FUNCTIONAL ANALYSIS LECTURE NOTES:

COMPACT SETS AND FINITE-DIMENSIONAL SPACES

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1. Compact Sets

Definition 1.1 (Compact and Totally Bounded Sets)**.** Let X be a metric space, and let $E \subseteq X$ be given.

(a) We say that E is *compact* if every open cover of E contains a finite subcover. That is, E is compact if whenever $\{U_\alpha\}_{\alpha \in I}$ is a collection of open sets whose union contains E, then there exist finitely many $\alpha_1, \dots, \alpha_N$ such that $E \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_N}$.

(b) We say that E is *sequentially compact* if every sequence $\{f_n\}_{n \in \mathbb{N}}$ of points of E contains a convergent subsequence $\{f_{n_k}\}_{k \in \mathbb{N}}$ whose limit belongs to E.

(c) We say that E is *totally bounded* if for every $\varepsilon > 0$ there exist finitely many points $f_1, \dots, f_N \in E$ such that

$$E \subseteq \bigcup_{k=1}^{N} B_\varepsilon(f_k),$$

where $B_\varepsilon(f_k)$ is the open ball of radius ε centered at f_k. That is, E is totally bounded if and only there exist finitely many points $f_1, \dots, f_N \in E$ such that every element of E is within ε of some f_k.

(d) We say that E is *complete* if every Cauchy sequence of points of E converges to a point in E.

Exercise 1.2. Suppose that X is a *complete* metric space. Show that, in this case,

$$E \text{ is complete} \iff E \text{ is closed.}$$

In a finite dimensionsional normed space, a set is compact if and only if it is closed and bounded. In infinite dimensional normed spaces, it is true all compact sets are closed and bounded, but the converse fails in general.

We have the following equivalent formulations of compactness for sets in metric spaces.

Theorem 1.3. Let E be a subset of a Banach space X. Then the following statements are equivalent.

(a) E is compact.

(b) E is sequentially compact.

(c) E is complete and totally bounded.

Proof. (a) $\Rightarrow$ (c). Exercise.

(c) $\Rightarrow$ (b). Assume that E is complete and totally bounded, and let $\{f_n\}_{n\in\mathbb{N}}$ be any sequence of points in E. Since E is covered by finitely many balls of radius $\frac{1}{2}$, one of those balls must contain infinitely many f_n, say $\{f_n^{(1)}\}_{n\in\mathbb{N}}$. Then we have

$$\forall\, m, n \in \mathbb{N}, \quad \mathrm{d}(f_m^{(1)}, f_n^{(1)}) \;<\; 1.$$

Since E is covered by finitely many balls of radius $\frac{1}{4}$, we can find a subsequence $\{f_n^{(2)}\}_{n\in\mathbb{N}}$ of $\{f_n^{(1)}\}_{n\in\mathbb{N}}$ such that

$$\forall\, m, n \in \mathbb{N}, \quad \mathrm{d}(f_m^{(1)}, f_n^{(1)}) \;<\; \frac{1}{2}.$$

By induction we keep constructing subsequences $\{f_n^{(k)}\}_{n\in\mathbb{N}}$ such that $\mathrm{d}(f_m^{(k)}, f_n^{(k)}) < \frac{1}{k}$ for all m, $n \in \mathbb{N}$.

Now consider the "diagonal subsequence" $\{f_n^{(n)}\}_{n\in\mathbb{N}}$. Given $\varepsilon > 0$, let N be large enough that $\frac{1}{N} < \varepsilon$. If $m \geq n > N$, then $f_m^{(m)}$ is one element of the sequence $\{f_k^{(n)}\}_{k\in\mathbb{N}}$, say $f_m^{(m)} = f_k^{(n)}$. Then

$$\mathrm{d}(f_m^{(m)}, f_n^{(n)}) \;=\; \mathrm{d}(f_k^{(n)}, f_n^{(n)}) \;<\; \frac{1}{n} \;<\; \varepsilon.$$

Thus $\{f_n^{(n)}\}_{n\in\mathbb{N}}$ is Cauchy, and therefore, since E is complete, it converges to some element of E.

