FUNCTIONAL ANALYSIS LECTURE NOTES:

ADJOINTS IN HILBERT SPACES

CHRISTOPHER HEIL

1. Adjoints in Hilbert Spaces

Recall that the dot product on $\mathbb{R}^n$ is given by $x \cdot y = x^{\mathbb{T}} y$, while the dot product on $\mathbb{C}^n$ is $x \cdot y = x^{\mathbb{T}} \bar{y}$.

Example 1.1. Let A be an $m \times n$ real matrix. Then $x \mapsto Ax$ defines a linear map of $\mathbb{R}^n$ into $\mathbb{R}^m$, and its transpose $A^{\mathbb{T}}$ satisfies

$$\forall\, x \in \mathbb{R}^n, \quad \forall\, y \in \mathbb{R}^m, \quad Ax \cdot y \;=\; (Ax)^{\mathbb{T}} y \;=\; x^{\mathbb{T}} A^{\mathbb{T}} y \;=\; x \cdot (A^{\mathbb{T}} y).$$

Similarly, if A is an $m \times n$ complex matrix, then its *Hermitian* or *adjoint* matrix $A^{\mathrm{H}} = \overline{A^{\mathbb{T}}}$ satisfies

$$\forall\, x \in \mathbb{C}^n, \quad \forall\, y \in \mathbb{C}^m, \quad Ax \cdot y \;=\; (Ax)^{\mathbb{T}} \bar{y} \;=\; x^{\mathbb{T}} A^{\mathbb{T}} \bar{y} \;=\; x \cdot (A^{\mathrm{H}} y).$$

Theorem 1.2 (Adjoint)**.** Let H and K be Hilbert spaces, and let $A\colon H \to K$ be a bounded, linear map. Then there exists a unique bounded linear map $A^*\colon K \to H$ such that

$$\forall\, x \in H, \quad \forall\, y \in K, \quad \langle Ax, y\rangle \;=\; \langle x, A^* y\rangle.$$

Proof. Fix $y \in K$. Then $Lx = \langle Ax, y\rangle$ is a bounded linear functional on H. By the Riesz Representation Theorem, there exists a unique vector $h \in H$ such that

$$\langle Ax, y\rangle \;=\; Lx \;=\; \langle x, h\rangle.$$

Define $A^* y = h$. Verify that this map A^* is linear (exercise). To see that it is bounded, observe that

$$\begin{aligned}
\|A^* y\| \;=\; \|h\| \;&=\; \sup_{\|x\|=1} |\langle x, h\rangle| \\
&=\; \sup_{\|x\|=1} |\langle Ax, y\rangle| \\
&\le\; \sup_{\|x\|=1} \|Ax\|\,\|y\| \\
&\le\; \sup_{\|x\|=1} \|A\|\,\|x\|\,\|y\| \;=\; \|A\|\,\|y\|.
\end{aligned}$$

These notes closely follow and expand on the text by John B. Conway, "A Course in Functional Analysis," Second Edition, Springer, 1990.

We conclude that A^* is bounded, and that $\|A^*\| \le \|A\|$.

Finally, we must show that A^* is unique. Suppose that $B \in \mathcal{B}(K, H)$ also satisfied $\langle Ax, y\rangle = \langle x, By\rangle$ for all $x \in H$ and $y \in K$. Then for each fixed y we would have that $\langle x, By - A^*y\rangle = 0$ for every x, which implies $By - A^*y = 0$. Hence $B = A^*$. □

Exercise 1.3 (Properties of the adjoint)**.**

(a) If $A \in \mathcal{B}(H, K)$ then $(A^*)^* = A$.

(b) If $A, B \in \mathcal{B}(H, K)$ and $\alpha, \beta \in \mathbb{F}$, then $(\alpha A + \beta B)^* = \bar{\alpha} A^* + \bar{\beta} B^*$.

(c) If $A \in \mathcal{B}(H_1, H_2)$ and $B \in \mathcal{B}(H_2, H_3)$, then $(BA)^* = A^*B^*$.

(d) If $A \in \mathcal{B}(H)$ is invertible in $\mathcal{B}(H)$ (meaning that there exists $A^{-1} \in \mathcal{B}(H)$ such that $AA^{-1} = A^{-1}A = I$), then A^* is invertible in $\mathcal{B}(H)$ and $(A^{-1})^* = (A^*)^{-1}$.

