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A Short Review of Cardinality

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Chapter 1
Cardinality

We will give a short review of the definition of cardinality and prove some facts about the cardinality of sets. Throughout, the set of natural numbers is denoted by

$$\mathbb{N} = \{1, 2, 3, \dots\},$$

the set of integers is

$$\mathbb{Z} = \{\dots, -1, 0, 1, \dots\},$$

the set of rational numbers is

$$\mathbb{Q} = \{m/n : m, n \in \mathbb{Z},\ n \neq 0\},$$

the set of real numbers is

$$\mathbb{R} = \{x : x \text{ is a real number}\},$$

and the set of complex numbers is

$$\mathbb{C} = \{a + ib : a, b \in \mathbb{R}\}.$$

1.1 The Definition of Cardinality

We say that two sets A and B *have the same cardinality* if there exists a bijection f that maps A onto B, i.e., if there is a function $f\colon A \to B$ that is both injective and surjective. Such a function f pairs each element of A with a unique element of B and vice versa, and therefore is sometimes called a *1-1 correspondence.*

Example 1.1.1. (a) The function $f\colon [0,2] \to [0,1]$ defined by $f(x) = x/2$ for $0 \le x \le 2$ is a bijection, so the intervals $[0,2]$ and $[0,1]$ have the same cardinality. This shows that a proper subset of a set can have the same cardinality as the set itself.

(b) The function $f\colon \mathbb{N} \to \{2,3,4,\dots\}$ defined by $f(n) = n+1$ for $n \in \mathbb{N}$ is a bijection, so the set of natural numbers $\mathbb{N} = \{1,2,3,\dots\}$ has the same cardinality as its proper subset $\{2,3,4,\dots\}$.

(c) The function $f\colon \mathbb{N} \to \mathbb{Z}$ defined by

$$f(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even,} \\ -\frac{n-1}{2}, & \text{if } n \text{ is odd,} \end{cases}$$

is a bijection, so the set of integers $\mathbb{Z}$ has the same cardinality as the set of natural numbers $\mathbb{N}$.

(d) If n is a finite positive integer, then there is no way to define a function $f\colon \{1,\dots,n\} \to \mathbb{N}$ that is a bijection. Hence $\{1,\dots,n\}$ and $\mathbb{N}$ do not have the same cardinality. Likewise, if $m \neq n$ are distinct positive integers, then $\{1,\dots,m\}$ and $\{1,\dots,n\}$ do not have the same cardinality. $\diamondsuit$

1.2 Finite, Countable, and Uncountable Sets

We use cardinality to define finite sets and infinite sets, as follows.

Definition 1.2.1 (Finite and Infinite Sets). Let X be a set. We say that X is *finite* if X is either empty or there exists an integer $n > 0$ such that X has the same cardinality as the set $\{1,\dots,n\}$. That is, a nonempty X is finite if there is a bijection of the form

$$f\colon \{1,\dots,n\} \to X.$$

In this case we say that X *has n elements*.

We say that X is *infinite* if it is not finite. $\diamondsuit$

We use the following terminology to further distinguish among sets based on cardinality.

Definition 1.2.2 (Countable and Uncountable Sets). We say that a set X is:

- *denumerable* or *countably infinite* if it has the same cardinality as the natural numbers, i.e., if there exists a bijection $f\colon \mathbb{N} \to X$,
- *countable* if X is either finite *or* countably infinite,
- *uncountable* if X is not countable. $\diamondsuit$

Every finite set is countable by definition, and parts (b) and (c) of Example 1.1.1 show that $\mathbb{N}$, $\mathbb{Z}$, and $\{2, 3, 4, \dots\}$ are countable. Here is another countable set.