(b) $\Rightarrow$ (c). Suppose that E is sequentially compact.

Exercise: Show that E is complete.

Suppose that E was not totally bounded. Then there is an $\varepsilon > 0$ such that E cannot be covered by finitely many ε-balls centered at points of E. Choose any $f_1 \in E$. Since E cannot be covered by a single ε-ball, E cannot be a subset of $B_\varepsilon(f_1)$. Hence there exists $f_2 \in E \setminus B_\varepsilon(f_1)$, i.e., $f_2 \in E$ and $\mathrm{d}(f_2, f_1) \geq \varepsilon$. But E cannot be covered by two ε-balls, so there must exist an $f_3 \in E \setminus \big(B_\varepsilon(f_1) \cup B_\varepsilon(f_2)\big)$. In particular, we have $\mathrm{d}(f_3, f_1)$, $\mathrm{d}(f_3, f_2) \geq \varepsilon$. Continuing in this way we obtain a sequence of points $\{f_n\}_{n\in\mathbb{N}}$ in E which has no convergent subsequence, which is a contradiction.

(b) $\Rightarrow$ (a). Assume that E is sequentially compact. Then by the implication (b) $\Rightarrow$ (c) proved above, we know that E is also complete and totally bounded.

Choose any open cover $\{U_\alpha\}_{\alpha\in I}$ of E. We must show that it contains a finite subcover. The main point towards proving this is the following claim.

Claim. There exists a number $\delta > 0$ such that if B is any ball of radius δ that intersects E, then there is an $\alpha \in I$ such that $B \subseteq U_\alpha$.

To prove the claim, suppose that $\{U_\alpha\}_{\alpha\in I}$ was an open cover of E such that no δ with the required property existed. Then for each $n \in \mathbb{N}$, we could find a ball B_n with radius $\frac{1}{n}$ that intersects E but is not contained in any U_α. Choose any $f_n \in B_n \cap E$. Since E is sequentially compact, there must be a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ that converges to an element of E, say $f_{n_k} \to f \in E$. But we must have $f \in U_\alpha$ for some α, and since U_α is open there must exist some $r > 0$ such that $B_r(f) \subseteq U_\alpha$. Now choose k large enough that we have both

$$\frac{1}{n_k} < \frac{r}{3} \qquad \text{and} \qquad \mathrm{d}(f, f_{n_k}) < \frac{r}{3}.$$

Then it follows that $B_{n_k} \subseteq B_r(f) \subseteq U_\alpha$, which is a contradiction. Hence the claim holds.

To finish the proof, we use the fact that E is totally bounded, and therefore can be covered by finitely many balls of radius δ. But by the claim, each of these balls is contained in some U_α, so E is covered by finitely many of the U_α. $\square$

Exercise 1.4. Show that if E is a totally bounded subset of a Banach space X, then its closure $\overline{E}$ is compact. A set whose closure is compact is said to be *precompact.*

Exercise 1.5. Prove that if H is an infinite-dimensional Hilbert space, then the closed unit sphere $\{f \in H : \|f\| \leq 1\}$ is not compact.

Exercise 1.6. Suppose that E is a compact subset of a Banach space X. Show that any continuous function $f\colon E \to \mathbb{C}$ must be bounded, i.e., $\sup_{x \in E} |f(x)| < \infty$.

2. Finite-Dimensional Normed Spaces

In this section we will prove some basic facts about finite-dimensional spaces.

First, recall that a finite-dimensional vector space has a finite basis, which gives us a natural notion of coordinates of a vector, which in turn yields a linear bijection of X onto $\mathbb{C}^d$ for some d.

Example 2.1 (Coordinates)**.** Let X be a finite-dimensional vector space over $\mathbb{C}$. Then X has a finite basis, say $\mathcal{B} = \{e_1, \dots, e_d\}$. Every element of X can be written uniquely in this basis, say,

$$x \;=\; c_1(x)\, e_1 + \cdots + c_d(x)\, e_d, \qquad x \in X.$$

Define the *coordinates* of x with respect to the basis $\mathcal{B}$ to be

$$[x]_{\mathcal{B}} \;=\; \begin{bmatrix} c_1(x) \\ \vdots \\ c_d(x) \end{bmatrix}.$$

Then the mapping $T\colon X \to \mathbb{C}^d$ given by $x \mapsto [x]_{\mathcal{B}}$ is, by definition of basis, a linear bijection of X onto $\mathbb{C}^d$.

