PUBLICATION LIST WITH SELECTED ABSTRACTS

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1. C. Heil, **"A Basis Theory Primer"**, electronic manuscript, 1987 (revised 1997), 93 pp. (Expanded into Publication #80, 2010).

2. C. E. Heil and D. F. Walnut, **Continuous and discrete wavelet transforms**, SIAM Review, **31** (1989), pp. 628–666. MR 91c:42032. Zbl. 0683.42031.

   Abstract. This paper is an expository survey of results on integral representations and discrete sum expansions of functions in $L^2(\mathbf{R})$ in terms of coherent states. Two types of coherent states are considered: Weyl–Heisenberg coherent states, which arise from translations and modulations of a single function, and affine coherent states, called "wavelets," which arise as translations and dilations of a single function. In each case it is shown how to represent any function in $L^2(\mathbf{R})$ as a sum or integral of these states. Most of the paper is a survey of literature, most notably the work of I. Daubechies, A. Grossmann, and J. Morlet. A few results of the authors are included.

3. C. Heil, *Wavelets and frames*, in: "Signal Processing, Part I: Signal Processing Theory," L. Auslander, T. Kailath, and S. Mitter, eds., IMA Vol. Math. Appl. **22**, Springer–Verlag, New York (1990), pp. 147–160. MR 91b:42050. Zbl. 0721.46006. (Short version of Publication #2.)

4. C. Heil and D. Walnut, *Gabor and wavelet expansions*, in: "Recent Advances in Fourier Analysis and its Applications" (Il Ciocco, 1989), J. S. Byrnes and J. L. Byrnes, eds., NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci. **315**, Kluwer Acad. Pub., Dordrecht (1990), pp. 441–454. MR 91j:42024. Zbl. 0736.42023.

5. C. Heil, *A discrete Zak transform*, The MITRE Corporation, Technical Report MTR-89W00128, December 1989.

6. J. J. Benedetto, C. Heil, and D. F. Walnut, **Differentiation and the Balian–Low theorem**, J. Fourier Anal. Appl., **1** (1995), pp. 355–402. MR 96f:42002. Zbl. 0887.42026.

   Abstract. The *Balian–Low theorem* (BLT) is a key result in time-frequency analysis, originally stated by Balian and, independently, by Low, as: If a Gabor system $\{e^{2\pi imbt}\, g(t-na)\}_{m,n\in\mathbf{Z}}$ with $ab=1$ forms an orthonormal basis for $L^2(\mathbf{R})$, then
   $$\left(\int_{-\infty}^{\infty} |t\, g(t)|^2\, dt\right) \left(\int_{-\infty}^{\infty} |\gamma\, \hat{g}(\gamma)|^2\, d\gamma\right) = +\infty.$$
   The BLT was later extended from orthonormal bases to exact frames.

   This paper presents a tutorial on Gabor systems, the BLT, and related topics, such as the Zak transform

and Wilson bases. Because of the fact that $(g')^\wedge(\gamma) = 2\pi i\gamma\,\hat{g}(\gamma)$, the role of differentiation in the proof of the BLT is examined carefully. The major new contributions of this paper are the construction of a complete Gabor system of the form $\{e^{2\pi i b_m t}\,g(t-a_n)\}$ such that $\{(a_n, b_m)\}$ has density strictly less than 1, an Amalgam BLT that provides distinct restrictions on Gabor systems $\{e^{2\pi i mbt}\,g(t-na)\}$ that form exact frames, and a new proof of the BLT for exact frames that does not require differentiation and relies only on classical real variable methods from harmonic analysis.

7. C. E. Heil, "Wiener Amalgam Spaces in Generalized Harmonic Analysis and Wavelet Theory," Ph.D. Thesis, University of Maryland, College Park, Maryland, May 1990.

8. C. Heil, *Applications of the fast wavelet transform*, in: "Advanced Signal-Processing Algorithms, Architectures, and Implementations" (San Diego, CA, 1990), Proc. SPIE **1348**, F. T. Luk, ed., SPIE, Bellingham, WA (1990), pp. 248-259.

9. D. Colella and C. Heil, **The characterization of continuous, four-coefficient scaling functions and wavelets**, IEEE Trans. Inform. Theory, Special Issue on Wavelet Theory and Multiresolution Signal Analysis, **38** (1992), pp. 876–881. MR 1162225. Zbl. 0743.42012.

10. D. Colella and C. Heil, **Characterizations of scaling functions: Continuous solutions**, SIAM J. Matrix Anal. Appl., **15** (1994), pp. 496–518. MR 95f:26004. Zbl. 0797.39006.

    Abstract. A dilation equation is a functional equation of the form $f(t) = \sum_{k=0}^{N} c_k\, f(2t-k)$, and any nonzero solution of such an equation is called a scaling function. Dilation equations play an important role in several fields, including interpolating subdivision schemes and wavelet theory. This paper obtains sharp bounds for the Hölder exponent of continuity of any continuous, compactly supported scaling function in terms of the joint spectral radius of two matrices determined by the coefficients $\{c_0, \dots, c_N\}$. The arguments lead directly to a characterization of all dilation equations that have continuous, compactly supported solutions.

11. C. Heil, *Methods of solving dilation equations*, in: "Probabilistic and Stochastic Methods in Analysis, with Applications" (Il Ciocco, 1991), J. S. Byrnes, et al., eds., NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci. **372**, Kluwer Acad. Pub., Dordrecht (1992), pp. 15–45. MR 93h:42023. Zbl. 0761.93040.

12. J. Benedetto, C. Heil, and D. Walnut, *Uncertainty principles for time-frequency operators*, in: "Continuous and Discrete Fourier Transforms, Extension Problems and Wiener-Hopf Equations," Oper. Theory Adv. Appl., **58**, I. Gohberg, ed., Birkhäuser, Basel (1992), pp. 1–25. MR 94a:42041. Zbl. 0790.42017.

13. C. Heil and D. Colella, *Dilation equations and the smoothness of compactly supported wavelets*, in: "Wavelets: Mathematics and Applications," J. J. Benedetto and M. W. Frazier, eds., CRC Press, Boca Raton, FL (1994), pp. 163–201. MR 94j:42049. Zbl. 0882.42025.

    Abstract. The construction of compactly supported wavelets with specified amounts of smoothness is an important problem in wavelet theory. This problem reduces to the construction of scaling functions, i.e., solutions f of dilation equations $f(t) = \sum_{k=0}^{N} c_k\, f(2t-k)$, with specified smoothness. This article characterizes all smooth, compactly supported scaling functions in terms of a joint spectral radius of two $N \times N$ matrices T_0, T_1 constructed from the coefficients $\{c_0, \dots, c_N\}$ of the dilation equation, restricted to an appropriate subspace of $\mathbf{C}^N$. The number of continuous derivatives of the scaling function and the range of Hölder exponents of continuity of the last continuous derivative are determined by the value of this joint spectral radius. Numerous examples are provided to illustrate the results.

14. C. Heil and G. Strang, *Continuity of the joint spectral radius: Application to wavelets*, in: "Linear

Algebra for Signal Processing" (Minneapolis, MN, 1992), A. Bojanczyk and G. Cybenko, eds., IMA Vol. Math. Appl. **69**, Springer–Verlag, New York (1995), pp. 51–61. MR 96h:15028. Zbl. 0823.15009.

Abstract. The joint spectral radius is the extension to two or more matrices of the (ordinary) spectral radius $\rho(A) = \max|\lambda_i(A)| = \lim \|A^m\|^{1/m}$. The extension allows matrix products Π_m taken in all orders, so that norms and eigenvalues are difficult to estimate. We show that the limiting process does yield a continuous function of the original matrices—this is their joint spectral radius. Then we describe the construction of wavelets from a dilation equation with coefficients c_k. We connect the continuity of those wavelets to the value of the joint spectral radius of two matrices whose entries are formed from the c_k.

15. C. Heil, Book review, in *Computers in Physics*, **6** (1992), p. 697. Review of: I. Daubechies, *Ten Lectures on Wavelets*, SIAM, Philadelphia (1992) and C. K. Chui, *An Introduction to Wavelets*, Academic Press, Boston (1992).

16. C. Heil, Book review, in *SIAM Review*, **35** (1993), pp. 666–669. Review of: I. Daubechies, *Ten Lectures on Wavelets*, SIAM, Philadelphia (1992).

17. C. Heil, *Some stability properties of wavelets and scaling functions*, in: "Wavelets and Their Applications" (Il Ciocco, 1992), J. S. Byrnes, et al., eds., NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci. **442**, Kluwer Acad. Pub., Dordrecht (1994), pp. 19–38. MR 96h:42025. Zbl. 0813.42022.

