminus U), \quad \text{all } E \subseteq \mathbb{R}.$$

Now, because ν^* is an outer measure it is subadditive. It therefore follows from $E = (E \cap U) \cup (E \setminus U)$ that

$$\nu^*(E) \le \nu^*(E \cap U) + \nu^*(E \setminus U).$$

It is the converse inequality that we must focus upon. And even there we can assume that $\nu^*(E) < \infty$ as the inequality is trivial otherwise.

Suppose first that $E = V$ is an open set, and fix $\varepsilon > 0$.
Then $U \cap V$ is open, so by definition of ν there exists some $f \in C_c(\mathbb{R})$ with $\operatorname{supp}(f) \subseteq U \cap V$ and $0 \le f \le 1$ everywhere such that

$$\langle f, \mu \rangle \ge \nu(U \cap V) - \varepsilon.$$

Let $K = \operatorname{supp}(f)$. Then $V \setminus K$ is open, so $\exists g \in C_c(\mathbb{R})$ with $\operatorname{supp}(g) \subseteq V \setminus K$ and $0 \le g \le 1$ such that

$$\langle g, \mu \rangle \ge \nu(V \setminus K) - \varepsilon.$$

Then we have $f + g \in C_c(\mathbb{R})$, $0 \le f + g \le 1$ since their supports are disjoint, and $\operatorname{supp}(f+g) \subseteq V$.

Therefore,

$$\nu(V) \geq \langle f+g, \mu \rangle$$

$$= \langle f, \mu \rangle + \langle g, \mu \rangle$$

$$\geq \nu(U \cap V) - \varepsilon + \nu(V \setminus K) - \varepsilon$$

$$= \nu^*(U \cap V) + \nu^*(V \setminus K) - 2\varepsilon$$

$$\geq \nu^*(V \cap U) + \nu^*(V \setminus U) - 2\varepsilon$$

Since ε is arbitrary and $\nu^*(V) = \nu(V)$ by Step 3, we obtain the desired inequality for V.

Now let E be an arbitrary subset of $\mathbb{R}$ with $\nu^*(E) < \infty$. By Step 2, $\exists$ an open $V \supseteq E$ such that $\nu(V) < \nu^*(E) + \varepsilon$.

Since ν^* is monotonic, we therefore have

$$\nu^*(E) > \nu(V) - \varepsilon$$

$$= \nu^*(V \cap U) + \nu^*(V \setminus U) - \varepsilon$$

$$\geq \nu^*(E \cap E) + \nu^*(E \setminus U) - \varepsilon.$$

Letting $\varepsilon \to 0$ therefore gives the desired inequality.

<u>Step 5.</u> The class of ν^*-measurable sets forms a σ-algebra, and we just showed that every open set is ν^*-measurable, so we conclude that every Borel set is ν^*-measurable.

Since ν^* <u>restricted to</u> the ν^*-measurable sets is a measure, it follows that $\nu^*|_{\mathcal{B}_\sigma}$ is a positive Borel measure. Further,

$\nu(U) = \nu^*|_{\mathcal{B}_\sigma}(U) = \nu^*(U)$ for all open sets, so we can extend ν to a positive Borel measure by declaring that $\nu(E) = \nu^*(E)$ for all $E \in \mathcal{B}_\sigma$.

Note that by Step 2, the measure ν is outer regular. Further, by definition it satisfies equation (A).

<u>Step 6.</u> We will show that ν satisfies equation (B).

Suppose that $K \subseteq \mathbb{R}$ is compact and $f \in C_c(\mathbb{R})$ satisfies $\chi_K \leq f$. Set

$$U_\varepsilon = \{ f > 1 - \varepsilon \} = f^{-1}(1-\varepsilon, \infty).$$

Since f is continuous, U_ε is open, and $K \subseteq U_\varepsilon$.