Note: By the Inverse Mapping Theorem, A is invertible in $\mathcal{B}(H)$ if and only if A is a bounded linear bijection.

Proposition 1.4. If $A \in \mathcal{B}(H, K)$, then $\|A\| = \|A^*\| = \|A^*A\|^{1/2} = \|AA^*\|^{1/2}$.

Proof. In the course of proving Theorem 1.2, we already showed that $\|A^*\| \le \|A\|$. If $f \in H$, then

$$\|Af\|^2 \;=\; \langle Af, Af\rangle \;=\; \langle A^*Af, f\rangle \;\le\; \|A^*Af\|\,\|f\| \;\le\; \|A^*\|\,\|Af\|\,\|f\|. \tag{1.1}$$

Hence $\|Af\| \le \|A^*\|\,\|f\|$ (even if $\|Af\| = 0$, this is still true). Since this is true for all f we conclude that $\|A\| \le \|A^*\|$. Therefore $\|A\| = \|A^*\|$.

Next, we have $\|A^*A\| \le \|A\|\,\|A^*\| = \|A\|^2$. But also, from the calculation in (1.1), we have $\|Af\|^2 \le \|A^*Af\|\,\|f\|$. Taking the supremum over all unit vectors, we obtain

$$\|A\|^2 \;=\; \sup_{\|f\|=1} \|Af\|^2 \;\le\; \sup_{\|f\|=1} \|A^*Af\|\,\|f\| \;=\; \|A^*A\|.$$

Consequently $\|A\|^2 = \|A^*A\|$. The final equality follows by interchanging the roles of A and A^*. □

Exercise 1.5. Prove that if $U \in \mathcal{B}(H, K)$, then U is an isomorphism if and only if U is invertible and $U^{-1} = U^*$.

Exercise 1.6. Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for a separable Hilbert space H. Then we know that every $f \in H$ can be written

$$f \;=\; \sum_{n=1}^{\infty} \langle f, e_n\rangle\, e_n.$$

If $\lambda = (\lambda_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N})$ is given, then

$$Lf \;=\; \sum_{n=1}^{\infty} \lambda_n \,\langle f, e_n\rangle\, e_n \tag{1.2}$$

is a bounded linear map of L into itself. Find L^*.

Exercise 1.7. Let (X, Ω, μ) be a measure space, and let $\phi \in L^\infty(X)$ be a fixed measurable function. Then $M_\phi \colon L^2(X) \to L^2(X)$ given by

$$M_\phi f \;=\; f\phi, \qquad f \in L^2(X)$$

is a bounded linear operator. Prove that the adjoint of M_ϕ is the multiplication operator $M_{\bar\phi}$.

Exercise 1.8. Let L and R be the left- and right-shift operators on $\ell^2(\mathbb{N})$, i.e.,

$$L(x_1, x_2, \dots) \;=\; (x_2, x_3, \dots) \qquad \text{and} \qquad R(x_1, x_2, \dots) \;=\; (0, x_1, x_2, \dots).$$

Prove that $L = R^*$.

Example 1.9. Let (X, Ω, μ) be a σ-finite measure space. An *integral operator* is an operator of the form

$$Lf(x) \;=\; \int_X k(x,y)\, f(y)\, d\mu(y). \tag{1.3}$$

Assume that k is chosen so that $L \colon L^2(X) \to L^2(X)$ is bounded. The adjoint is the unique operator $L^* \colon L^2(X) \to L^2(X)$ which satisfies

$$\langle Lf, g\rangle \;=\; \langle f, L^*g\rangle, \qquad f, g \in L^2(X).$$

To find L^*, let $A \colon L^2(X) \to L^2(X)$ be the integral operator with kernel $\overline{k(y,x)}$, i.e.,

$$Af(x) \;=\; \int_X \overline{k(y,x)}\, f(y)\, d\mu(y).$$

Then, given any f and $g \in L^2(X)$, we have

$$\begin{aligned}
\langle f, L^*g\rangle \;=\; \langle Lf, g\rangle \;&=\; \int_X Lf(x)\, \overline{g(x)}\, d\mu(x) \\
&=\; \int_X \int_X k(x,y)\, f(y)\, d\mu(y)\, \overline{g(x)}\, d\mu(x) \\
&=\; \int_X f(y) \int_X k(x,y)\, \overline{g(x)}\, d\mu(x)\, d\mu(y) \\
&=\; \int_X f(y)\, \overline{\int_X \overline{k(x,y)}\, g(x)\, d\mu(x)}\, d\mu(y) \\
&=\; \int_X f(y)\, \overline{Ag(y)}\, d\mu(y) \\
&=\; \langle f, Ag\rangle.
\end{aligned}$$

By uniqueness of the adjoint, we must have $L^* = A$.