We already know how to construct many norms on $\mathbb{C}^d$. In particular, if $w_1, \dots, w_d > 0$ are fixed weights and $1 \le p \le \infty$, then

$$\|x\|_{p,w} \;=\; \begin{cases} \left(\displaystyle\sum_{k=1}^{d} |x_k|^p \, w(k)^p\right)^{1/p}, & 0 < p < \infty, \\ \sup\limits_{k} \, |x_k| \, w(k), & p = \infty, \end{cases}$$

defines a norm on $\mathbb{C}^d$ (and $\mathbb{C}^d$ is complete in this norm). If X is an n-dimensional vector space, then by transferring these norms on $\mathbb{C}^d$ to X we obtain a multitude of norms for X.

Exercise 2.2 (ℓ^p_w Norms on X)**.** Let X be a finite-dimensional vector space over $\mathbb{C}$ and let $\mathcal{B} = \{e_1, \dots, e_d\}$ be any basis. Fix any $1 \le p \le \infty$ and any weight $w \colon \{1, \dots, d\} \to (0, \infty)$. Using the notation of Example 2.1, given $x \in X$ define

$$\|x\|_{p,w} \;=\; \big\|[x]_{\mathcal{B}}\big\|_{p,w} \;=\; \begin{cases} \left(\displaystyle\sum_{k=1}^{d} |c_k(x)|^p \, w(k)^p\right)^{1/p}, & 1 \le p < \infty, \\ \max\limits_{k} |c_k(x)| \, w(k), & p = \infty. \end{cases}$$

Note that while we use the same symbol $\|\cdot\|_{p,w}$ to denote a function on X and on $\mathbb{C}^d$, by context it has different meanings depending on whether it is being applied to an element of X or to an element of $\mathbb{C}^d$.

Prove the following.

(a) $\|\cdot\|_{p,w}$ is a norm on X.

(b) $x \mapsto [x]_{\mathcal{B}}$ is a isometric isomorphism of X onto $\mathbb{C}^d$ (using the norm $\|\cdot\|_{p,w}$ on X and the norm $\|\cdot\|_{p,w}$ on $\mathbb{C}^d$).

(c) Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence of vectors in X and let $x \in X$. Prove that $x_n \to x$ with respect to the norm $\|\cdot\|_{p,w}$ on X if and only if the coordinate vectors $[x_n]_{\mathcal{B}}$ converge componentwise to the coordinate vector $[x]_{\mathcal{B}}$.

(d) X is complete in the norm $\|\cdot\|_{p,w}$.

We will need the following fact about the unit circle with respect to the ℓ^∞-norm on X.

Exercise 2.3. With X as before, let

$$D \;=\; \{x \in X : \|x\|_\infty = 1\}$$

denote the ℓ^∞-unit circle in X. Show that D is compact (with respect to the norm $\|\cdot\|_\infty$).

Hints: Suppose that $\{x_n\}_{n \in \mathbb{N}}$ is a sequence of vectors in D. Then for each n, we have $|c_k(x_n)| = 1$ for some $k \in \{1, \dots, n\}$. Hence there must be some k such that $|c_k(x_{n_j})| = 1$ for infinitely many n_j. Since $\{c_1(x_{n_j})\}_{j \in \mathbb{N}}$ is an infinite sequence of scalars in the compact set $\{c \in \mathbb{C} : |c| \le 1\}$, we can select a subsequence whose first coordinates converge. Repeat for each coordinate, and remember that the kth coordinate is always 1. Hence there exists a subsequence that converges to $x \in D$ with respect to the ℓ^∞-norm, and therefore D is sequentially compact.

Now we can show that *all* norms on a finite-dimensional space are equivalent.

Theorem 2.4. If X is a finite-dimensional vector space over $\mathbb{C}$, then any two norms on X are equivalent.

