

LIMITS

Imprecise discussion of limits. The intuitive idea of what we mean when we say $\lim_{x \rightarrow c} f(x) = L$ is that as x approaches closer and closer to c , the function values $f(x)$ approach closer and closer to L . But this is not a *precise* statement, it does not tell you how to figure out whether a function has a limit or what that limit is. It's like saying "a hot object in a cold room will cool down"—from that statement you get no precise information about how fast the object is cooling, or when it will reach a particular temperature, etc. By contrast, Newton's law of cooling is a *precise* statement about how the rate of change of the temperature relates to the difference between the temperature of the object and the temperature of the surrounding medium, and this *precise* statement does allow you to calculate how long it will take for an object to cool down, etc. We need a similarly precise definition of the meaning of limit in order to be able to deal *quantitatively* with limits.

Precise definition of limit. We must state EXACTLY what it means to write

$$\lim_{x \rightarrow c} f(x) = L.$$

The EXACT, PRECISE meaning of these symbols is that:

Given any number $\varepsilon > 0$, there is a number $\delta > 0$ such that:
if $0 < |x - c| < \delta$ then $|f(x) - L| < \varepsilon$.

Thus, to *prove* that a limit is a certain value L , we must demonstrate that given *any* number $\varepsilon > 0$ it is possible to find a number $\delta > 0$ so that a particular something happens. It's not enough to prove that something for a single value of ε ; we must be able to demonstrate that *no matter what ε is specified, there does exist a corresponding δ so that: if $0 < |x - c| < \delta$ then $|f(x) - L| < \varepsilon$.*

Example. Let me prove that $\lim_{x \rightarrow 3} (5x - 7) = 8$. I'm going to give the proof only—I'm not going to explain *how* I got the proof, I just want you to see if you can follow the proof and understand each step in the proof that I make. What we must show is that:

Given any number $\varepsilon > 0$, there is a number $\delta > 0$ such that:
if $0 < |x - 3| < \delta$ then $|(5x - 7) - 8| < \varepsilon$.

We must *prove* this fact, i.e., we must make a logical argument that proceeds step by step to demonstrate that the above statement is true. So, I have to show that given any number $\varepsilon > 0$, there is a number δ so that some particular thing happens. I have to convince you that no matter what number $\varepsilon > 0$ that you consider, there is a corresponding number $\delta > 0$ so that: if $0 < |x - 3| < \delta$ then $|(5x - 7) - 8| < \varepsilon$. Here is my proof, see if you believe that it proves what I claim that it does. If you don't believe that I've done it, you need to email me your questions about it right away.

Proof that $\lim_{x \rightarrow 3} (5x - 7) = 8$.

Let $\varepsilon > 0$ be any number. I have to convince you that there is a number $\delta > 0$ so that:

$$\text{if } 0 < |x - 3| < \delta \quad \text{then} \quad |(5x - 7) - 8| < \varepsilon.$$

I hope to convince you that the number $\delta = \varepsilon/5$ has this property. How I came up with this value of δ is what scratchwork is for, but the only issue as far as the proof is concerned is whether I can convince you that $\delta = \varepsilon/5$ does indeed have the right property.

So, let me try to show you that the number $\delta = \varepsilon/5$ has the right property. What we have to show is that if x is any number such that $0 < |x - 3| < \delta$, then $f(x) = 5x - 7$ satisfies $|(5x - 7) - 8| < \varepsilon$. Now, I can't just *claim* that $f(x)$ has the property, I have to *demonstrate* to you that it does have this property, by using the fact that x has the property $0 < |x - 3| < \delta$. Here is the calculation: *If x satisfies $0 < |x - 3| < \delta$, then:*

$$\begin{aligned} |(5x - 7) - 8| &= |5x - 15| && \text{(arithmetic)} \\ &= 5|x - 3| && \text{(arithmetic)} \\ &< 5\delta && \text{(because } |x - 3| < \delta) \\ &= 5\frac{\varepsilon}{5} && \text{(because } \delta = \varepsilon/5) \\ &= \varepsilon && \text{(arithmetic).} \end{aligned}$$

Thus, I've shown that *if $0 < |x - 3| < \delta$, then $|(5x - 7) - 8| < \varepsilon$* . That's exactly what I needed to show, so the proof is done.

Proof that $\lim_{x \rightarrow 3} (5x - 7) = 8$ (short version).

My proof above contained a lot of extra words that I included just to try to explain what was going on. Here is a shorter version that contains only the essential argument and nothing else:

Let $\varepsilon > 0$ be any number. Let $\delta = \varepsilon/5$. If x satisfies $0 < |x - 3| < \delta$, then:

$$|(5x - 7) - 8| = |5x - 15| = 5|x - 3| < 5\delta = 5\frac{\varepsilon}{5} = \varepsilon.$$

Done.

Sample incorrect proof. Here is an actual proof written by a student, and my comments on what is wrong with it.

