D.10 The Dual of $C_0(\mathbb{R})$

In this section we will see that the dual space of $C_0(\mathbb{R})$ can be identified with the space of complex Radon measures on the real line.

Radon measures, as discussed so far, are positive by definition. We extend the definition to signed and complex measures as follows. As usual, ν^+, ν^- denote the positive and negative parts of a signed measure ν, and ν_r, ν_i denote the real and imaginary parts of a complex measure ν.

Definition D.80. A signed Borel measure ν on $\mathbb{R}$ is a *signed Radon measure* on $\mathbb{R}$ if ν^+, ν^- are Radon measures.

A complex Borel measure ν on $\mathbb{R}$ is a *complex Radon measure* on $\mathbb{R}$ if ν_r, ν_i are signed Radon measures.

Because of the properties of the real line, these notions simplify as follows.

Lemma D.81. *The following statements are equivalent.*

(a) *ν is a bounded signed Borel measure on $\mathbb{R}$.*

(b) *ν is a bounded signed Radon measure on $\mathbb{R}$.*

The following statements are also equivalent.

(a') *ν is a complex Borel measure on $\mathbb{R}$.*

(b') *ν is a complex Radon measure on $\mathbb{R}$.*

Proof. A measure ν is a bounded signed Borel measure if and only if ν^+, ν^- are bounded positive Borel measures. By Theorem D.78, this happens if and only if ν^+, ν^- are bounded Radon measures, which is equivalent to ν being a bounded signed Radon measure.

A similar argument applies to complex measures, noting that all complex measures are bounded. □

Consequently, the Banach space $M_b(\mathbb{R})$ of all complex Borel measures on $\mathbb{R}$ introduced in Definition D.61 coincides with the space of all complex Radon measures on $\mathbb{R}$. For other domains, the distinction between these two spaces becomes important.

The following exercise shows that if ν is a complex Radon measure (which is necessary bounded), then it induces a linear functional on $C_c(\mathbb{R})$ that is continuous with respect to the uniform topology, and hence this linear functional extends to a continuous linear functional on $C_0(\mathbb{R})$.

Exercise D.82. Assume ν is a complex Radon measure, and let $\bar{\nu}$ be the complex conjugate measure defined in Problem D.18. Define a functional μ on $C_c(\mathbb{R})$ by

$$\langle f, \mu \rangle = \int f \, d\bar{\nu}, \qquad f \in C_c(\mathbb{R}),$$

and prove the following statements.

(a) μ is bounded on $C_c(\mathbb{R})$ with respect to the L^∞-norm, and furthermore

$$|\langle f, \mu\rangle| \;\le\; \|f\|_\infty \|\nu\|, \quad f \in C_c(\mathbb{R}), \tag{D.11}$$

where $\|\nu\| = |\nu|(\mathbb{R})$ is the norm of the measure ν.

(b) The operator norm of μ is $\|\mu\| = |\nu|(\mathbb{R}) = \|\nu\|$.

(c) μ extends to a bounded linear functional on $C_0(\mathbb{R})$, also defined by $\langle f, \mu\rangle = \int f\, d\bar{\nu}$ for $f \in C_0(\mathbb{R})$.

~~Although we will not prove it,~~ the Riesz Representation Theorem states that every bounded linear functional is induced from a complex Radon measure, as follows.

Theorem D.83 (Riesz Representation Theorem II). *Given $\nu \in M_b(\mathbb{R})$, define $\mu_\nu \colon C_0(\mathbb{R}) \to \mathbb{C}$ by*

$$\langle f, \mu_\nu\rangle \;=\; \int f\, d\bar{\nu}, \qquad f \in C_0(\mathbb{R}).$$

Then $\mathcal{U} \colon \nu \mapsto \mu_\nu$ is an antilinear isometry of $M_b(\mathbb{R})$ onto $C_0(\mathbb{R})^*$.