Abstract. The property of continuity of an arbitrary scaling function is known to be unstable with respect to the coefficients in the associated dilation equation. That is, if f is a continuous function which is a solution of the dilation equation $f(x) = \sum_{k=0}^{N} c_k f(2x-k)$ then a dilation equation with slightly perturbed coefficients $\{\tilde{c}_0, \dots, \tilde{c}_N\}$ need not have a continuous solution. The convergence of the Cascade Algorithm, an iterative method for solving a dilation equation, is likewise unstable in general. This paper establishes a condition under which stability does occur: both continuity and uniform convergence of the Cascade Algorithm are shown to be stable for those initial choices of coefficients $\{c_k\}$ such that the integer translates of the scaling function f are ℓ^∞ linearly independent. In particular, this applies to those scaling functions which can be used to construct orthogonal or biorthogonal wavelets. We show by example that this ℓ^∞ linear independence condition is not necessary for stability to occur.

18. C. Heil, J. Ramanathan, and P. Topiwala, **Linear independence of time-frequency translates**, Proc. Amer. Math. Soc., **124** (1996), pp. 2787–2795. MR 96k:42039. Zbl. 0859.42023.

Abstract. The refinement equation $\varphi(t) = \sum_{k=N_1}^{N_2} c_k\, \varphi(2t-k)$ plays a key role in wavelet theory and in subdivision schemes in approximation theory. Viewed as an expression of linear dependence among the time-scale translates $|a|^{1/2}\varphi(at-b)$ of $\varphi \in L^2(\mathbf{R})$, it is natural to ask if there exist similar dependencies among the time-frequency translates $e^{2\pi i b t} f(t+a)$ of $f \in L^2(\mathbf{R})$. In other words, what is the effect of replacing the group representation of $L^2(\mathbf{R})$ induced by the affine group with the corresponding representation induced by the Heisenberg group? This paper proves that there are no nonzero solutions to lattice-type generalizations of the refinement equation to the Heisenberg group. Moreover, it is proved that for each arbitrary finite collection $\{(a_k, b_k)\}_{k=1}^N$, the set of all functions $f \in L^2(\mathbf{R})$ such that $\{e^{2\pi i b_k t} f(t+a_k)\}_{k=1}^N$ is independent is an open, dense subset of $L^2(\mathbf{R})$. It is conjectured that this set is all of $L^2(\mathbf{R}) \setminus \{0\}$.

19. C. Heil, J. Ramanathan, and P. Topiwala, **Singular values of compact pseudodifferential operators**, J. Functional Anal., **150** (1997), pp. 426–452. MR 98k:47102. Zbl. 0990.35139.

Abstract. This paper investigates the asymptotic decay of the singular values of compact operators arising from the Weyl correspondence. The motivating problem is to find sufficient conditions on a symbol which insure that the corresponding operator has singular values with a prescribed rate of decay. The problem is approached by using a Gabor frame expansion of the symbol to construct an approximating

finite rank operator. This establishes a variety of sufficient conditions for the associated operator to be in a particular Schatten class. In particular, an improvement of a sufficient condition of Daubechies for an operator to be trace-class is obtained. In addition, a new development and improvement of the Calderón–Vaillancourt theorem in the context of the Weyl correspondence is given. Additional results of this type are then obtained by interpolation.

20. O. Christensen and C. Heil, **Perturbations of Banach frames and atomic decompositions**, Math. Nachr., **185** (1997), pp. 33–47. MR 98m:42061. Zbl. 0868.42013.

    Abstract. Banach frames and atomic decompositions are sequences which have basis-like properties but which need not be bases. In particular, they allow elements of a Banach space to be written as linear combinations of the frame or atomic decomposition elements in a stable manner. In this paper we prove several functional-analytic properties of these decompositions, and show how these properties apply to Gabor and wavelet systems. We first prove that frames and atomic decompositions are stable under small perturbations. This is inspired by corresponding classical perturbation results for bases, including the Paley–Wiener basis stability criteria and the perturbation theorem of Kato. We introduce new and weaker conditions which ensure the desired stability. We then prove duality properties of atomic decompositions and consider some consequences for Hilbert frames. Finally, we demonstrate how our results apply in the practical case of Gabor systems in weighted L^2 spaces. Such systems can form atomic decompositions for $L^2_w(\mathbf{R})$, but cannot form Hilbert frames for $L^2_w(\mathbf{R})$ unless the weight is trivial.

21. C. Heil, J. Ramanathan, and P. Topiwala, *Asymptotic singular value decay of time-frequency localization operators*, in: "Wavelet Applications in Signal and Image Processing II" (San Diego, CA, 1994), Proc. SPIE **2303**, A. F. Laine and M. A. Unser, eds., SPIE, Bellingham, WA (1994), pp. 15–24. (Conference announcement for Publication #19.)

22. C. Heil, G. Strang, and V. Strela, **Approximation by translates of refinable functions**, Numerische Math., **73** (1996), pp. 75–94. MR 97c:65033. Zbl. 0857.65015.

    Abstract. The functions $f_1(x), \dots, f_r(x)$ are *refinable* if they are combinations of the rescaled and translated functions $f_i(2x-k)$. This is very common in scientific computing on a regular mesh. The space V_0 of approximating functions with meshwidth $h=1$ is a subspace of V_1 with meshwidth $h=1/2$. These refinable spaces have refinable basis functions. The accuracy of the computations depends on p, the *order of approximation*, which is determined by the degree of polynomials $1, x, \dots, x^{p-1}$ that lie in V_0.

    Most refinable functions (such as scaling functions in the theory of wavelets) have no simple formulas. The functions $f_i(x)$ are known only through the coefficients c_k in the refinement equation—scalars in the traditional case, $r \times r$ matrices for multiwavelets. The scalar "sum rules" that determine p are well known. We find the conditions on the matrices c_k that yield approximation of order p from V_0. These are equivalent to the Strang–Fix conditions on the Fourier transforms $\hat{f}_i(\omega)$, but for refinable functions they can be explicitly verified from the c_k.

23. C. Heil and D. Colella, **Matrix refinement equations: Existence and uniqueness**, J. Fourier Anal. Appl., **2** (1996), pp. 363–377. MR 97k:39021. Zbl. 0904.39017.

    Abstract. Matrix refinement equations are functional equations of the form $f(x) = \sum_{k=0}^{N} c_k\, f(2x-k)$, where the coefficients c_k are matrices and f is a vector-valued function. Refinement equations play a key role in both wavelet theory and approximation theory. Existence and uniqueness properties of scalar refinement equations (where the coefficients c_k are scalars) are known. This paper considers analogous questions for matrix refinement equations. Conditions for existence and uniqueness of compactly supported distributional solutions are given in terms of the convergence properties of an infinite product of the matrix $\Delta = \frac{1}{2}\sum c_k$ with itself. Important fundamental differences between solutions to matrix equations and scalar refinement equations are examined. In particular, it is shown that "constrained" solutions to the matrix refinement equation can exist when the infinite product diverges. The existence of constrained solutions is related to the eigenvalue structure of Δ; solutions are obtained from the convergence of Δ on its 1-eigenspace.

24. C. Heil and D. Colella, *Sobolev regularity for scaling functions via ergodic theory*, in: "Approximation Theory VIII," Vol. 2 (College Station, TX, 1995), C. K. Chui and L. L. Schumaker, eds., World Scientific, Singapore (1995), pp. 151–158. MR 98e:42033. Zbl. 0927.42019.

    Abstract. The refinement equation $f(x) = \sum_{k=0}^{N} c_k\, f(2x-k)$ plays a key role in wavelet theory and in subdivision schemes in approximation theory. This paper explores the relationship of the refinement equation to the mapping $\tau(x) = 2x \bmod 1$. A simple necessary condition for the existence of integrable solutions is obtained by considering the periodic cycles of τ. Another simple necessary condition for the existence of an integrable solution satisfying $(1+|\gamma|^2)^{s/2}\, \hat{f}(\gamma) \in L^p(\mathbf{R})$ is obtained by considering the ergodic property of τ. In particular, taking $p = 2$ results in a necessary condition for the solution f to lie in the Sobolev space H^s.

25. P. N. Heller, V. Strela, G. Strang, P. Topiwala, C. Heil, and L. S. Hills, *Multiwavelet filter banks for data compression*, in: ISCAS '95, Proc. International Symposium on Circuits and Systems (Seattle, WA, 1995), Vol. 3, IEEE, Piscataway, NJ (1995), pp. 1796–1799. (Conference announcement for Publication #28.)

26. C. Heil, *Wavelets*, Section 7.13.6 in the CRC Standard Mathematical Tables and Formulae, 30th Edition, D. Zwillinger, ed., CRC Press, Boca Raton, FL (1996), pp. 663–667; Section 7.15.5 in the CRC Standard Mathematical Tables and Formulae, 31st Edition, D. Zwillinger, ed., CRC Press, Boca Raton, FL (2003), pp. 723–726.

27. C. A.Cabrelli, C. Heil, and U. M. Molter, **Self-similarity and multiwavelets in higher dimensions**, Memoirs Amer. Math. Soc., Vol. **170**, No. 807 (2004), viii+82 pp. MR 2005d:42037. Zbl. 1063.42024.