Proof. Let $\mathcal{B} = \{e_1, \dots, e_d\}$ be any basis for X, and let $\|\cdot\|_\infty$ be the norm on X defined in Exercise 2.2. Since equivalence of norms is an equivalence relation, it suffices to show that an arbitrary norm $\|\cdot\|$ on X is equivalent to $\|\cdot\|_\infty$.

Using the notation of Example 2.1, given $x \in X$ write x uniquely as

$$x \;=\; c_1(x)\, e_1 + \cdots + c_d(x)\, e_d.$$

Then

$$\|x\| \;\le\; \sum_{k=1}^{d} |c_k(x)|\, \|e_k\| \;\le\; \biggl(\sum_{k=1}^{d} \|e_k\|\biggr) \bigl(\max_k |c_k(x)|\bigr) \;=\; C_2\, \|x\|_\infty, \tag{2.1}$$

where $C_2 = \sum_{k=1}^{d} \|e_k\|$ is a nonzero constant independent of x.

It remains to show that there is a constant $C_1 > 0$ such that $C_1\, \|x\|_\infty \le \|x\|$ for every x. Let

$$D \;=\; \{x \in X : \|x\|_\infty = 1\}$$

denote the ℓ^∞-unit circle in X. By Exercise 2.3, we know that D is compact with respect to the norm $\|\cdot\|_\infty$. Also, by equation (2.1), we have that $x \mapsto \|x\|$ is continuous with respect to the norm $\|\cdot\|_\infty$. By Exercise 1.6, a real-valued continuous function on a compact set must be bounded. Hence, there must exist constants $0 \le m \le M < \infty$ such that

$$m \;\le\; \|x\| \;\le\; M \quad \text{for all } x \in D, \tag{2.2}$$

and we can take $m = \inf_{x \in D} \|x\|$ and $M = \sup_{x \in D} \|x\|$. Suppose that $m = 0$. Then there exist $x_n \in D$ such that $\|x_n\| \to 0$. Since D is compact with respect to the norm $\|\cdot\|_\infty$, there exists $x \in D$ and a subsequence $\{x_{n_k}\}_{k \in \mathbb{N}}$ such that $x_{n_k} \to x$ with respect to $\|\cdot\|_\infty$, i.e., $\|x - x_{n_k}\|_\infty \to 0$. Since $\|\cdot\|$ is continuous with respect to $\|\cdot\|_\infty$, this implies that $\|x_{n_k}\| \to \|x\|$. Hence $\|x\| = 0$, which implies $x = 0$ and therefore $\|x\|_\infty = 0$. This contradicts the fact that $x \in D$, so we must have $m > 0$. Therefore, since $x \in D$ if and only if $\|x\|_\infty = 1$, we can rewrite equation (2.2) as

$$\forall\, x \in X, \quad m\, \|x\|_\infty \;\le\; \|x\| \;\le\; M\, \|x\|_\infty,$$

so $\|\cdot\| \asymp \|\cdot\|_\infty$. □

Consequently, from now on we need not specify the norm on a finite-dimensional vector space X—we can take any norm that we like whenever we need it.

Exercise 2.5. Let $\mathbb{C}^{m \times n}$ be the space of all $m \times n$ matrices with complex entries. $\mathbb{C}^{m \times n}$ is naturally isomorphic to $\mathbb{C}^{mn}$. Prove that if $\|\cdot\|_a$ is any norm on $\mathbb{C}^{m \times n}$, $\|\cdot\|_b$ is any norm on $\mathbb{C}^{n \times k}$, and $\|\cdot\|_c$ is any norm on $\mathbb{C}^{m \times k}$, then there exists a constant $C > 0$ such that

$$\forall\, A \in \mathbb{C}^{m \times n}, \quad \forall\, B \in \mathbb{C}^{n \times k}, \quad \|AB\|_c \;\le\; C\, \|A\|_a\, \|B\|_b.$$

Proposition 2.6. If M is a subspace of a finite-dimensional vector space X, then M is closed.