Thus, $C_0(\mathbb{R})^* \cong M_b(\mathbb{R})$. We often write $C_0(\mathbb{R})^* = M_b(\mathbb{R})$, meaning equality in the sense of the identification given in Theorem D.83. Since $C_c(\mathbb{R})$ is dense in $C_0(\mathbb{R})$ in the uniform topology, this implies that $C_c(\mathbb{R})^* \cong M_b(\mathbb{R})$ with respect to the uniform topology on $C_c(\mathbb{R})$. Therefore, a complementary development of complex Radon measures could have started by *declaring* a complex Radon measure to be an element of the dual space of $C_c(\mathbb{R})$ or $C_0(\mathbb{R})$ with respect to the uniform topology, and then showing that every such linear functional induces a corresponding complex measure.

Although we will not discuss this further, we close this appendix by commenting that the space of unbounded complex Radon measures can be defined to be the dual space of $C_c(\mathbb{R})$ with respect to the inductive limit topology on $C_c(\mathbb{R})$. Indeed, Theorem D.76 shows that the positive Radon measures correspond exactly to the positive linear functionals on $C_c(\mathbb{R})$ that are continuous with respect to this inductive limit topology.

Additional Problems

D.26. Show directly that if ν is an unbounded Radon measure on $\mathbb{R}$, then there exist $f_n \in C_c(\mathbb{R})$ with $f_n \ge 0$ and $\|f_n\|_\infty \le 1$ such that $\langle f_n, \nu\rangle \to \infty$ as $n \to \infty$.

D.27. Given $f_n,\ f \in C_0(\mathbb{R})$, show that $f_n \xrightarrow{\text{w}} f$ if and only if $f_n(x) \to f(x)$ pointwise for each $x \in \mathbb{R}$ and $\sup \|f_n\|_\infty < \infty$ (weak convergence of sequences is discussed in Section C.16).

D.28. Given $g = (g_k)_{k\in\mathbb{N}} \in \ell^1$, define $\mu_g \colon c_0 \to \mathbb{C}$ by $\langle f, \mu_g\rangle = \sum f_k \overline{g_k}$. Show that $g \mapsto \mu_g$ is an antilinear isometric isomorphism of ℓ^1 onto $c_0{}^*$. Thus $c_0{}^* \cong \ell^1$. Compare this to Problem D.22, which showed that $M_b(\mathbb{N}) \cong \ell^1$. Thus we see directly that the Riesz Representation Theorem holds for c_0, i.e., $c_0{}^* \cong M_b(\mathbb{N})$.

Riesz Representation Theorem II

Our goal is to characterize the dual space of $C_0(\mathbb{R})$. We have already characterized the positive linear functionals on $C_c(\mathbb{R})$, and we will make use of this to characterize arbitrary bounded linear functionals on $C_0(\mathbb{R})$.

Our first lemma is a kind of "Jordan Decomposition" for real-valued functionals on $C_0(\mathbb{R})$.

Lemma
Let $\mathbb{F} = \mathbb{R}$, so all functionals are real-valued.

If $\mu \in C_0(\mathbb{R})^*$ then $\exists$ positive functionals $\mu^+, \mu^- \in C_0(\mathbb{R})^*$ such that $\mu = \mu^+ - \mu^-$.

Proof:
In this proof, all functions f, g, h, etc. will implicitly belong to $C_0(\mathbb{R})$.

Fix $\mu \in C_0(\mathbb{R})^*$, and recall μ is real-valued.

For $f \geq 0$ define

$$\langle f, \mu^+ \rangle = \sup\{\langle g, \mu \rangle : 0 \leq g \leq f\}.$$

Note that μ^+ satisfies

$$|\langle f, \mu^+ \rangle| \leq \sup\{|\langle g, \mu \rangle| : 0 \leq g \leq f\}$$

$$\leq \sup\{\|g\|_\infty \|\mu\| : 0 \leq g \leq f\}$$

$$= \|f\|_\infty \|\mu\| \qquad (*)$$

Step 1. We will show that μ^+ is additive on nonnegative functions.