    Abstract. Let A be a dilation matrix, an $n \times n$ expansive matrix that maps a full-rank lattice $\Gamma \subset \mathbf{R}^n$ into itself. Let Λ be a finite subset of Γ, and for $k \in \Lambda$ let c_k be $r \times r$ complex matrices. The refinement equation corresponding to A, Γ, Λ, and $c = \{c_k\}_{k\in\Lambda}$ is $f(x) = \sum_{k\in\Lambda} c_k\, f(Ax-k)$. A solution $f\colon \mathbf{R}^n \to \mathbf{C}^r$, if one exists, is called a refinable vector function or a vector scaling function of multiplicity r. In this manuscript we characterize the existence of compactly supported L^p or continuous solutions of the refinement equation, in terms of the p-norm joint spectral radius of a finite set of finite matrices determined by the coefficients c_k. We obtain sufficient conditions for the L^p convergence ($1 \le p \le \infty$) of the Cascade Algorithm $f^{(i+1)}(x) = \sum_{k\in\Lambda} c_k\, f^{(i)}(Ax-k)$, and necessary conditions for the uniform convergence of the Cascade Algorithm to a continuous solution. We also characterize those compactly supported vector scaling functions which give rise to a multiresolution analysis for $L^2(\mathbf{R}^n)$ of multiplicity r, and provide conditions under which there exist corresponding multiwavelets whose dilations and translations form an orthonormal basis for $L^2(\mathbf{R}^n)$.

28. V. Strela, P. N. Heller, G. Strang, P. Topiwala, and C. Heil, **The application of multiwavelet filterbanks to image processing**, IEEE Trans. Image Processing, **8** (1999), pp. 548–563.

    Abstract. Multiwavelets are a new addition to the body of wavelet theory. Realizable as matrix-valued filter banks leading to wavelet bases, multiwavelets offer simultaneous orthogonality, symmetry, and short support, which is not possible with scalar 2-channel wavelet systems. After reviewing this recently developed theory, we examine the use of multiwavelets in a filter bank setting for discrete-time signal and image processing. Multiwavelets differ from scalar wavelet systems in requiring two or more input streams to the multiwavelet filter bank. We describe two methods (repeated row and approximation/deapproximation) for obtaining such a vector input stream from a one-dimensional signal. Algorithms for symmetric extension of signals at boundaries are then developed, and naturally integrated with approximation-based preprocessing. We describe an additional algorithm for multiwavelet processing of two-dimensional signals, two rows at a time, and develop a new family of multiwavelets (the constrained pairs) that is well-suited to this approach. This suite of novel techniques is then applied to two basic signal processing problems, denoising via wavelet-shrinkage, and data compression. After developing the approach via model problems in one dimension, we applied multiwavelet processing to images, frequently obtaining performance superior to the comparable scalar wavelet transform.

29. C. Heil, *Existence and accuracy for matrix refinement equations*, Z. Angew. Math. Mech., Special issue on Applied Stochastics and Optimization, **76** (1996), pp. 251–254. Zbl. 0925.42018. (Expository summary of Publications #22 and #23.)

30. C. Heil, **The Wiener transform on the Besicovitch spaces**, Proc. Amer. Math. Soc., **127** (1999), pp. 2065–2071. MR 99j:42007. Zbl. 0943.42005.

    Abstract. In his fundamental research on generalized harmonic analysis, Wiener proved that the integrated Fourier transform defined by $Wf(\gamma) = \int f(t)\,(e^{-2\pi i\gamma t} - \chi_{[-1,1]}(t))/(-2\pi it)\,dt$ is an isometry from a nonlinear space of functions of bounded average quadratic power into a nonlinear space of functions of bounded quadratic variation. We consider this *Wiener transform* on the larger, linear, *Besicovitch spaces* $\mathcal{B}_{p,q}(\mathbf{R})$ defined by the norm $\|f\|_{\mathcal{B}_{p,q}} = \bigl(\int_0^\infty \bigl(\frac{1}{2T}\int_{-T}^{T} |f(t)|^p\,dt\bigr)^{q/p} \frac{dT}{T}\bigr)^{1/q}$. We prove that W maps $\mathcal{B}_{p,q}(\mathbf{R})$ continuously into the homogeneous Besov space $\dot{B}^{1/p'}_{p',q}(\mathbf{R})$ for $1 < p \le 2$ and $1 < q \le \infty$, and is a topological isomorphism when $p = 2$.

31. C. Cabrelli, C. Heil, and U. Molter, **Accuracy of lattice translates of several multidimensional refinable functions**, J. Approx. Th., **95** (1998), pp. 5–52. MR 99g:42038. Zbl. 0911.41008.

    Abstract. Complex-valued functions $f_1, \dots, f_r$ on $\mathbf{R}^d$ are *refinable* if they are linear combinations of finitely many of the rescaled and translated functions $f_i(Ax - k)$, where the translates k are taken along a lattice $\Gamma \subset \mathbf{R}^d$ and A is a *dilation matrix* that expansively maps Γ into itself. Refinable functions satisfy a *refinement equation* $f(x) = \sum_{k\in\Lambda} c_k\, f(Ax - k)$, where Λ is a finite subset of Γ, the c_k are $r \times r$ matrices, and $f(x) = (f_1(x), \dots, f_r(x))^{\mathrm{T}}$. The *accuracy* of f is the highest degree p such that all multivariate polynomials q with degree$(q) < p$ are exactly reproduced from linear combinations of translates of $f_1, \dots, f_r$ along the lattice Γ. In this paper, we determine the accuracy p from the matrices c_k. Moreover, we determine explicitly the coefficients $y_{\alpha,i}(k)$ such that $x^\alpha = \sum_{i=1}^r \sum_{k\in\Gamma} y_{\alpha,i}(k)\, f_i(x+k)$. These coefficients are multivariate polynomials $y_{\alpha,i}(x)$ of degree $|\alpha|$ evaluated at lattice points $k \in \Gamma$.

32. C. Cabrelli, C. Heil, and U. Molter, **Accuracy of several multidimensional refinable distributions**, J. Fourier Anal. Appl., **6** (2000), pp. 483–502. MR 2001g:65179. Zbl. 0960.42016.

    Abstract. Compactly supported distributions $f_1, \dots, f_r$ on $\mathbf{R}^d$ are refinable if each f_i is a finite linear combination of the rescaled and translated distributions $f_j(Ax - k)$, where the translates k are taken along a lattice $\Gamma \subset \mathbf{R}^d$ and A is a dilation matrix that expansively maps Γ into itself. Refinable distributions satisfy a refinement equation $f(x) = \sum_{k\in\Lambda} c_k\, f(Ax - k)$, where Λ is a finite subset of Γ, the c_k are $r \times r$ matrices, and $f = (f_1, \dots, f_r)^{\mathrm{T}}$. The accuracy of f is the highest degree p such that all multivariate polynomials q with degree$(q) < p$ are exactly reproduced from linear combinations of translates of $f_1, \dots, f_r$ along the lattice Γ. We determine the accuracy p from the matrices c_k. Moreover, we determine explicitly the coefficients $y_{\alpha,i}(k)$ such that $x^\alpha = \sum_{i=1}^r \sum_{k\in\Gamma} y_{\alpha,i}(k)\, f_i(x + k)$. These coefficients are multivariate polynomials $y_{\alpha,i}(x)$ of degree $|\alpha|$ evaluated at lattice points $k \in \Gamma$.

33. K. Gröchenig and C. Heil, **Modulation spaces and pseudodifferential operators**, Integral Equations Operator Theory, **34** (1999), pp. 439–457. MR 2001a:47051. Zbl. 0936.35209.

    Abstract. We use methods from time-frequency analysis to study boundedness and trace-class properties of pseudodifferential operators. As natural symbol classes, we use the modulation spaces on $\mathbf{R}^{2d}$, which quantify the notion of the time-frequency content of a function or distribution. We show that if a symbol σ lies in the modulation space $M_{\infty,1}(\mathbf{R}^{2d})$, then the corresponding pseudodifferential operator is bounded on $L^2(\mathbf{R}^d)$ and, more generally, on the modulation spaces $M_{p,p}(\mathbf{R}^d)$ for $1 \le p \le \infty$. If σ lies in the modulation space $M^s_{2,2}(\mathbf{R}^{2d}) = L^2_s(\mathbf{R}^{2d}) \cap H^s(\mathbf{R}^{2d})$, i.e., the intersection of a weighted L^2-space and a Sobolev space, then the corresponding operator lies in a specified Schatten class. These results hold for both the Weyl and the Kohn-Nirenberg correspondences. Using recent embedding theorems of Lipschitz and Fourier spaces into modulation spaces, we show that these results improve on the classical Calderòn–Vaillancourt boundedness theorem and on Daubechies' trace-class results.

34. J. J. Benedetto, C. Heil, and D. F. Walnut, *Gabor systems and the Balian–Low theorem*, in: "Gabor Analysis and Algorithms: Theory and Applications," H. G. Feichtinger and T. Strohmer, eds., Birkhäuser, Boston (1998), pp. 85–122. MR 98j:42016. Zbl. 0890.42007.

35. O. Christensen, B. Deng, and C. Heil, **Density of Gabor frames**, Appl. Comput. Harmon. Anal., **7** (1999), pp. 292–304. MR 2000j:42043. Zbl. 0960.42007.