Example 1.2.3. Consider $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N} = \bigl\{(j,k) : j,k \in \mathbb{N}\bigr\}$, the set of all ordered pairs of positive integers. We display the elements of $\mathbb{N}^2$ as the following table of pairs (with additional arrows that we will shortly explain):

$$
\begin{array}{ccccccccccc}
(1,1) & \to & (1,2) & & (1,3) & \to & (1,4) & & (1,5) & \to & (1,6) \\
 & \swarrow & & \nearrow & & \swarrow & & \nearrow & & \swarrow & \\
(2,1) & & (2,2) & & (2,3) & & (2,4) & & (2,5) & & \\
\downarrow & \nearrow & & \swarrow & & \nearrow & & \swarrow & & & \\
(3,1) & & (3,2) & & (3,3) & & (3,4) & & & & \\
 & \swarrow & & \nearrow & & \swarrow & & & & & \\
(4,1) & & (4,2) & & (4,3) & & & & & & \\
\downarrow & \nearrow & & \swarrow & & & & & & & \\
(5,1) & & (5,2) & & & & & & & & \\
 & \swarrow & & \nearrow & & & & & & & \\
(6,1) & & (6,2) & & & & & & & & \\
\downarrow & \nearrow & & & & & & & & & \\
(7,1) & & & & & & & & & &
\end{array}
$$

The first line of pairs in this table includes all ordered pairs whose first component is 1, the second line of pairs lists those whose first component is 2, and so forth. We define a bijection $f \colon \mathbb{N} \to \mathbb{N}^2$ by following the arrows in the table:

$$
\begin{aligned}
f(1) &= (1,1), \\
f(2) &= (1,2), \\
f(3) &= (2,1), \\
f(4) &= (3,1), \\
f(5) &= (2,2), \\
f(6) &= (1,3), \\
f(7) &= (1,4), \\
f(8) &= (2,3), \\
&\ \vdots
\end{aligned}
$$

In other words, once $f(n)$ has been defined to be a particular ordered pair (j,k), then we let $f(n+1)$ be the ordered pair that (j,k) points to next. In this way the outputs $f(1), f(2), f(3), \dots$ give us a list of every ordered pair in $\mathbb{N}^2$. Thus $\mathbb{N}$ and $\mathbb{N}^2$ have the same cardinality, so $\mathbb{N}^2$ is denumerable and hence countable. $\diamondsuit$

In Example 1.2.3 we showed how to create a *list* of the all of the elements of $\mathbb{N}^2$. In the same way, if X is any countable set then we can create a list of the elements of X. There are two possibilities. First, a countable X set might be finite, in which case there exists a bijection $f\colon \{1,2,\dots,n\} \to X$ for some positive integer n. Since f is surjective, we therefore have

$$X \;=\; \mathrm{range}(f) \;=\; \bigl\{f(1),\; f(2),\; \dots,\; f(n)\bigr\}.$$

Thus the function f gives us a way to list the n elements of X. On the other hand, if X is countably infinite then there is a bijection $f\colon \mathbb{N} \to X$, and hence

$$X \;=\; \mathrm{range}(f) \;=\; \bigl\{f(1),\; f(2),\; f(3),\; \dots\bigr\}.$$

Thus the elements of X have been again been listed in some order. For example, Example 1.2.3 shows that we can list the elements of $\mathbb{N}^2$ in the following order:

$$\mathbb{N}^2 \;=\; \bigl\{(1,1),\;\; (1,2),\;\; (2,1),\;\; (3,1),\;\; (2,2),\;\; (1,3),\;\; (1,4),\;\; (2,3),\;\; \dots\bigr\}.$$

It may seem more natural to depict $\mathbb{N}^2$ as a "two-dimensional" table, but because $\mathbb{N}^2$ is countable we can also make a "one-dimensional" *list* of all of the elements of $\mathbb{N}^2$.