Fix $f_1, f_2 \geq 0$. Given any $0 \leq g_1 \leq f_1$, $0 \leq g_2 \leq f_2$, we have

$$\langle g_1, \mu \rangle + \langle g_2, \mu \rangle = \langle g_1 + g_2, \mu \rangle \leq \langle f_1 + f_2, \mu^+ \rangle.$$

Taking the suprema over all $0 \leq g_1 \leq f_1$, we have

$$\langle f_1, \mu^+ \rangle + \langle g_2, \mu \rangle \leq \langle f_1 + f_2, \mu^+ \rangle,$$

and then taking the suprema over $0 \leq g_2 \leq f_2$ gives

$$\langle f_1, \mu^+ \rangle + \langle f_2, \mu^+ \rangle \le \langle f_1 + f_2, \mu^+ \rangle$$

To establish the opposite inequality, choose any $0 \le g \le f_1 + f_2$. Set $g_1 = \min\{g, f_1\}$ and $g_2 = g - g_1$. Then $0 \le g_1 \le f_1$, $0 \le g_2 \le f_2$, so

$$\langle g, \mu \rangle = \langle g_1, \mu \rangle + \langle g_2, \mu \rangle \quad (\mu \text{ is linear})$$

$$\le \langle f_1, \mu^+ \rangle + \langle f_2, \mu^+ \rangle.$$

Taking the supremum over all $0 \le g \le f_1 + f_2$ gives

$$\langle f_1 + f_2, \mu^+ \rangle \le \langle f_1, \mu^+ \rangle + \langle f_2, \mu^+ \rangle.$$

<u>Step 2</u>. Exercise: $\langle cf, \mu^+ \rangle = c \langle f, \mu^+ \rangle$ for $f \ge 0$, $c \ge 0$.

<u>Step 3</u>. We extend μ^+ to $C_0(\mathbb{R})$.

Given an arbitrary $f \in C_0(\mathbb{R})$, write

$f = f^+ - f^-$ (recall f is real-valued),

and define

$$\langle f, \mu^+ \rangle = \langle f^+, \mu^+ \rangle - \langle f^-, \mu^+ \rangle$$

This is a real number, and we claim that

$\langle f, \mu^+ \rangle = \langle g, \mu^+ \rangle - \langle h, \mu^+ \rangle$ whenever $g, h \geq 0$

satisfy $f = g - h$. For, in this case we have

$f = f^+ - f^- = g - h$, so

$$f^+ + h = g + f^- \geq 0$$

and therefore

$$\langle f^+, \mu^+ \rangle + \langle h, \mu^+ \rangle$$

$$= \langle f^+ + h, \mu^+ \rangle \quad (\mu^+ \text{ linear on nonneg functions})$$

$$= \langle g + f^-, \mu^+ \rangle$$

$$= \langle g, \mu^+ \rangle + \langle f^-, \mu^+ \rangle.$$

Rearranging gives

$$\langle f, \mu^+ \rangle \stackrel{\text{def}}{=} \langle f^+, \mu^+ \rangle - \langle f^-, \mu^- \rangle = \langle g, \mu^+ \rangle - \langle h, \mu^+ \rangle.$$

Step 4: We show μ^+ is additive on $C_0(\mathbb{R})$.

Fix any $f, g \in C_0(\mathbb{R})$. Then

$$f+g = (f^+ + g^+) - (f^- + g^-)$$

and $f^+ + g^+ \geq 0$, $f^- + g^- \geq 0$, so

$$\langle f+g, \mu^+ \rangle = \langle f^+ + g^+, \mu^+ \rangle - \langle f^- + g^-, \mu^+ \rangle \quad \text{(Step 3)}$$

$$= \langle f^+, \mu^+ \rangle + \langle g^+, \mu^+ \rangle - \langle f^-, \mu^+ \rangle - \langle g^-, \mu^+ \rangle$$

$$= \langle f, \mu^+ \rangle + \langle g, \mu^+ \rangle.$$

Step 5

Exercise: $\langle cf, \mu^+ \rangle = c \langle f, \mu^+ \rangle$ for $f \in C_0(\mathbb{R})$, $c \in \mathbb{R}$,

so μ^+ is linear on $C_0(\mathbb{R})$.