Suppose $g \in C_c(\mathbb{R})$, $0 \leq g \leq 1$, and $\operatorname{supp}(g) \subseteq U_\varepsilon$. Then $f \geq (1-\varepsilon) g$, so $f - (1-\varepsilon) g \geq 0$. Since μ is a positive linear functional we conclude that

$$\langle f - (1-\varepsilon) g, \mu \rangle \geq 0$$

and therefore

$$\langle f, \mu \rangle \geq (1-\varepsilon) \langle g, \mu \rangle.$$

Hence

$$\nu(K) \leq \nu(U_\varepsilon) \leq \langle g, \mu \rangle \leq \frac{1}{1-\varepsilon} \langle f, \mu \rangle.$$

Letting $\varepsilon \to 0$ we obtain $\nu(K) \leq \langle f, \mu \rangle$, so

$$\nu(K) \leq \inf\{\langle f,\mu\rangle : f \in C_c(\mathbb{R}),\ f \geq \chi_K\}.$$

On the other hand, if $U \supseteq K$ is open, then by Urysohn's Lemma $\exists f \in C_c(\mathbb{R})$ with $f = 1$ on K, $\operatorname{supp}(f) \subseteq U$, and $0 \leq f \leq 1$ everywhere. By definition of ν we therefore have

$$\langle f,\mu\rangle \leq \nu(U).$$

Hence

$$\begin{aligned}
&\inf\{\langle f,\mu\rangle : f \in C_c(\mathbb{R}),\ f \geq \chi_K\} \\
&\qquad \leq \inf\{\nu(U) : \text{open } U \supseteq K\} \\
&\qquad = \nu(K),
\end{aligned}$$

the last equality following from the fact that ν is outer regular.

Step 7. We show that ν is a Radon measure.

We already know that ν is outer regular.

Since $\langle f, \mu \rangle$ is a finite nonnegative real number for all $f \geq 0$, it follows from equation (B) that $\nu(K) < \infty$ for all compact K. Hence ν is locally finite.

Finally we must show that ν is inner regular on each open set U. Given $\varepsilon > 0$, by definition of ν there exists some $f \in C_c(\mathbb{R})$ with $\mathrm{supp}(f) \subseteq U$, $0 \leq f \leq 1$, and $\langle f, \mu \rangle \geq \nu(U) - \varepsilon$. Let $K = \mathrm{supp}(f)$. If $g \in C_c(\mathbb{R})$ and $g \geq \chi_K$ then $g - f \geq 0$, so $\langle f, \mu \rangle \leq \langle g, \mu \rangle$. Hence

$$\begin{aligned}
\nu(K) &= \inf \{ \langle g, \mu \rangle : g \in C_c(\mathbb{R}),\ g \geq \chi_K \} \\
&\geq \langle f, \mu \rangle \\
&\geq \nu(U) - \varepsilon.
\end{aligned}$$

Step 8. We will show that $\langle f, \mu \rangle = \int f \, d\nu$ for $f \in C_c(\mathbb{R})$.

Suppose first that $f \in C_c(\mathbb{R})$ and $0 \le f \le 1$.

Fix $N \in \mathbb{N}$ and set

$$K_j = \left\{ f \ge \frac{j}{N} \right\}, \qquad j = 1, \dots, N$$

and

$$K_0 = \operatorname{supp}(f).$$

Note that each set K_j is compact. Define

$$f_j = \min\left\{ \left(f - \frac{j-1}{N}\right)^+, \ \frac{1}{N} \right\}, \qquad j = 1, \dots, N.$$

Note that $f_j \in C_c(\mathbb{R})$ and $0 \le f_j \le \frac{1}{N}$.

