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Five Mini-Courses on Analysis

Metrics, Norms, Inner Products, and Topology

Lebesgue Measure and Integral

Operator Theory and Functional Analysis

Borel and Radon Measures

Topological Vector Spaces

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