Step 6 If $f \in C_0(\mathbb{R})$ then by equation $(*)$

$$|\langle f, \mu^+ \rangle| \leq |\langle f^+, \mu^+ \rangle| + |\langle f^-, \mu^+ \rangle|$$

$$\leq \|f^+\|_\infty \|\mu\| + \|f^-\|_\infty \|\mu\|$$

$$\leq 2 \|f\|_\infty \|\mu\|,$$

so μ^+ is bounded on $C_0(\mathbb{R})$.

Step 7. Set $\mu^- = \mu^+ - \mu$.

We have $\mu^- \in C_0(\mathbb{R})^*$ since μ, μ^+ are bounded.

By construction, μ^+ is a positive functional (just consider $g=0$ in the definition of μ^+). Also, if $f \geq 0$ then

$$\begin{aligned}\langle f, \mu^- \rangle &= \sup\{\langle g, \mu \rangle : 0 \leq g \leq f\} - \langle f, \mu \rangle \\ &= \sup\{\langle g - f, \mu \rangle : 0 \leq g \leq f\} \\ &\geq 0 \quad (\text{consider } g = f),\end{aligned}$$

so μ^- is positive as well. $\blacksquare$

Recall that

$$M_b(\mathbb{R}) = \{\text{complex Borel measures}\}$$
$$= \{\text{complex Radon measures}\}$$

All measures in $M_b(\mathbb{R})$ are bounded, and $M_b(\mathbb{R})$ is a Banach space with respect to the total variation norm:

$$\|\nu\| = |\nu|(\mathbb{R}).$$

Our goal is to show that the dual of $C_c(\mathbb{R})$ with respect to the uniform norm is isomorphic to $M_b(\mathbb{R})$:

$$C_c(\mathbb{R})^* = C_0(\mathbb{R})^* \cong M_b(\mathbb{R})$$

(Since $C_c(\mathbb{R})$ is dense in $C_0(\mathbb{R})$ w.r.t. $\|\cdot\|_\infty$, their duals w.r.t. this topology are the same).

Some properties of the total variation are recalled on the next page.

We will need the following exercise in order to define the total variation of a complex measure.

Exercise D.55. Let ν be a complex Borel measure on $\mathbb{R}$, and define $\mu = |\nu_r| + |\nu_i|$, so μ is a positive bounded Borel measure. Show there exists a function $f \in L^1(\mu)$ such that $d\nu = f\,d\mu$.

The total variation of a complex measure is a little more awkward to define than it is for a signed measure. By Exercise D.55, we know that if ν is a complex measure, then there exists at least one positive measure μ and one function $f \in L^1(\mu)$ such that $d\nu = f\,d\mu$. We will define the total variation of ν to be the measure $d|\nu| = |f|\,d\mu$, but of course we need to know that this is well-defined. The following theorem shows that this definition is indeed independent of the choice of μ and f.

Theorem D.56. *Let ν be a complex Borel measure on $\mathbb{R}$. If μ_1, μ_2 are bounded positive measures and $f_1 \in L^1(\mu_1)$, $f_2 \in L^1(\mu_2)$ are such that $f_1\,d\mu_1 = d\nu = f_2\,d\mu_2$, then $|f_1|\,d\mu_1 = |f_2|\,d\mu_2$.*

Proof. Define $\mu = \mu_1 + \mu_2$. Then since $\mu_1 \ll \mu$, there exists a function $g_1 \in L^1(\mu)$ such that $d\mu_1 = g_1\,d\mu$. Likewise, there exists $g_2 \in L^1(\mu)$ such that $d\mu_2 = g_2\,d\mu$. Because $\mu_1, \mu_2, \mu \geq 0$, we have $g_1, g_2 \geq 0$ μ-a.e.

Thus, we have $d\nu = f_1 d\mu_1$ and $d\mu_1 = g_1\,d\mu$. Exercise: Show that Exercise D.45 generalizes to complex measures, and use this to show that $d\nu = f_1 g_1\,d\mu$, and similarly $d\nu = f_2 g_2\,d\mu$ (see also Problem D.6).