If $x \in K_j$ then

$$f(x) - \frac{j-1}{N} \ge \frac{j}{N} - \frac{j-1}{N} = \frac{1}{N},$$

so $f_j(x) = \frac{1}{N}$. Thus $f_j \ge \frac{1}{N}$ on K_j, or

$$\frac{1}{N} \chi_{K_j} \le f_j.$$

By equation (B), the fact that $\chi_{K_j} \leq Nf_j$ implies that

$$\nu(K_j) \leq N \langle f_j, \mu \rangle.$$

If $U \supseteq K_{j-1}$ is open, then $\operatorname{supp}(Nf_j) \subseteq K_{j-1} \subseteq U$, so by definition of ν we have

$$N\langle f_j, \mu \rangle \leq \nu(U).$$

By outer regularity, this implies

$$N\langle f_j, \mu\rangle \leq \nu(K_{j-1}).$$

Thus

$$\frac{1}{N}\nu(K_j) \leq \langle f_j, \mu\rangle \leq \frac{1}{N}\nu(K_{j-1})$$

so summing over j yields

$$\frac{1}{N}\sum_{j=1}^{N} \nu(K_j) \leq \langle f, \mu\rangle \leq \frac{1}{N}\sum_{j=0}^{N-1} \nu(K_j). \qquad (**)$$

Comparing equations (*) & (**), we see that

$$\left| \langle f, \mu \rangle - \int f\, d\nu \right| \leq \frac{\nu(K_0) - \nu(K_N)}{N} \leq \frac{\nu(\operatorname{supp}(f))}{N}.$$

If $f_j(x) \neq 0$ then we must have

$$f(x) - \frac{j-1}{N} > 0$$

and hence $x \in K_{j-1}$. Therefore

$$f_j \leq \frac{1}{N} \chi_{j-1}.$$

Exercise: Similar reasoning shows that

$$f = \sum_{j=1}^{N} f_j.$$

Therefore

$$\begin{aligned}
\frac{1}{N}\sum_{j=1}^{N} \nu(K_j) &= \frac{1}{N}\sum_{j=1}^{N} \int \chi_{K_j}\, d\nu \\
&\leq \sum_{j=1}^{N} \int f_j \, d\nu \\
&= \int f \, d\nu \qquad (*) \\
&= \sum_{j=1}^{N} \int f_j \, d\nu \\
&\leq \frac{1}{N}\sum_{j=1}^{N} \int \chi_{K_{j-1}}\, d\nu \\
&= \frac{1}{N}\sum_{j=0}^{N-1} \nu(K_j).
\end{aligned}$$

Since N is arbitrary, we conclude that

$\langle f, \mu \rangle = \int f \, d\nu.$

Finally, a general $f \in C_c(\mathbb{R})$ can be written as $f = ag - bh$ where $0 \le g, h \le 1$, so the equality $\langle f, \mu \rangle = \int f \, d\nu$ holds for f.

Step 9. Uniqueness.

Suppose ρ is another Radon measure that has the property that $\langle f, \mu \rangle = \int f \, d\rho$ for $f \in C_c(\mathbb{R})$.

If $f \in C_c(\mathbb{R})$ is any function with $\mathrm{supp}(f) \subseteq U$ and $0 \le f \le 1$, then

$$0 \le \langle f, \mu \rangle = \int f \, d\rho \le \int \chi_U \, d\rho = \rho(U).$$

On the other hand, if K is any compact subset of U then by Urysohn's Lemma $\exists$ some $f \in C_c(\mathbb{R})$

such that $f = 1$ on K, $\text{supp}(f) \subseteq U$, and $0 \leq f \leq 1$.

Hence

$$0 \leq \rho(K) = \int \chi_K \, d\rho \leq \int f \, d\rho = \langle f, \mu \rangle.$$

Since ρ is inner regular on U, we therefore have $\rho(U) \leq \langle f, \mu \rangle$. Hence

$$\rho(U) = \inf\{\langle f, \mu \rangle : f \in C_c(\mathbb{R}),\ \text{supp}(f) \subseteq U,\ 0 \leq f \leq 1\}$$
$$= \nu(U).$$

Thus ρ & ν agree on the open sets, & hence $\rho = \nu$ since they are both Borel measures. ∎