Proof. Let $\|\cdot\|$ be any norm on X. Suppose that $x_n \in M$ and that $x_n \to y \in X$. Suppose that $y \notin M$, and define

$$M_1 \;=\; \mathrm{span}\{M, y\} \;=\; \{m + cy : m \in M, c \in \mathbb{C}\}.$$

Then M_1 is finite-dimensional subspace of X, and every element $z \in M_1$ can be written *uniquely* as $z = m(z) + c(z)y$ where $m(z) \in M$ and $c(z)$ is a scalar. For $z \in M_1$ define

$$\|z\|_{M_1} \;=\; \|m(z)\| + |c(z)|.$$

Exercise: Show that $\|\cdot\|_{M_1}$ is a norm on M_1.

Since $\|\cdot\|$ is also a norm on M_1 and all norms on a finite-dimensional space are equivalent, we conclude that there is a constant $C > 0$ such that $\|z\|_{M_1} \le C\,\|z\|$ for all $z \in M_1$. Since $x_n \in M$, we have

$$1 \;\le\; \|x_n\| + 1 \;=\; \|y - x_n\|_{M_1} \;\le\; C\,\|y - x_n\| \;\to\; 0.$$

This is a contradiction, so we must have $y \in M$. □

The proof of the preceding proposition can be easily extended to give the following.

Exercise 2.7. Show that if X is a normed linear space, then any finite-dimensional subspace M of X must be closed.

Exercise 2.8. Let X be a finite-dimensional normed space, and let Y be a normed linear space. Prove that if $L\colon X \to Y$ is linear, then L is bounded.

The following lemma will be needed for Exercise 2.10.

Lemma 2.9 (F. Riesz's Lemma)**.** Let M be a proper, closed subspace of a normed space X. Then for each $\varepsilon > 0$, there exists $g \in X$ with $\|g\| = 1$ such that

$$\mathrm{dist}(g, M) \;=\; \inf_{f \in M} \|g - f\| \;>\; 1 - \varepsilon.$$

Proof. Choose any $u \in X \setminus M$. Since M is closed, we have

$$a \;=\; \mathrm{dist}(u, M) \;=\; \inf_{f \in M} \|u - f\| \;>\; 0.$$

Fix $\delta > 0$ small enough that $\frac{a}{a+\delta} > 1 - \varepsilon$. By definition of infimum, there exists $v \in M$ such that $a \le \|u - v\| < a + \delta$. Set

$$g \;=\; \frac{u - v}{\|u - v\|},$$

and note that $\|g\| = 1$. Given $f \in M$ we have $h = v + \|u - v\|\, f \in M$, so

$$\|g - f\| \;=\; \left\|\frac{u - v - \|u - v\|\, f}{\|u - v\|}\right\| \;=\; \frac{\|u - h\|}{\|u - v\|} \;>\; \frac{a}{a + \delta} \;>\; 1 - \varepsilon.$$ □

Exercise 2.10. Let X be a normed linear space. Let $B = \{x \in X : \|x\| \leq 1\}$ be the closed unit ball in X. Prove that if B is compact, then X is finite-dimensional.

Hints: Suppose that X is infinite-dimensional. Given any nonzero e_1 with $\|e_1\| \leq 1$, by Lemma 2.9 there exists $e_2 \in X \setminus \operatorname{span}\{e_1\}$ with $\|e_2\| \leq 1$ such that $\|e_2 - e_1\| > \frac{1}{2}$. Continue in this way to construct vectors e_k such that $\{e_1, \dots, e_n\}$ are independent for any n. Conclude that X is infinite-dimensional.

Definition 2.11. We say that a normed linear space X is *locally compact* if for each $f \in X$ there exists a compact $K \subseteq X$ *with nonempty interior* K° such that $f \in K^\circ$.

In other words, X is locally compact if for every $f \in X$ there is a neighborhood of f that is contained in a compact subset of X. For example, $\mathbb{C}^n$ is locally compact.

With this terminology, we can reword Exercise 2.10 as follows.

Exercise 2.12. Let X be a normed linear space.

(a) Prove that if X is locally compact, then X is finite-dimensional.

(b) Prove that if X is infinite-dimensional, then no nonempty open subset of X has compact closure.