    Abstract. A Gabor system is a set of time-frequency shifts $S(g,\Lambda) = \{e^{2\pi ibx}g(x-a)\}_{(a,b)\in\Lambda}$ of a function $g \in L^2(\mathbf{R}^d)$. We prove that if a finite union of Gabor systems $\bigcup_{k=1}^r S(g_k,\Lambda_k)$ forms a frame for $L^2(\mathbf{R}^d)$ then the lower and upper Beurling densities of $\Lambda = \bigcup_{k=1}^r \Lambda_k$ satisfy $D^-(\Lambda) \ge 1$ and $D^+(\Lambda) < \infty$. This extends recent work of Ramanathan and Steger. Additionally, we prove the conjecture that no collection $\bigcup_{k=1}^r \{g_k(x-a)\}_{a\in\Gamma_k}$ of pure translates can form a frame for $L^2(\mathbf{R}^d)$.

36. C. A. Cabrelli, C. Heil, and U. M. Molter, *Polynomial reproduction by refinable functions*, in: "Advances in Wavelets" (Hong Kong, 1997), K.-S. Lau, ed., Springer–Verlag, Singapore (1999), pp. 121–161. MR 2000g:42040. (Exposition related to Publications #22, 31, 32).

37. K. Gröchenig, C. Heil, and D. Walnut, **Nonperiodic sampling and the local three squares theorem**, Ark. Mat., **38** (2000), pp. 77–92. MR 2001g:42015. Zbl. 1012.42003.

    Abstract. This paper presents an elementary, real-variable proof of the following theorem: *Given* $\{r_j\}_{j=1}^m$ *with* $m = d+1$, *fix* $R \ge \sum_{j=1}^m r_j$ *and let* $Q = [-R,R]^d$. *Then any* $f \in L^2(Q)$ *is completely determined by its averages over cubes of side* r_j *that are completely contained in* Q *and have edges parallel to the coordinate axes if and only if* r_j/r_k *is irrational for* $j \ne k$. When $d = 2$ this theorem is known as the local three squares theorem and is an example of a Pompeiu-type theorem. The proof of the theorem combines ideas in multisensor deconvolution and the theory of sampling on unions of rectangular lattices having incommensurate densities with a theorem of Young on sequences biorthogonal to exact sequences of exponentials.

38. R. Ashino, C. Heil, M. Nagase, and R. Vaillancourt, **Microlocal filtering with multiwavelets**, Comput. Math. Appl., **41** (2001), pp. 111–133. MR 2002a:94005. Zbl. 1005.94005.

39. K. Gröchenig and C. Heil, **Gabor meets Littlewood–Paley: Gabor expansions in** $L^p(\mathbf{R}^d)$, Studia Math., **146** (2001), pp. 15–33. MR 2002b:42028. Zbl. 0970.42021.

    Abstract. It is known that Gabor expansions do not converge unconditionally in L^p and that L^p cannot be characterized in terms of the magnitudes of Gabor coefficients. By using a combination of Littlewood–Paley and Gabor theory, we show that L^p can nevertheless be characterized in terms of Gabor expansions, and that the partial sums of Gabor expansions converge in L^p-norm.

40. C. A. Cabrelli, C. Heil, and U. M. Molter, *Multiwavelets in* $\mathbb{R}^n$ *with an arbitrary dilation matrix*, in: "Wavelets and Signal Processing," L. Debnath, ed., Birkhäuser, Boston (2003), pp. 23–39.

MR 2004e:42050. Zbl. 1045.42020. (Exposition related to Publication #27.)

41. R. Ashino, C. Heil, M. Nagase, and R. Vaillancourt, *Multiwavelets, pseudodifferential operators and microlocal analysis*, in: "Wavelet Analysis and Applications" (Guangzhou, 1999), D. Deng et al., eds., AMS/IP Stud. Adv. Math., **25**, American Mathematical Society, Providence, RI (2002), pp. 9–20. MR 2002k:42065. Zbl. 0997.42021.

42. B. Deng and C. Heil, *Density of Gabor Schauder bases*, in: "Wavelet Applications in Signal and Image Processing VIII" (San Diego, CA, 2000), Proc. SPIE **4119**, A. Aldroubi et al., eds., SPIE, Bellingham, WA (2000), pp. 153–164.

    Abstract. A Gabor system is a fixed set of time-frequency shifts $G(g,\Lambda) = \{e^{2\pi i b\cdot x} g(x-a)\}_{(a,b)\in\Lambda}$ of a function $g \in L^2(\mathbf{R}^d)$. We prove that if $G(g,\Lambda)$ forms a Schauder basis for $L^2(\mathbf{R}^d)$ then the upper Beurling density of Λ satisfies $D^+(\Lambda) \le 1$. We also prove that if $G(g,\Lambda)$ forms a Schauder basis for $L^2(\mathbf{R}^d)$ and if g lies in a the modulation space $M^{1,1}(\mathbf{R}^d)$, which is a dense subset of $L^2(\mathbf{R}^d)$, or if $G(g,\Lambda)$ possesses at least a lower frame bound, then Λ has uniform Beurling density $D(\Lambda) = 1$. We use related techniques to show that if $g \in L^1(\mathbf{R}^d) \cap L^2(\mathbf{R}^d)$ then no collection $\{g(x-a)\}_{a\in\Gamma}$ of pure translates of g can form a Schauder basis for $L^2(\mathbf{R}^d)$. We also extend these results to the case of finitely many generating functions $g_1, \dots, g_r$.

43. C. A. Cabrelli, C. Heil, and U. M. Molter, *Necessary conditions for the existence of multivariate multiscaling functions*, in: "Wavelet Applications in Signal and Image Processing VIII" (San Diego, CA, 2000), Proc. SPIE **4119**, A. Aldroubi et al., eds., SPIE, Bellingham, WA (2000), pp. 395–406. (Exposition related to Publication #27.)

44. R. Balan, P. G. Casazza, C. Heil, and Z. Landau, **Deficits and excesses of frames**, Adv. Comput. Math., Special Issue on Frames, **18** (2003), pp. 93–116. MR 2004a:42040. Zbl. 1029.42030.

    Abstract. The excess of a sequence in a Hilbert space is the greatest number of elements that can be removed yet leave a set with the same closed span. We study the excess and the dual concept of the deficit of Bessel sequences and frames. In particular, we characterize those frames for which there exist infinitely many elements that can be removed from the frame yet still leave a frame, and we show that all overcomplete Weyl-Heisenberg and wavelet frames have this property.

45. R. Ashino, C. Heil, M. Nagase, and R. Vaillancourt, *Microlocal analysis and multiwavelets*, in: "Geometry, Analysis and Applications" (Varanasi, India, 2000), R. S. Pathak, ed., World Scientific, Singapore (2001), pp. 293–302. MR 2002f:42038. Zbl. 0986.42024.

46. C. Heil, Book review, in *SIAM Review*, **43** (2001), pp. 722–724. Review of: D. W. Kammler, *A First Course in Fourier Analysis*, Prentice Hall, Upper Saddle River, NJ (2000).

47. C. Heil and D. F. Walnut, Editors, "Fundamental Papers in Wavelet Theory," Princeton University Press, Princeton, NJ, 2006 (xix+878 pp.). MR 2229251. Zbl. 1113.42001.

    Jacket summary. This book traces the prehistory and initial development of wavelet theory, a discipline that has had a profound impact on mathematics, physics, and engineering. Interchanges between these fields during the last 15 years have led to a number of advances in applications such as image compression, turbulence, machine vision, radar, and earthquake prediction.

    This book contains the seminal papers that presented the ideas from which wavelet theory developed, as well as those major papers that developed the theory into its current form. These papers originated in a variety of journals from different disciplines, making it difficult for the researcher to obtain a complete view of wavelet theory and its origins. Additionally, some of the most significant papers have heretofore been available only in French or German.

Heil and Walnut bring together these documents in a book that allows researchers a complete view of wavelet theory's origins and development.

48. R. Balan, P. G. Casazza, C. Heil, and Z. Landau, **Excesses of Gabor frames**, Appl. Comput. Harmon. Anal., **14** (2003), pp. 87–106. MR 2004e:42058. Zbl. 1028.42021.

Abstract. A Gabor system for $L^2(\mathbf{R}^d)$ has the form $\mathcal{G}(g,\Lambda) = \{e^{2\pi i b\cdot x} g(x-a)\}_{(a,b)\in\Lambda}$, where $g \in L^2(\mathbf{R}^d)$ and Λ is a sequence of points in $\mathbf{R}^{2d}$. We prove that, with only a mild restriction on the generator g and for nearly arbitrary sets of time-frequency shifts Λ, an overcomplete Gabor frame has infinite excess, and in fact there exists an infinite subset that can be removed yet leave a frame. The proof of this result yields an interesting connection between the density of Λ and the excess of the frame.

49. K. Gröchenig, C. Heil, and K. Okoudjou, **Gabor analysis in weighted amalgam spaces**, Sampl. Theory Signal Image Process., **1** (2003), pp. 225–259. MR 2005e:42118. Zbl. 1044.42025.