Student line 1: Let $\varepsilon > 0$.

Comment. Good start.

Student line 2: If $0 < |x - c| < \delta$ then $|f(x) - L| < \varepsilon$.

Comment. Problem 1: What is δ ? This makes no sense unless δ has been given a value. Problem 2: This line is what we have to PROVE. The student wrote it down as if it was a fact. If we knew that this line was true then we would be done! The whole point is to PROVE that if $0 < |x - c| < \delta$ then $|f(x) - L| < \varepsilon$! If you like, you could write “WE MUST SHOW THAT if $0 < |x - c| < \delta$ then $|f(x) - L| < \varepsilon$ ”, but you cannot simply write it down as if it was a fact.

Student line 3: Choose $\delta = \varepsilon/5$.

Comment. Yes, but you should have told me what δ was BEFORE you tried to use it in line 2.

Student line 4: If $|x - 3| < \delta = \varepsilon/5$ then $|5x - 15| < \varepsilon$.

Comment. Again, this is what must be PROVED. If we knew that this line was true, we would be done. You can't write this down as a fact, you have to PROVE that it is a true fact. You have to *convince me* that if x satisfies $0 < |x - 3| < \delta$ then $f(x) = 5x - 7$ satisfies $|(5x - 7) - 8| < \varepsilon$. This is what the student tries to do in the next line, but he shouldn't have claimed as he did on this line that the conclusion $|5x - 15| < \varepsilon$ was true until he has *proved it*.

Student line 5: \Rightarrow and $|5x - 15| < 5|x - 3| = \delta \Rightarrow |5x - 15| < 5\varepsilon/5 = \varepsilon$.

Comment. Problem 1: What does “ \Rightarrow and” mean? “implies and”? The wording does not make sense. What you write must make sense—if you said exactly what you wrote out loud, would another person understand what you meant? Problem 2: Where did $5|x - 3| = \delta$ come from? We have assumed that x satisfies $|x - 3| < \delta$, how could it follow from this that $5|x - 3| = \delta$? (It doesn't.)

A more difficult proof. Let me prove something more difficult now. I'll try to prove that $\lim_{x \rightarrow 1} x^3 = 1$. Again, let me just give the proof without the scratchwork. The point

here is: Can you follow the proof? Do you believe that it really shows what it is supposed to? Here it is, with extra explanations at each step.

Let $\varepsilon > 0$ be any given value. I have to convince you that there is a number $\delta > 0$ so that:

$$\text{if } 0 < |x - 1| < \delta \quad \text{then} \quad |x^3 - 1| < \varepsilon.$$

I claim that the number $\delta = \min(\varepsilon/7, 1)$ has this property. So, what I have to show you is that for this value of δ , if x has the property $0 < |x - 1| < \delta$, then x^3 has the property that $|x^3 - 1| < \varepsilon$.

So, let's look at those x 's which have the property that $0 < |x - 1| < \delta$, and let's try to see whether x^3 will have the right property or not. Look at our x . We know that $0 < |x - 1| < \delta$. We also know that δ is the smaller of the two numbers $\varepsilon/7$ and 1. So, δ could at most be 1, and maybe is smaller. But in any case, we have $|x - 1| < 1$ (and maybe smaller than that). Therefore, x must be somewhere in the range $0 < x < 2$ (and maybe even in a smaller range than that). Hence $0 < x^2 < 4$, and so

$$|x^2 + x + 1| \leq |x^2| + |x| + |1| < 4 + 2 + 1 = 7.$$

Thus, for all of these x 's, we know that $|x^2 + x + 1| < 7$. Therefore, for these x , the ones that satisfy $0 < |x - 1| < \delta$, we have

$$\begin{aligned} |x^3 - 1| &= |x - 1| |x^2 + x + 1| && \text{(factoring)} \\ &< 7|x - 1| && \text{(because } |x^2 + x + 1| < 7 \text{ for those } x\text{'s)} \\ &< 7\delta && \text{(because } |x - 1| < \delta) \\ &\leq 7\frac{\varepsilon}{7} && \text{(because } \delta \leq \varepsilon/7) \\ &= \varepsilon && \text{(arithmetic).} \end{aligned}$$

Thus, I've shown that if $0 < |x - 1| < \delta$, then $|x^3 - 1| < \varepsilon$. That's exactly what I needed to show, so the proof is done.

Scratchwork. HOW did I come up with this proof? That's what scratchwork is for. But the scratchwork isn't part of the proof, it's just what I do in order to figure out how to get the proof. After I've done the scratchwork, I can throw it away, it has no more use. Only the PROOF is important. Most of you are able to do a fairly decent job on the scratchwork; rather, it seems to be the *proof* that is giving trouble. If you are having trouble doing the scratchwork, get help now. But also be absolutely sure that you understand *how* to write the proof, and—more importantly—*WHY* the proof is written the way it is. If you don't understand this, *get help now*.