Exercise: Justify the interchange in the order of integration in the above calculation, i.e., provide hypotheses under which the calculations above are justified.

Exercise 1.10. Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for a separable Hilbert space H. Define $T\colon H \to \ell^2(\mathbb{N})$ by $T(f) = \{\langle f, e_n\rangle\}_{n\in\mathbb{N}}$. Find a formula for $T^*\colon \ell^2(\mathbb{N}) \to H$.

Definition 1.11. Let $A \in \mathcal{B}(H)$.

(a) We say that A is *self-adjoint* or *Hermitian* if $A = A^*$.

(b) We say that A is *normal* if $AA^* = A^*A$.

Example 1.12. A real $n \times n$ matrix A is self-adjoint if and only if it is symmetric, i.e., if $A = A^{\mathrm{T}}$. A complex $n \times n$ matrix A is self-adjoint if and only if it is Hermitian, i.e., if $A = A^{\mathrm{H}}$.

Exercise 1.13. Show that every self-adjoint operator is normal. Show that every unitary operator is normal, but that a unitary operator need not be self-adjoint. For $H = \mathbb{C}^n$, find examples of matrices that are not normal. Are the left- and right-shift operators on $\ell^2(\mathbb{N})$ normal?

Exercise 1.14. (a) Show that if A, $B \in \mathcal{B}(H)$ are self-adjoint, then AB is self-adjoint if and only if $AB = BA$.

(b) Give an example of self-adjoint operators A, B such that AB is not self-adjoint.

(c) Show that if A, $B \in \mathcal{B}(H)$ are self-adjoint then $A + A^*$, AA^*, A^*A, $A + B$, ABA, and BAB are all self-adjoint. What about $A - A^*$ or $A - B$? Show that $AA^* - A^*A$ is self-adjoint.

Exercise 1.15. (a) Let $\lambda = (\lambda_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N})$ be given and let L be defined as in equation 1.2. Show that L is normal, find a formula for L^*, and prove that L is self-adjoint if and only if each λ_n is real.

(b) Determine a necessary and sufficient condition on ϕ so that the multiplication operator M_ϕ defined in Exercise 1.7 is self-adjoint.

(c) Determine a necessary and sufficient condition on the kernel k so that the integral operator L defined in equation(1.3) is self-adjoint.

The following result gives a useful condition for telling when an operator on a *complex* Hilbert space is self-adjoint.

Proposition 1.16. Let H be a complex Hilbert space (i.e., $\mathbb{F} = \mathbb{C}$), and let $A \in \mathcal{B}(H)$ be given. Then:

$$A \text{ is self-adjoint} \quad \Longleftrightarrow \quad \langle Af, f\rangle \in \mathbb{R} \quad \forall\, f \in H.$$

Proof. $\Rightarrow$. Assume $A = A^*$. Then for any $f \in H$ we have

$$\overline{\langle Af, f\rangle} \;=\; \langle f, Af\rangle \;=\; \langle A^*f, f\rangle \;=\; \langle Af, f\rangle.$$

Therefore $\langle Af, f\rangle$ is real.