Abstract. Gabor frames $\{e^{2\pi i n\beta\cdot x} g(x-k\alpha)\}_{n,k\in\mathbf{Z}^d}$ provide series representations not only of functions in $L^2(\mathbf{R}^d)$ but of the entire range of spaces $M^{p,q}_\nu$ known as the modulation spaces. Membership of a function or distribution f in the modulation space is characterized by a sequence-space norm of the Gabor coefficients of f depending only on the magnitudes of those coefficients, and the Gabor series representation of f converges unconditionally in the norm of the modulation space. This paper shows that Gabor expansions also converge in the entire range of amalgam spaces $W(L^p, L^q_\nu)$, which are not modulation spaces in general but, along with the modulation spaces, play important roles in time-frequency analysis and sampling theory. It is shown that membership of a function or distribution in the amalgam space is characterized by an appropriate sequence space norm of the Gabor coefficients. However, this sequence space norm depends on the phase of the Gabor coefficients as well as their magnitudes, and the Gabor expansions converge conditionally in general. Additionally, some converse results providing necessary conditions on g are obtained.

50. C. Heil, *Integral operators, pseudodifferential operators, and Gabor frames*, in: "Advances in Gabor Analysis," H. G. Feichtinger and T. Strohmer, eds., Birkhäuser, Boston (2003), pp. 153–169. MR 1955935. Zbl. 1036.42030. (Exposition and new results related to Publications #19, 33.)

Abstract. This chapter illustrates the use of Gabor frame analysis to derive results on the spectral properties of integral and pseudodifferential operators. In particular, we obtain a sufficient condition on the kernel of an integral operator or the symbol of a pseudodifferential operator which implies that the operator is trace-class. This result significantly improves a sufficient condition due to Daubechies and Hörmander.

51. K. Gröchenig, D. Han, C. Heil, and G. Kutyniok, **The Balian–Low theorem for symplectic lattices in higher dimensions**, Appl. Comput. Harmon. Anal., **13** (2002), pp. 169–176. MR 2003i:42041. Zbl. 1017.42027.

Abstract. The Balian–Low theorem expresses the fact that time-frequency concentration is incompatible with non-redundancy for Gabor systems that form orthonormal or Riesz bases for $L^2(\mathbf{R})$. We extend the Balian–Low theorem for Riesz bases to higher dimensions, obtaining a weak form valid for all sets of time-frequency shifts which form a lattice in $\mathbf{R}^{2d}$, and a strong form valid for symplectic lattices in $\mathbf{R}^{2d}$. For the orthonormal basis case, we obtain a strong form valid for general non-lattice sets which are symmetric with respect to the origin.

52. R. Ashino, S. J. Desjardins, C. Heil, M. Nagase, and R. Vaillancourt, **Smooth tight frame wavelets and image analysis in Fourier space**, Comput. Math. Appl., **45** (2003), pp. 1551–1579.
MR 2004i:42030. Zbl. 1044.42027.

Abstract. General results on microlocal analysis and tight frames in $\mathbb{R}^2$ are summarized. To perform microlocal analysis of tempered distributions, orthogonal multiwavelets, whose Fourier transforms consist of characteristic functions of squares or sectors of annuli, are constructed in the Fourier domain and are shown to satisfy a multiresolution analysis with several choices of scaling functions. To have good localization in both the x and Fourier domains, redundant smooth tight wavelet frames, with frame bounds equal to one, called Parseval wavelet frames, are obtained in the Fourier domain by properly tapering the above characteristic functions. These nonorthogonal frame wavelets can be generated by two-scale equations from a multiresolution analysis. A natural formulation of the problem is by means of pseudodifferential operators. Singularities, which are added to smooth images, can be localized in position and direction by means of the frame coefficients of the filtered images computed in the Fourier domain. Using Plancherel's theorem, the frame expansion of the filtered images is obtained in the x domain. Subtracting this expansion from the scarred images restores the original images.

53. R. Ashino, S. J. Desjardins, C. Heil, M. Nagase, and R. Vaillancourt, **Microlocal analysis, smooth frames and denoising in Fourier space**, Asian Information-Science-Life, **1** (2002), pp. 153–160.

54. C. Heil and G. Kutyniok, **Density of weighted wavelet frames**, J. Geometric Analysis, **13** (2003), pp. 479–493. MR 2004d:42065. Zbl. 1029.42031.

Abstract. If $\psi \in L^2(\mathbf{R})$, Λ is a discrete subset of the affine group $\mathbf{A} = \mathbf{R}^+ \times \mathbf{R}$, and $w\colon \Lambda \to \mathbf{R}^+$ is a weight function, then the weighted wavelet system generated by ψ, Λ, and w is $\mathcal{W}(\psi, \Lambda, w) = \{w(a,b)^{1/2}\, a^{-1/2}\psi\big(\frac{x}{a} - b\big) : (a,b) \in \Lambda\}$. In this paper we define lower and upper weighted densities $\mathcal{D}^-_w(\Lambda)$ and $\mathcal{D}^+_w(\Lambda)$ of Λ with respect to the geometry of the affine group, and prove that there exist necessary conditions on a weighted wavelet system in order that it possesses frame bounds. Specifically, we prove that if $\mathcal{W}(\psi, \Lambda, w)$ possesses an upper frame bound, then the upper weighted density is finite. Further, for the unweighted case $w = 1$, we prove that if $\mathcal{W}(\psi, \Lambda, 1)$ possesses a lower frame bound and $\mathcal{D}^+_w(\Lambda^{-1}) < \infty$, then the lower density is strictly positive. We apply these results to oversampled affine systems (which include the classical affine and the quasi-affine systems as special cases), to co-affine wavelet systems, and to systems consisting only of dilations, obtaining some new results relating density to the frame properties of these systems.

55. R. Ashino, S. J. Desjardins, C. Heil, M. Nagase, and R. Vaillancourt, *Image restoration through microlocal analysis with smooth tight wavelet frames*, in: "Theoretical Development and Feasibility of Mathematical Analysis on the Computer" (Japanese) (Kyoto, 2002), Sūrikaisekikenkyūsho Kōkyūroku No. 1286 (2002), pp. 101–118.

56. R. Balan, P. G. Casazza, C. Heil, and Z. Landau, **Density, overcompleteness, and localization of frames, I. Theory**, J. Fourier Anal. Appl., **12** (2006), pp. 105–143. MR 2007b:42041. Zbl. 1096.42014.

Abstract. Frames have applications in numerous fields of mathematics and engineering. The fundamental property of frames which makes them so useful is their overcompleteness. In most applications, it is this overcompleteness that is exploited to yield a decomposition that is more stable, more robust, or more compact than is possible using nonredundant systems. This work presents a quantitative framework for describing the overcompleteness of frames. It introduces notions of localization and approximation between two frames $\mathcal{F} = \{f_i\}_{i\in I}$ and $\mathcal{E} = \{e_j\}_{j\in G}$ (G a discrete abelian group), relating the decay of the expansion of the elements of $\mathcal{F}$ in terms of the elements of $\mathcal{E}$ via a map $a\colon I \to G$. A fundamental set of equalities are shown between three seemingly unrelated quantities: the relative measure of $\mathcal{F}$, the relative measure of $\mathcal{E}$ — both of which are determined by certain averages of inner products of frame elements with their corresponding dual frame elements — and the density of the set $a(I)$ in G. Fundamental new results are obtained on the excess and overcompleteness of frames, on the relationship between frame bounds and density, and on the structure of the dual frame of a localized frame. In a subsequent paper, these results are applied to the case of Gabor frames, producing an array of new results as well as clari-

fying the meaning of existing results.

The notion of localization and related approximation properties introduced in this paper are a spectrum of ideas that quantify the degree to which elements of one frame can be approximated by elements of another frame. A comprehensive examination of the interrelations among these localization and approximation concepts is presented.

57. C. Heil, *An introduction to weighted Wiener amalgams*, in: "Wavelets and their Applications" (Chennai, January 2002), M. Krishna, R. Radha and S. Thangavelu, eds., Allied Publishers, New Delhi (2003), pp. 183–216.

    Abstract. Wiener amalgam spaces are a class of spaces of functions or distributions defined by a norm which amalgamates a local criterion for membership in the space with a global criterion. Feichtinger has developed an extensive theory of amalgams allowing a wide range of Banach spaces to serve as local or global components in the amalgam. We present an introduction to a more limited case, namely, the weighted amalgams $W(L^p, L^q_w)$ over the real line, where local and global components are defined solely by integrability criteria. We derive some of the basic properties of these weighted amalgams, and give one application of amalgam spaces to time-frequency analysis, proving the Amalgam Balian-Low Theorem.

58. K. Gröchenig and C. Heil, *Modulation spaces as symbol classes for pseudodifferential operators*, in: "Wavelets and their Applications" (Chennai, January 2002), M. Krishna, R. Radha and S. Thangavelu, eds., Allied Publishers, New Delhi (2003), pp. 151–169.