The uniqueness statement in the Lebesgue–Radon–Nikodym Theorem therefore implies that $f_1 g_1 = f_2 g_2$ μ-a.e. Consequently,

$$|f_1|\,g_1 \;=\; |f_1 g_1| \;=\; |f_2 g_2| \;=\; |f_2|\,g_2 \;\mu\text{-a.e.},$$

and hence

$$|f_1|\,d\mu_1 \;=\; |f_1|\,g_1\,d\mu \;=\; |f_2|\,g_2\,d\mu \;=\; |f_2|\,d\mu_2. \qquad \square$$

Definition D.57 (Total Variation of a Complex Measure). Let ν be a complex Borel measure on $\mathbb{R}$. Then the *total variation* $|\nu|$ of ν is the positive measure $d|\nu| = |f|\,d\mu$, where μ is any positive measure and f is any function in $L^1(\mu)$ such that $d\nu = f\,d\mu$.

Next we give some properties of complex measures.

Exercise D.58. Let ν be a complex Borel measure on $\mathbb{R}$. Show that the following statements hold.

(a) $|\nu(E)| \leq |\nu|(E)$ for all $E \in \mathcal{B}_\sigma$.

(b) $\nu \ll |\nu|$, and there exists g with $|g| = 1$ $|\nu|$-a.e. such that $d\nu = g\,d|\nu|$.

(c) If $f \in L^1(\nu)$, then $\left|\int f\,d\nu\right| \leq \int |f|\,d|\nu|$.

The representation $d\nu = g\,d|\nu|$ in part (b) of the preceding exercise is called the *polar decomposition* of ν.

The following equivalent reformulations of the total variation measure are often easier to employ in practice than Definition D.57.

Exercise D.59. Let ν be a complex Borel measure on $\mathbb{R}$. Prove the following equivalent characterizations of $|\nu|$.

(a) $|\nu|(E) = \sup\left\{\sum_{k=1}^{n} |\nu(E_k)| : n \in \mathbb{N}, E_k \in \mathcal{B}_\sigma, E = \bigcup_{k=1}^{n} E_k \text{ disjointly}\right\}.$

(b) $|\nu|(E) = \sup\left\{\sum_{k=1}^{\infty} |\nu(E_k)| : E_k \in \mathcal{B}_\sigma, E = \bigcup_{k=1}^{\infty} E_k \text{ disjointly}\right\}.$

(c) $|\nu|(E) = \sup\left\{\left|\int_E f\,d\nu\right| : |f| \le 1 \ |\nu|\text{-a.e.}\right\}.$

Exercise D.60. Show that if ν is a complex measure, then $E \in \mathcal{B}_\sigma$ is a null set for ν (i.e., $\nu(A) = 0$ for every Borel set $A \subseteq E$) if and only if $|\nu|(E) = 0$.

For a complex measure ν, we say that a property holds ν-*almost everywhere* if it holds except possibly on a null set for ν. Thus "ν-almost everywhere" is the same as "$|\nu|$-almost everywhere."

The space $M_b(\mathbb{R})$ of all complex Borel measures is a Banach space.

Definition D.61 (Space of Complex Borel Measures). We set

$$M_b(\mathbb{R}) = \{\nu : \nu \text{ is a complex Borel measure on } \mathbb{R}\},$$

and define the norm of a complex measure ν to be

$$\|\nu\| = |\nu|(\mathbb{R}). \tag{D.4}$$

Exercise D.62. Show that $\|\cdot\|$ as defined in equation (D.4) is a norm on $M_b(\mathbb{R})$, and that $M_b(\mathbb{R})$ is a Banach space with respect to this norm.

We identify some particular subspaces of $M_b(\mathbb{R})$.

Exercise D.63. Show that if μ be a positive Borel measure on $\mathbb{R}$ and $d\nu = f\,d\mu$ where $f \in L^1(\mu)$, then $\|\nu\| = \|f\|_1 = \int |f|\,d\mu$.