$\Leftarrow$. Assume that $\langle Af, f\rangle$ is real for all f. Choose any f, $g \in H$. Then

$$\langle A(f+g), f+g\rangle \;=\; \langle Af, f\rangle + \langle Af, g\rangle + \langle Ag, f\rangle + \langle Ag, g\rangle.$$

Since $\langle A(f+g), f+g\rangle$, $\langle Af, f\rangle$, and $\langle Ag, g\rangle$ are all real, we conclude that $\langle Af, g\rangle + \langle Ag, f\rangle$ is real. Hence it equals its own complex conjugate, i.e.,

$$\langle Af, g\rangle + \langle Ag, f\rangle \;=\; \overline{\langle Af, g\rangle + \langle Ag, f\rangle} \;=\; \langle g, Af\rangle + \langle f, Ag\rangle. \tag{1.4}$$

Similarly, since

$$\langle A(f+ig), f+ig\rangle \;=\; \langle Af, f\rangle - i\langle Af, g\rangle + i\langle Ag, f\rangle + \langle Ag, g\rangle$$

we see that

$$-i\langle Af, g\rangle + i\langle Ag, f\rangle \;=\; \overline{-i\langle Af, g\rangle + i\langle Ag, f\rangle} \;=\; i\langle g, Af\rangle - i\langle f, Ag\rangle.$$

Multiplying through by i yields

$$\langle Af, g\rangle - \langle Ag, f\rangle \;=\; -\langle g, Af\rangle + \langle f, Ag\rangle. \tag{1.5}$$

Adding (1.4) and (1.5) together, we obtain

$$2\langle Af, g\rangle \;=\; 2\langle f, Ag\rangle \;=\; 2\langle A^*f, g\rangle.$$

Since this is true for every f and g, we conclude that $A = A^*$. □

Example 1.17. The preceding result is false for real Hilbert spaces. After all, if $\mathbb{F} = \mathbb{R}$ then $\langle Af, f\rangle$ is real for every f no matter what A is. Therefore, any non-self-adjoint operator provides a counterexample. For example, if $H = \mathbb{R}^n$ then any non-symmetric matrix A is a counterexample.

The next result provides a useful way of calculating the operator norm of a self-adjoint operator.

Proposition 1.18. If $A \in \mathcal{B}(H)$ is self-adjoint, then

$$\|A\| \;=\; \sup_{\|f\|=1} |\langle Af, f\rangle|.$$

Proof. Set $M = \sup_{\|f\|=1} |\langle Af, f\rangle|$.

By Cauchy–Schwarz and the definition of operator norm, we have

$$M \;=\; \sup_{\|f\|=1} |\langle Af, f\rangle| \;\le\; \sup_{\|f\|=1} \|Af\|\,\|f\| \;\le\; \sup_{\|f\|=1} \|A\|\,\|f\|\,\|f\| \;=\; \|A\|.$$

To get the opposite inequality, note that if f is any nonzero vector in H then $f/\|f\|$ is a unit vector, so $\big\langle A\frac{f}{\|f\|}, \frac{f}{\|f\|}\big\rangle \le M$. Rearranging, we see that

$$\forall\, f \in H, \quad \langle Af, f\rangle \;\le\; M\,\|f\|^2. \tag{1.6}$$

Now choose any $f,\, g \in H$ with $\|f\| = \|g\| = 1$. Then, by expanding the inner products, canceling terms, and using the fact that $A = A^*$, we see that

$$\begin{aligned}\big\langle A(f+g),\, f+g\big\rangle - \big\langle A(f-g),\, f-g\big\rangle &= 2\,\langle Af, g\rangle + 2\,\langle Ag, f\rangle \\ &= 2\,\langle Af, g\rangle + 2\,\langle g, Af\rangle \\ &= 4\,\mathrm{Re}\,\langle Af, g\rangle.\end{aligned}$$

Therefore, applying (1.6) and the Parallelogram Law, we have

$$\begin{aligned}4\,\mathrm{Re}\,\langle Af, g\rangle &\le |\langle A(f+g), f+g\rangle| + |\langle A(f-g), f-g\rangle| \\ &\le M\,\|f+g\|^2 + M\,\|f-g\|^2 \\ &= 2M\left(\|f\|^2 + \|g\|^2\right) \;=\; 4M.\end{aligned}$$

That is, $\mathrm{Re}\,\langle Af, g\rangle \le M$ for every choice of unit vectors f and g. Write $\langle Af, g\rangle = |\langle Af, g\rangle|\, e^{i\theta}$. Then $e^{i\theta} g$ is another unit vector, so

$$M \;\ge\; \mathrm{Re}\,\langle Af, e^{-i\theta} g\rangle \;=\; \mathrm{Re}\, e^{i\theta}\langle Af, g\rangle \;=\; |\langle Af, g\rangle|.$$

Hence

$$\|Af\| \;=\; \sup_{\|g\|=1} |\langle Af, g\rangle| \;\le\; M.$$

Since this is true for every unit vector f, we conclude that $\|A\| \le M$. □

The following corollary is a very useful consequence.