    Abstract. We investigate the Weyl calculus of pseudodifferential operators with the methods of time-frequency analysis. As symbol classes we use the modulation spaces, which are the function spaces associated to the short-time Fourier transform and the Wigner distribution. We investigate the boundedness and Schatten-class properties of pseudodifferential operators, and furthermore we study their mapping properties between modulation spaces.

59. K. Gröchenig and C. Heil, **Counterexamples for boundedness of pseudodifferential operators**, Osaka J. Math., **41** (2004), pp. 681–691. MR 2005i:35300. Zbl. 02134874.

    Abstract. We give a complete classification of the boundedness properties on L^2 of pseudodifferential operators with symbols in the modulation spacess $M^{p,q}$, $1 \le p, q \le \infty$.

60. C. Heil, P. E. T. Jorgensen and D. R. Larson, Editors, "Wavelets, Frames, and Operator Theory" (College Park, 2003), Contemporary Math., Vol. 345, Amer. Math. Soc., Providence, RI, 2004 (xii+342 pp.). MR 2004m:42001. Zbl. 1052.42002.

61. C. Heil and G. Kutyniok, **The Homogeneous Approximation Property for wavelet frames**, J. Approx. Theory, **147** (2007), pp. 28–46. MR 2008g:42031. Zbl. 1133.42049.

    Abstract. An irregular wavelet frame has the form $\mathcal{W}(\psi,\Lambda) = \{a^{-1/2}\psi(\frac{x}{a} - b)\}_{(a,b)\in\Lambda}$, where $\psi \in L^2(\mathbf{R})$ and Λ is an arbitrary sequence of points in the affine group $\mathbb{A} = \mathbb{R}^+ \times \mathbb{R}$. Such irregular wavelet frames are poorly understood, yet they arise naturally, e.g., from sampling theory or the inevitability of perturbations. This paper proves that irregular wavelet frames satisfy a Homogeneous Approximation Property, which essentially states that the rate of approximation of a wavelet frame expansion of a function f is invariant under time-scale shifts of f, even though Λ is not required to have any structure—it is only required that the wavelet ψ have a modest amount of time-scale concentration. It is shown that the Homogeneous Approximation Property has several implications on the geometry of Λ, and in particular a relationship between the affine Beurling density of the frame and the affine Beurling density of any other Riesz basis of wavelets is derived. This further yields necessary conditions for the existence of wavelet frames, and insight into the fundamental question of why there is no Nyquist density phenomenon for wavelet frames, as there is for Gabor frames that are generated from time-frequency shifts.

62. Á. Bényi, K. Gröchenig, C. Heil, and K. Okoudjou, **Modulation spaces and a class of bounded multilinear pseudodifferential operators**, J. Operator Theory, **54** (2005), pp. 389–401. MR 2006h:47072. Zbl. 1106.47041.

Abstract. We show that multilinear pseudodifferential operators with symbols in the modulation space $M^{\infty,1}$ are bounded on products of modulation spaces. In particular, $M^{\infty,1}$ includes non-smooth symbols. Several multilinear Calderón–Vaillancourt-type theorems are then obtained by using certain embeddings of classical function spaces into modulation spaces.

63. C. Heil, Editor, "Harmonic Analysis and Applications," In honor of John J. Benedetto, Birkhäuser, Boston, 2006 (xxviii+374 pp.). MR 2007d:42002. Zbl. 1095.00007.

Jacket summary. John J. Benedetto has had a profound influence not only on the direction of harmonic analysis and its applications, but also on the entire community of people involved in the field. This self-contained volume in honor of John covers a wide range of topics in harmonic analysis and related areas, including weighted-norm inequalities, frame theory, wavelet theory, time-frequency analysis, and sampling theory. The invited chapters pay tribute to Johns many achievements and express an appreciation for both the mathematical and personal inspiration he has given to so many students, coauthors, and colleagues.

Although the scope of the book is broad, chapters are clustered by topic to provide authoritative expositions that will be of lasting interest. The original papers collected here are written by prominent, well-respected researchers and professionals in the field of harmonic analysis. The book is divided into the following five sections:

- Classical harmonic analysis
- Frame theory
- Time-frequency analysis
- Wavelet theory
- Sampling theory and shift-invariant spaces

"Harmonic Analysis and Applications" is an excellent reference for graduate students, researchers, and professionals in pure and applied mathematics, engineering, and physics.

Contributors: A. Aldroubi, L. Baggett, G. Benke, C. Cabrelli, P. G. Casazza, O. Christensen, W. Czaja, M. Fickus, J.-P. Gabardo, K. Grchenig, K. Guo, E. Hayashi, C. Heil, H. P. Heinig, J. A. Hogan, J. Kovačević, D. Labate, J.D. Lakey, D. Larson, M. T. Leon, S. Li, W.-Q Lim, A. Lindner, U. Molter, A. M. Powell, B. Rom, E. Schulz, T. Sorrells, D. Speegle, K. F. Taylor, J. Tremain, D. Walnut, G. Weiss, E. Wilson, G. Zimmermann.

64. R. Ashino, S. J. Desjardins, C. Heil, M. Nagase, and R. Vaillancourt, *Pseudodifferential operators, microlocal analysis and image restoration*, in: "Advances in Pseudo-Differential Operators," R. Ashino, P. Boggiatto, and M.-W. Wong, eds., Birkhäuser, Boston (2004), pp. 187–202. MR 2006a:35326. Zbl. 1075.35125.

65. C. Heil, *Linear independence of finite Gabor systems*, in: "Harmonic Analysis and Applications," C. Heil, ed., Birkhäuser, Boston (2006), pp. 171–206. MR 2007d:42057. Zbl. 1129.42421.

Abstract. This chapter is an introduction to an open conjecture in time-frequency analysis on the linear independence of a finite set of time-frequency shifts of a given L^2 function. Background and motivation for the conjecture are provided in the form of a survey of related ideas, results, and open problems in frames, Gabor systems, and other aspects of time-frequency analysis, especially those related to independence. The partial results that are known to hold for the conjecture are also presented and discussed.

66. R. Balan, P. G. Casazza, C. Heil, and Z. Landau, **Density, overcompleteness, and localization of frames, II. Gabor systems**, J. Fourier Anal. Appl., **12** (2006), pp. 307–344. MR 2007b:42042. Zbl. 1097.42022.

Abstract. This work develops a quantitative framework for describing the overcompleteness of a large class of frames. A previous paper introduced notions of localization and approximation between two frames $\mathcal{F} = \{f_i\}_{i\in I}$ and $\mathcal{E} = \{e_j\}_{j\in G}$ (G a discrete abelian group), relating the decay of the expansion of the elements of $\mathcal{F}$ in terms of the elements of $\mathcal{E}$ via a map $a\colon I \to G$. This paper shows that those abstract results yield an array of new implications for irregular Gabor frames. Additionally, various Nyquist density results for Gabor frames are recovered as special cases, and in the process both their meaning and implications are clarified. New results are obtained on the excess and overcompleteness of Gabor frames, on the relationship between frame bounds and density, and on the structure of the dual frame of an irregular Gabor frame. More generally, these results apply both to Gabor frames and to systems of Gabor molecules, whose elements share only a common envelope of concentration in the time-frequency plane.

The notions of localization and related approximation properties are a spectrum of ideas that quantify the degree to which elements of one frame can be approximated by elements of another frame. In this paper, a comprehensive examination of the interrelations among these localization and approximation concepts is made, with most implications shown to be sharp.

67. R. Balan, P. G. Casazza, C. Heil, and Z. Landau, **Density, overcompleteness, and localization of frames**, Electron. Res. Announc. Amer. Math. Soc., **12** (2006), pp. 71–86. (Announcement and summary of Publications #56 and #66.) MR 2007b:42042. Zbl. 1142.42313.

68. C. Heil and G. Kutyniok, **Density of frames and Schauder bases of windowed exponentials**, Houston J. Math., **34** (2008), pp. 565–600. MR 2009f:42042. Zbl. 1214.42070.

    Abstract. This paper proves that every frame of windowed exponentials satisfies a Strong Homogeneous Approximation Property with respect to its canonical dual frame, and a Weak Homogeneous Approximation Property with respect to an arbitrary dual frame. As a consequence, a simple proof of the Nyquist density phenomenon satisfied by frames of windowed exponentials with one or finitely many generators is obtained. The more delicate cases of Schauder bases and exact systems of windowed exponentials are also studied. New results on the relationship between density and frame bounds for frames of windowed exponentials are obtained. In particular, it is shown that a tight frame of windowed exponentials must have uniform Beurling density.

69. R. Balan, P. G. Casazza, C. Heil, and Z. Landau, *Excess of Parseval frames*, in: "Wavelets XI" (San Diego, CA, 2005), Proc. SPIE **5914**, M. Papadakis et al., eds., SPIE, Bellingham, WA (2005), pp. 39–46. (Exposition related to Publication #45.)