Consequently, if μ is a positive measure then we have $L^1(\mu) \subseteq M_b(\mathbb{R})$. More precisely, if we define $d\nu_g = g\,d\mu$ for each $g \in L^1(\mu)$, then Exercise D.63 shows that $\mathcal{U}_\mu \colon g \mapsto \nu_g$ is an isometric embedding of $L^1(\mu)$ into $M_b(\mathbb{R})$. If μ is σ-finite, then range$(\mathcal{U}_\mu) = \{\nu \in M_b(\mathbb{R}) : \nu \ll \mu\}$. In particular, Lebesgue measure is a positive Borel measure, and hence if we identify $f \in L^1(\mathbb{R})$ with $f\,dx \in M_b(\mathbb{R})$ then we have $L^1(\mathbb{R}) \subseteq M_b(\mathbb{R})$.

Riesz Representation Theorem II

Given $\nu \in M_b(\mathbb{R})$, define $\mu_\nu : C_0(\mathbb{R}) \to \mathbb{C}$ by

$$\langle f, \mu_\nu \rangle = \int f \, d\bar{\nu}, \qquad f \in C_0(\mathbb{R}).$$

Then $U : \nu \longmapsto \mu_\nu$ is an antilinear isometry of $M_b(\mathbb{R})$ onto $C_0(\mathbb{R})^*$.

Proof:

Given $\nu \in M_b(\mathbb{R})$, we have that μ_ν is linear, and it is bounded on $C_0(\mathbb{R})$ because

$$\begin{aligned} |\langle f, \mu_\nu \rangle| &= \left| \int f \, d\bar{\nu} \right| \\ &\leq \int |f| \, d|\bar{\nu}| \\ &\leq \|f\|_\infty \int d|\nu| \\ &= \|f\|_\infty \, |\nu|(\mathbb{R}) \\ &= \|f\|_\infty \, \|\nu\|. \end{aligned}$$

This shows that $\|\mu_\nu\| \leq \|\nu\|$.

To obtain the opposite inequality, recall that

$\bar{\nu} \ll |\nu|$, so $\exists$ a function g such that

$|g| = 1$ $|\nu|$-a.e. and $d\bar{\nu} = \bar{g}\, d|\nu|$, i.e.,

$$\overline{\nu(E)} = \int_E \bar{g}\, d|\nu|, \qquad E \in \mathcal{B}_\sigma.$$

Fix $\varepsilon > 0$. Since $|\nu|$ is bounded, Lusin's Theorem implies $\exists\, \varphi \in C_c(\mathbb{R})$ such that $\|\varphi\|_\infty \le 1$ and $\varphi = \bar{g}$ except on a set E with $|\nu|(E) < \varepsilon$.

Hence

$$\begin{aligned}
\|\nu\| = |\nu|(\mathbb{R}) &= \int d|\nu| \\
&= \int |g|^2\, d|\nu| \\
&= \int g\,\bar{g}\, d|\nu| \\
&= \int g\, d\bar{\nu} \\
&= \left| \int \varphi\, d\bar{\nu} + \int (g-\varphi)\, d\bar{\nu} \right| \\
&\le \left| \int \varphi\, d\bar{\nu} \right| + \int_E |g-\varphi|\, d|\nu| \\
&\le |\langle \varphi, \mu_\nu \rangle| + \|g-\varphi\|_\infty\, |\nu|(E)
\end{aligned}$$

$$\leq \|\varphi\|_\infty \|\mu_\nu\| + 2\varepsilon.$$

$$\leq \|\mu_\nu\| + 2\varepsilon.$$

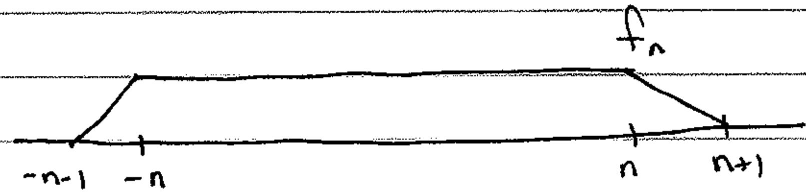
Letting $\varepsilon \to 0$ yields $\|\nu\| \leq \|\mu_\nu\|$.