1.3 Uncountability of the Real Line

Now we will show that there exist infinite sets that are not countable. Let S be the open interval $(0,1)$, which is the set of all real numbers that lie strictly between zero and one:

$$S \;=\; (0,1) \;=\; \bigl\{x \in \mathbb{R} : 0 < x < 1\bigr\}.$$

We will use an argument by contradiction to prove that S is not countable. First we recall that every real number can be written in decimal form. In particular, if $0 < x < 1$ then we can write

$$x \;=\; 0.d_1 d_2 d_3 \dots \;=\; \sum_{k=1}^{\infty} \frac{d_k}{10^k},$$

where each digit d_k is an integer between 0 and 9. Some numbers have two decimal representations, for example

$$\frac{1}{2} \;=\; 0.5000\dots \;=\; \frac{5}{10} \;+\; \sum_{k=2}^{\infty} \frac{0}{10^k},$$

but also

$$\frac{1}{2} \;=\; 0.4999\ldots \;=\; \frac{4}{10} \;+\; \sum_{k=2}^{\infty} \frac{9}{10^k}. \qquad (1.1)$$

Any number whose decimal representation ends in infinitely many zeros also has a decimal representation that ends in infinitely many nines, but all other real numbers have a unique decimal representation.

Suppose that S was countable. In this case there would exist a bijection $f\colon \mathbb{N} \to S$, and therefore we could make a list of all the elements of S. If we let $x_n = f(n)$, then we can write S as follows:

$$S \;=\; \mathrm{range}(f) \;=\; \{f(1),\; f(2),\; f(3),\; \ldots\} \;=\; \{x_1,\, x_2,\, x_3,\, \ldots\}.$$

This is a list of every real number between 0 and 1. Further, we can write each x_n in decimal form, say,

$$x_n \;=\; 0.d_1^n d_2^n d_3^n \ldots,$$

where each digit d_k^n is an integer between 0 and 9.

Now we will create another sequence of digits between 0 and 9. In fact, in order to avoid difficulties arising from the fact that some numbers have two decimal representations, we will always choose digits that are between 1 and 8. To start, let e_1 be any integer between 1 and 8 that does not equal d_1^1 (the first digit of the first number x_1). For example, if the decimal representation of x_1 happened to be $x_1 = 0.72839172\ldots$, then we let e_1 be any digit other than 0, 7, or 9 (so we might take $e_1 = 5$ in this case). Then we let e_2 be any integer between 1 and 8 that does not equal d_2^2 (the second digit of the second number x_2), and so forth. This gives us digits $e_1, e_2, \ldots$, and we let x be the real number whose decimal expansion has exactly those digits:

$$x \;=\; 0.e_1e_2e_3\ldots \;=\; \sum_{k=1}^{\infty} \frac{e_k}{10^k}.$$

Then x is a real number between 0 and 1, so x is one of the real numbers in the set S. Yet $x \neq x_1$, because the first digit of x (which is e_1) is not equal to the first digit of x_1 (why not—what if x_1 has two decimal representations?). Similarly $x \neq x_2$, because their second digits are different, and so forth. Hence x does not equal any element of S, which is a contradiction. Therefore S cannot be a countable set.

1.4 Facts about Cardinality

Here are some properties of countable and uncountable sets (the proof is assigned as Problem 1.5.4).

Lemma 1.4.1. *If X and Y are sets, then the following statements hold.*

(a) *If X is countable and $Y \subseteq X$, then Y is countable.*

(b) *If X is uncountable and $Y \supseteq X$, then Y is uncountable.*

(c) *If X is countable and there exists an injection $f\colon Y \to X$, then Y is countable.*

(d) *If X is uncountable and there exists an injection $f\colon X \to Y$, then Y is uncountable.* ♢

Example 1.4.2. (a) Let $\mathbb{Q}^+ = \{r \in \mathbb{Q} : r > 0\}$ be the set of all positive rational numbers. If $r \in \mathbb{Q}^+$, then there is a unique way to write r as a fraction in lowest terms. That is, $r = m/n$ for a unique choice of positive integers m and n that have no common factors. Therefore, by setting $f(r) = (m, n)$ we can define an injective map of $\mathbb{Q}^+$ into $\mathbb{N}^2$. Since $\mathbb{N}^2$ is countable and f is injective, we apply Lemma 1.4.1 and conclude that $\mathbb{Q}^+$ is countable.