70. C. Heil, *The Density Theorem and the Homogeneous Approximation Property for Gabor frames*, in: "Representations, Wavelets, and Frames," P. E. T. Jorgensen, K. D. Merrill, and J. A. Packer, eds., Birkhäuser, Boston (2008), pp. 71–102. MR 2009k:42063. Zbl. 1213.42141.

    Abstract. The Density Theorem for Gabor Frames is a fundamental result in time-frequency analysis. Beginning with Baggett's proof that a rectangular lattice Gabor system $\{e^{2\pi i\beta nt}g(t-\alpha k)\}_{n,k\in\mathbf{Z}}$ must be incomplete in $L^2(\mathbf{R})$ whenever $\alpha\beta > 1$, the necessary conditions for a Gabor system to be complete, a frame, a Riesz basis, or a Riesz sequence have been extended to arbitrary lattices and beyond. The first partial proofs of the Density Theorem for irregular Gabor frames were given by Landau in 1993 and by Ramanathan and Steger in 1995. A key fact proved by Ramanathan and Steger is that irregular Gabor frames possess a certain Homogeneous Approximation Property (HAP), and that the Density Theorem is a consequence of this HAP. This chapter provides a brief history of the Density Theorem and a detailed account of the proofs of Ramanathan and Steger. Furthermore, we show that the techniques of Ramanathan and Steger can be used to give a full proof of a general version of Density Theorem for irregular Gabor frames in higher dimensions and with finitely many generators.

71. C. Heil, *History and evolution of the Density Theorem for Gabor frames*, J. Fourier Anal. Appl., **13** (2007), pp. 113–166. MR 2008b:42058. Zbl. 1133.42043.

   Abstract. The Density Theorem for Gabor Frames is one of the fundamental results of time-frequency analysis. This expository survey attempts to reconstruct the long and very involved history of this theorem and to present its context and evolution, from the one-dimensional rectangular lattice setting, to arbitrary lattices in higher dimensions, to irregular Gabor frames, and most recently beyond the setting of Gabor frames to abstract localized frames. Related fundamental principles in Gabor analysis are also surveyed, including the Wexler–Raz biorthogonality relations, the Duality Principle, the Balian–Low Theorem, the Walnut and Janssen representations, and the Homogeneous Approximation Property. An extended bibliography is included.

72. C. Heil and G. Kutyniok, *Convolution and Wiener amalgam spaces on the affine group*, in: "Recent Advances in Computational Sciences," P. E. T. Jorgensen, X. Shen, C.-W. Shu, and N. Yan, eds., World Scientific, Singapore (2008), pp. 209–217. MR 2010c:46063. Zbl. 1161.43004.

73. C. Heil and D. R. Larson, *Operator theory and modulation spaces*, in: "Frames and Operator Theory in Analysis and Signal Processing" (San Antonio, 2006), D. R. Larson et al., eds., Contemp. Math., Vol. 451, Amer. Math. Soc., Providence, RI (2008), pp. 137–150. MR 2008h:42072. Zbl. 1186.47031.

   Abstract. This is a "problems" paper. We isolate some connections between operator theory and the theory of modulation spaces that were stimulated by a question of Feichtinger's regarding integral and pseudodifferential operators. We discuss several problems inspired by this question, and give a reformulation of the original question in operator-theoretic terms. A detailed discussion of the background and context for these problems is included, along with a solution of the problem for the case of finite-rank operators.

74. C. Heil and A. M. Powell, **Gabor Schauder bases and the Balian–Low Theorem**, J. Math. Physics, **47** (2006), pp. 113506-1–113506-21. MR 2008h:42007. Zbl. 1112.42004.

   Abstract. The Balian–Low Theorem is a strong form of the uncertainty principle for Gabor systems which form orthonormal or Riesz bases for $L^2(\mathbf{R})$. In this paper we investigate the Balian–Low Theorem in the setting of Schauder bases. We prove that new weak versions of the Balian–Low Theorem hold for Gabor Schauder bases, but we constructively demonstrate that several variants of the BLT can fail for Gabor Schauder bases that are not Riesz bases. We characterize a class of Gabor Schauder bases in terms of the Zak transform and product $\mathcal{A}_2$ weights; the Riesz bases correspond to the special case of weights that are bounded away from zero and infinity.

75. A. Aldroubi, C. Cabrelli, C. Heil, K. Kornelson, and U. Molter, **Invariance of a shift-invariant space**, J. Fourier Anal. Appl., **16** (2010), pp. 60–75. MR 2011a:42052. Zbl. 1194.42042.

   Abstract. A shift-invariant space is a space of functions that is invariant under integer translations. Such spaces are often used as models for spaces of signals and images in mathematical and engineering applications. This paper characterizes those shift-invariant subspaces S that are also invariant under additional (non-integer) translations. For the case of finitely generated spaces, these spaces are characterized in terms of the generators of the space. As a consequence, it is shown that principal shift-invariant spaces with a compactly supported generator cannot be invariant under any non-integer translations.

76. C. Heil, **"Introduction to Harmonic Analysis"**, book manuscript in preparation.

77. C. Heil, Y. Y. Koo, and J. K. Lim, **Duals of frame sequences**, Acta Appl. Math., **107** (2009), pp. 75–90. MR 2010c:4206. Zbl. 1178.42031.

   Abstract. Frames provide unconditional basis-like, but generally nonunique, representations of vectors in a Hilbert space $\mathcal{H}$. The redundancy of frame expansions allows the flexibility of choosing different dual sequences to employ in frame representations. In particular, oblique duals, Type I duals, and Type II duals have been introduced in the literature because of the special properties that they possess. This paper proves that all Type I and Type II duals are oblique duals, but not conversely, and characterizes the existence of oblique and Type II duals in terms of direct sum decompositions of $\mathcal{H}$, as well as characterizing when the Type I, Type II, and oblique duals will be unique. These results are also applied to the case of shift-generated sequences that are frames for shift-invariant subspaces of $L^2(\mathbf{R}^d)$.

78. S. Bishop, C. Heil, Y. Y. Koo, and J. K. Lim, **Invariances of frame sequences under perturbations**, Linear Algebra Appl., **432** (2010), pp. 1501–1514. MR 2011b:42100. Zbl. 1185.42031.

Abstract. This paper determines the exact relationships that hold among the major Paley–Wiener perturbation theorems for frame sequences. It is shown that major properties of a frame sequence such as excess, deficit, and rank remain invariant under Paley–Wiener perturbations, but need not be preserved by compact perturbations. For localized frames, which are frames with additional structure, it is shown that the frame measure function is also preserved by Paley–Wiener perturbations.

79. C. Heil and A. M. Powell, **Regularity for complete and minimal Gabor systems on a lattice**, Illinois J. Math., **53** (2010), pp. 1077–1094. MR 2011m:42063. Zbl. 1207.42025.

Abstract. Nonsymmetrically weighted extensions of the Balian–Low theorem are proved for Gabor systems $\mathcal{G}(g,1,1)$ that are complete and minimal in $L^2(\mathbf{R})$. For $g \in L^2(\mathbf{R})$, it is proven that if $3 < p \leq 4 \leq q < \infty$ satisfy $3/p + 1/q = 1$ and $\int |x|^p\,|g(x)|^2\,dx < \infty$ and $\int |\xi|^q\,|\hat{g}(\xi)|^2\,d\xi < \infty$ then $\mathcal{G}(g,1,1) = \{e^{2\pi inx} g(x-k)\}_{k,n\in\mathbf{Z}}$ cannot be complete and minimal in $L^2(\mathbf{R})$. For the endpoint case $(p,q) = (3,\infty)$, it is proved that if $g \in L^2(\mathbf{R})$ is compactly supported and $\int |\xi|^3\,|\hat{g}(\xi)|^2\,d\xi < \infty$ then $\mathcal{G}(g,1,1)$ is not complete and minimal in $L^2(\mathbf{R})$. These theorems extend the work of Daubechies and Janssen from the case $(p,q) = (4,4)$. Further refinements and optimal examples are also provided.

80. C. Heil, **"A Basis Theory Primer"**, Expanded Edition, Birkhäuser/Springer, New York, 2011 (xxvi + 534 pp.). MR 2012b:46022. Zbl. 1227.46001. Supplementary material:
   - Solutions manual available for instructors, 281 pp.

Jacket summary. The classical subject of bases in Banach spaces has taken on a new life in the modern development of applied harmonic analysis. This textbook is a self-contained introduction to the abstract theory of bases and redundant frame expansions and its use in both applied and classical harmonic analysis. The four parts of the text take the reader from classical functional analysis and basis theory to modern time-frequency and wavelet theory.

- Part I develops the functional analysis that underlies most of the concepts presented in the later parts of the text.
- Part II presents the abstract theory of bases and frames in Banach and Hilbert spaces, including the classical topics of convergence, Schauder bases, biorthogonal systems, and unconditional bases, followed by the more recent topics of Riesz bases and frames in Hilbert spaces.
- Part III relates bases and frames to applied harmonic analysis, including sampling theory, Gabor analysis, and wavelet theory.
- Part IV deals with classical harmonic analysis and Fourier series, emphasizing the role played by bases, which is a different viewpoint from that taken in most discussions of Fourier series.