This shows that $U: \nu \longmapsto \mu_\nu$ is an antilinear isometry. To show it is surjective, fix $\mu \in C_0(\mathbb{R})^*$.

If $\mathbb{F} = \mathbb{R}$ then by the earlier lemma we can write $\mu = \mu^+ - \mu^-$ where $\mu^\pm \in C_0(\mathbb{R})^*$ are positive linear functionals. By RRT I, $\exists$ positive Radon measures ν^+, ν^- such that

$$\langle f, \mu^\pm \rangle = \int f \, d\nu^\pm, \qquad f \in C_c(\mathbb{R}).$$

Let $f_n \in C_c(\mathbb{R})$ be



Then $f_n \nearrow 1$ so by ~~the~~ Monotone Convergence Thm,

$$\|\nu^+\| = \int d\nu^+ \qquad \text{since } \nu^+ \geq 0$$

$$= \lim_{n\to\infty} \int f_n \, d\nu^+ \qquad \text{MCT}$$

$$= \lim_{n\to\infty} \langle f_n, \mu^+ \rangle$$

$$\leq \lim_{n\to\infty} \|f_n\|_\infty \|\mu^+\|$$

$$= \|\mu^+\| < \infty.$$

Hence ν^+ is a bounded Radon measure, & similarly for ν^-, so $\nu = \nu^+ - \nu^- \in M_b(\mathbb{R})$.

Finally, for $f \in C_c(\mathbb{R})$ we have

$$\langle f, \mu \rangle = \langle f, \mu^+ \rangle - \langle f, \mu^- \rangle$$

$$= \int f \, d\nu^+ - \int f \, d\nu^-$$

$$= \int f \, d\nu,$$

and this extends to $f \in C_0(\mathbb{R})$ by density. Thus

$\mu = \mu_\nu$.

If $\mathbb{F} = \mathbb{C}$ then we can ~~[illegible]~~ write

$\mu = \mu_r + i\mu_i$ where $\mu_r, \mu_i : C_0(\mathbb{R}) \longrightarrow \mathbb{R}$.

If $f \in C_0(\mathbb{R})$ is real-valued then

$$|\langle f, \mu_r \rangle| \leq \sqrt{|\langle f, \mu_r \rangle|^2 + |\langle f, \mu_i \rangle|^2}$$

$$= \left| \langle f, \mu_r \rangle + i \langle f, \mu_i \rangle \right|$$

$$= |\langle f, \mu \rangle|$$

$$\leq \|f\|_\infty \, \|\mu\|.$$

Hence $\mu_r \in C_0(\mathbb{R})^*$ with respect to <u>real scalars</u>.

By the previous case, $\exists$ bounded signed Radon measure ν_r such that

$$\langle f, \mu_r \rangle = \int f \, d\nu_r, \quad f \in C_0(\mathbb{R}), \ f \text{ real-valued}$$

A similar argument applies to ν_i. Then

$\nu = \nu_r + i\nu_i$ is a complex Radon measure.

Given a complex-valued $f \in C_0(\mathbb{R})$, write

$f = f_r + i f_i$ where $f_r, f_i \in C_0(\mathbb{R})$ are real-valued. Then

$$\langle f, \mu \rangle = \langle f_r + i f_i, \mu \rangle$$

$$= \langle f_r, \mu \rangle + i \langle f_i, \mu \rangle \qquad \mu \text{ is } \mathbb{C}\text{-linear}$$

$$= \langle f_r, \mu_r \rangle + i \langle f_r, \mu_i \rangle + i \left(\langle f_i, \mu_r \rangle + i \langle f_i, \mu_i \rangle \right)$$

$$= \int f_r \, d\overline{\nu}_r + i \int f_r \, d\overline{\nu}_i + i \left(\int f_i \, d\overline{\nu}_i + i \int f_i \, d\overline{\nu}_i \right)$$

$$= \int f_r \, d\nu + i \int f_i \, d\nu$$

$$= \int f \, d\nu$$

Hence the measure we seek is $\overline{\nu}$, i.e. $\mu = \mu_{\overline{\nu}}$. $\blacksquare$