Corollary 1.19. Assume that $A \in \mathcal{B}(H)$.

(a) If $\mathbb{F} = \mathbb{R}$, $A = A^*$, and $\langle Af, f\rangle = 0$ for every f, then $A = 0$.

(b) If $\mathbb{F} = \mathbb{C}$ and $\langle Af, f\rangle = 0$ for every f, then $A = 0$.

Proof. Assume the hypotheses of either statement (a) or statement (b). In the case of statement (a), we have by hypothesis that A is self-adjoint. In the case of statement (b), we can conclude that A is self-adjoint because $\langle Af, f\rangle = 0$ is real for every f. Hence in either case we can apply Proposition 1.18 to conclude that

$$\|A\| \;=\; \sup_{\|f\|=1} |\langle Af, f\rangle| \;=\; 0.$$ □

Lemma 1.20. If $A \in \mathcal{B}(H)$, then the following statements are equivalent.

(a) A is normal, i.e., $AA^* = A^*A$.

(b) $\|Af\| = \|A^*f\|$ for every $f \in H$.

Proof. (b) $\Rightarrow$ (a). Assume that (b) holds. Then for every f we have

$$\begin{aligned} \big\langle (A^*A - AA^*)f,\, f\big\rangle \;&=\; \langle A^*Af, f\rangle - \langle AA^*f, f\rangle \\ &=\; \langle Af, Af\rangle - \langle A^*f, A^*f\rangle \\ &=\; \|Af\|^2 - \|A^*f\|^2 \;=\; 0. \end{aligned}$$

Since $A^*A - AA^*$ is self-adjoint, it follows from Corollary 1.19 that $A^*A - AA^* = 0$.

(a) $\Rightarrow$ (b). Exercise. □

Corollary 1.21. If $A \in \mathcal{B}(H)$ is normal, then $\ker(A) = \ker(A^*)$.

Exercise 1.22. Suppose that $A \in \mathcal{B}(H)$ is normal. Prove that A is injective if and only if range(A) is dense in H.

Exercise 1.23. If $A \in \mathcal{B}(H)$, then the following statements are equivalent.

(a) A is an isometry, i.e., $\|Af\| = \|f\|$ for every $f \in H$.

(b) $A^*A = I$.

(c) $\langle Af, Ag\rangle = \langle f, g\rangle$ for every $f,\, g \in H$.

Exercise 1.24. If $H = \mathbb{C}^n$ and $A,\, B$ are $n \times n$ matrices, then $AB = I$ implies $BA = I$. Give a counterexample to this for an infinite-dimensional Hilbert space. Consequently, the hypothesis $A^*A = I$ in the preceding result does not imply that $AA^* = I$.

Exercise 1.25. If $A \in \mathcal{B}(H)$, then the following statements are equivalent.

(a) $A^*A = AA^* = I$.

(b) A is unitary, i.e., it is a surjective isometry.

(c) A is a normal isometry.

The following result provides a very useful relationship between the range of A^* and the kernel of A.

Theorem 1.26. Let $A \in \mathcal{B}(H, K)$.

(a) $\ker(A) = \mathrm{range}(A^*)^\perp$.

(b) $\ker(A)^\perp = \overline{\mathrm{range}(A^*)}$.

(c) A is injective if and only if range(A^*) is dense in H.

Proof. (a) Assume that $f \in \ker(A)$ and let $h \in \mathrm{range}(A^*)$, i.e., $h = A^*g$ for some $g \in K$. Then since $Af = 0$, we have $\langle f, h \rangle = \langle f, A^*g \rangle = \langle Af, g \rangle = 0$. Thus $f \in \mathrm{range}(A^*)^\perp$, so $\ker(A) \subseteq \mathrm{range}(A^*)^\perp$.

Now assume that $f \in \mathrm{range}(A^*)^\perp$. Then for any $h \in H$ we have $\langle Af, h \rangle = \langle f, A^*h \rangle = 0$. But this implies $Af = 0$, so $f \in \ker(A)$. Thus $\mathrm{range}(A^*)^\perp \subseteq \ker(A)$.

(b), (c) Exercises. □