Key features:

- Self-contained presentation with clear proofs is accessible to graduate students, pure and applied mathematicians, and engineers interested in the mathematical underpinnings of applications.
- Extensive exercises complement the text and provide opportunities for learning-by-doing, making the text suitable for graduate-level courses; hints for selected exercises are included at the end of the book.

• A separate solutions manual is available for instructors upon request at
`www.birkhauser-science.com/978-9-8176-4686-8/` .

• No other text develops the ties between classical basis theory and its modern uses in applied harmonic analysis.

A Basis Theory Primer is suitable for independent study or as the basis for a graduate-level course. Instructors have several options for building a course around the text depending on the level and background of their students.

81. G. J. Yoon and C. Heil, **Duals of weighted exponentials**, Acta Appl. Math., **119** (2012), pp. 97–112. MR 2915572. Zbl. 06118580.

Abstract. The paper considers the basis and frame properties of the system of weighted exponentials $\mathcal{E}(g,\mathbf{Z}\backslash F) = \{e^{2\pi inx}g(x)\}_{n\in\mathbf{Z}\backslash F}$ in $L^2(\mathbf{T})$, where $g \in L^2(\mathbf{T})\backslash\{0\}$ and $F \subset \mathbf{Z}$. It is shown that many of the frame properties of $\mathcal{E}(g,\mathbf{Z}\backslash F)$ are affected by the cardinalities of F and the behavior of the zeros of g.

82. R. Tinaztepe and C. Heil, **Modulation spaces, BMO, and the Balian–Low Theorem**, Sampl. Theory Signal Image Process., **11** (2012), pp. 25–41.

Abstract. The modulation spaces $M^{p,q}_m(\mathbf{R}^d)$ quantify the time-frequency concentration of functions and distributions. The first main result of this paper proves proves embeddings of certain modulation spaces into VMO, the space of functions with vanishing mean oscillation. The second main result proves that the Zak transform maps certain modulation spaces on $\mathbf{R}^d$ into modulation spaces on $\mathbf{R}^{2d}$. These two results allow us to give a Balian–Low-type of uncertainty principle for Gabor systems in the setting of modulation spaces.

83. S. Bishop, C. Heil, Y. Y. Koo, and J. K. Lim, *Duals and invariances of frame sequences*, in: "Wavelets XIII" (San Diego, CA, 2009), Proc. SPIE **7446**, V. Goyal et al., eds., SPIE, Bellingham, WA (2009), pp. 74460K1–74460K8.

84. C. Heil, **"A Brief Guide to Metrics, Norms, and Inner Products"**, electronic manuscript, 2016, 64 pp. (A much expanded and improved version is available in Publication #90.) Supplementary material:

    • Solutions manual available for instructors.

Summary. This short manuscript introduces metric spaces, normed spaces, and inner product spaces. Topics include:

• Metrics: Convergence, completeness, compactness, continuity, exercises.

• Norms: Properties of norms, Banach spaces, infinite series, span and closed span, Schauder bases, exercises.

• Inner products: Properties, Hilbert spaces, orthogonal complements and the orthogonal projection, orthonormal sequences, exercises.

85. C. Heil, *WHAT IS . . . a frame?*, Notices Amer. Math. Soc., **60** (2013), pp. 748–750. MR 3076247.

MathSciNet review. This short expository article explains the salient points of frame theory, lists a number of practical applications where frame theory plays an important role, and describes three open problems: the Feichtinger Conjecture, the Paving Conjecture, and the Linear Independence of Time-Frequency Translates Conjecture. There are also four well-chosen references for further reading. Reviewed by R. A. Zalik.

86. C. Heil, **"Introduction to Real Analysis"**, Springer, Cham, 2019 (xvii + 400 pp.). MR 3967720. Zbl. 07093485. Supplementary material:

    • Online course guide, 131 pp.

- Online Chapter 0 (Expanded Notation and Preliminaries), 49 pp.
- Online Alternative Chapter 1 (An Introduction to Norms and Banach Spaces), 62 pp.
- Online Chapter 10 (Abstract Measure Theory), 27 pp.
- Selected Solutions for Students, 39 pp.
- Solutions manual available for instructors, 426 pp.

From the preface. This text is an introduction to real analysis. There are several classic analysis texts that I keep close by on my bookshelf and refer to often. However, I find it difficult to use any of these as the textbook for teaching a first course on analysis. They tend to be dense and, in the classic style of mathematical elegance and conciseness, they develop the theory in the most general setting, with few examples and limited motivation. These texts are valuable resources, but I suggest that they should be the second set of books on analysis that you pick up. I hope that this text will be the analysis text that you read first. The definitions, theorems, and other results are motivated and explained; the why and not just the what of the subject is discussed. Proofs are completely rigorous, yet difficult arguments are motivated and discussed . . . .

87. M. D. Weir and J. Haas, with the assistance of C. Heil:
    - **"Thomas' Calculus"**, 13th Edition, Pearson, Boston, 2014 (xiv + 1032 pp. + appendices);
    - **"Thomas' Calculus: Early Transcendentals"**, 13th Edition, Pearson, Boston, 2014 (xiv + 1044 pp. + appendices);
    - **"University Calculus: Early Transcendentals"**, 3rd Edition, Pearson, Boston, 2016 (vii + 912 pp. + appendices).

88. C. Heil and D. Speegle, *The HRT Conjecture and the Zero Divisor Conjecture for the Heisenberg group*, in: "Excursions in Harmonic Analysis, Volume 3," R. Balan et al., eds., Birkhäuser/ Springer, Cham (2015), pp. 159–176. MR 3380410. Zbl. 06492117.

    Abstract. This chapter reports on the current status of the HRT Conjecture (also known as the Linear Independence of Time-Frequency Shifts Conjecture), and discusses its relationship with a longstanding conjecture in algebra known as the Zero Divisor Conjecture.

89. C. Heil, D. Jacobs, and R. Tinaztepe, **Smoothness of refinable function vectors on $\mathbf{R}^n$**, Int. J. Wavelets Multiresolut. Inf. Process., **15** (2017) 1750051 (16 pages). MR 3690431. Zbl. 06787544.

    Abstract. Let A be a dilation matrix, an $n \times n$ expansive matrix that maps $\mathbf{Z}^n$ into itself. Let Λ be a finite subset of $\mathbf{Z}^n$, and for $k \in \Lambda$ let c_k be $r \times r$ complex matrices. The refinement equation corresponding to A, $\mathbf{Z}^n$, Λ, and $c = \{c_k\}_{k\in\Lambda}$ is $f(x) = \sum_{k\in\Lambda} c_k\, f(Ax - k)$. A solution $f\colon \mathbf{R}^n \to \mathbf{C}^r$, if one exists, is called a refinable vector function or a vector scaling function of multiplicity r. This paper characterizes the higher-order smoothness of compactly supported solutions of the refinement equation, in terms of the p-norm joint spectral radius of a finite set of finite matrices determined by the coefficients c_k.

90. C. Heil, **"Metrics, Norms, Inner Products, and Operator Theory"**, Birkhäuser/Springer, Cham, 2018 (xxi + 359 pp.). MR 3838450. Zbl. 06764326. Supplementary material:
    - Online Chapter 8 (Integral Operators), 17 pp.
    - Extra online material, 103 pp.
    - Solutions manual available for instructors, 273 pp.

91. J. Haas, C. Heil, and M. D. Weir:
    - **"Thomas' Calculus"**, 14th Edition, Pearson, Boston, 2018 (xviii + 1048 pp. + appendices);
    - **"Thomas' Calculus: Early Transcendentals"**, 14th Edition, Pearson, Boston,

2018 (xviii + 1062 pp. + appendices);

- J. Haas, C. Heil, P. Bogacki, and M. D. Weir: **"University Calculus: Early Transcendentals"**, 4th Edition, Pearson, Hoboken, NJ, 2020 (xviii + 938 pp. + appendices).

92. C. Heil, *Reflections on a Theorem of Boas and Pollard*, in: "Harmonic Analysis and Applications," M. Rassias, ed., Springer, to appear (11 pages).

    Abstract. Inspired by an elegant theorem of Boas and Pollard (and related results by Kazarian, Price, Talalyan, Zink, and others), we discuss multiplicative completion of redundant systems in Hilbert and Banach function spaces.

93. C. Heil, *Foreword*, in: "New Trends in Applied Harmonic Analysis," Volume 2, A. Aldroubi, C. Cabrelli, S. Jaffard, and U. Molter, eds., Birkhäuser/Springer, 2019, pp. ix–xiii.

94. C. Heil, *Absolute Continuity and the Banach–Zaretsky Theorem*, in: "Excursions in Harmonic Analysis," R. Balan et al., ed., Springer, New York (2019), submitted (25 pages).

    Abstract. The Banach–Zaretsky Theorem is a fundamental but often overlooked result that characterizes the functions that are absolutely continuous. This chapter presents basic results on differentiability, absolute continuity, and the Fundamental Theorem of Calculus with an emphasis on the role of the Banach–Zaretsky Theorem.