A similar argument shows that $\mathbb{Q}^-$, the set of negative rational numbers, is countable. Problem 1.5.5 tells us that a union of finitely many (or countably many) countable sets is countable, so it follows that $\mathbb{Q} = \mathbb{Q}^+ \cup \mathbb{Q}^- \cup \{0\}$ is countable.

(b) We saw above that $S = (0, 1)$ is uncountable. Since $\mathbb{R}$ contains S, Lemma 1.4.1 implies that $\mathbb{R}$ is uncountable. Also, every real number is a complex number, so the real line $\mathbb{R}$ is a subset of the complex plane $\mathbb{C}$. Therefore $\mathbb{C}$ is uncountable as well.

(c) Let $I = \mathbb{R} \backslash \mathbb{Q}$ be the set of irrational real numbers. Since $\mathbb{R} = I \cup \mathbb{Q}$, if I was countable then $\mathbb{R}$ would be the union of two countable sets, which is countable by Problem 1.5.5. This is a contradiction, so the set of irrationals must be uncountable.

Thus $\mathbb{Q}$ is countable while I is uncountable. This may seem counterintuitive since between any two rational numbers there is an irrational, and between any two irrational numbers there is a rational number! ♢

1.5 Problems

1.5.1. Prove equation (1.1).

1.5.2. Prove that if A, B, and C are sets, then the following statements hold.

(a) A has the same cardinality as A.

(b) If A has the same cardinality as B, then B has the same cardinality as A.

(c) If A has the same cardinality as B and B has the same cardinality as C, then A has the same cardinality as C.

1.5.3. Prove that the closed interval $[0,1]$ and the open interval $(0,1)$ have the same cardinality by exhibiting a bijection $f\colon [0,1] \to (0,1)$.

Hint: Do not try to create a *continuous* function f.

1.5.4. Prove Lemma 1.4.1.

1.5.5. (a) Show that if X and Y are countable sets, then their union $X \cup Y$ is countable.

(b) Prove that the union of finitely many countable sets $X_1, \dots, X_n$ is countable.

(c) Suppose that $X_1, X_2, \dots$ are countably many sets, each of which is countable. Prove that $\cup_{k=1}^{\infty} X_k = X_1 \cup X_2 \cup \cdots$ is countable. Thus the union of countably many countable sets is countable.

Hint: Consider the table that appeared in Section 1.2 depicting the elements of $\mathbb{N}^2$.

1.5.6. Let $\mathcal{F}$ be the set of all functions that map real numbers to real numbers, i.e., $\mathcal{F}$ is the set of all functions of the form $f\colon \mathbb{R} \to \mathbb{R}$.

Suppose that $\mathbb{R}$ and $\mathcal{F}$ had the same cardinality. Then there would exist a bijection $G\colon \mathbb{R} \to \mathcal{F}$. This function G maps a real number x to a function in $\mathcal{F}$. Let us call this function f_x. That is, to simplify the notation we will write f_x instead of $G(x)$. Each real number x corresponds to a function f_x, and since G is onto we know that every function is G equals some function f_x for some x.

Now define a function $g\colon \mathbb{R} \to \mathbb{R}$ by setting

$$g(x) \;=\; f_x(x) \,+\, 1, \qquad \text{for } x \in \mathbb{R}.$$

For example, if f_x happened to take the value 100 at x, then we would declare that $g(x)$ is $g(x) = f_x(x) + 1 = 101$. For each number x, we look at the function f_x, take the output of f_x at the point x, and add 1 to that to get the value that we assign to $g(x)$. Prove that g is a function that maps real numbers to real numbers, and therefore g belongs to $\mathcal{F}$. Also prove that there is no $x \in \mathbb{R}$ such that $g = f_x$. That is, show that no matter what x that we choose, $g(t)$ and $f_x(t)$ cannot be equal for every $t \in \mathbb{R}$, so g and f_x cannot be the same function. Explain why this is a contradiction, and use this to show that $\mathbb{R}$ and $\mathcal{F}$ do *not* have the same cardinality.