FUNCTIONAL ANALYSIS LECTURE NOTES:

REFLEXIVITY OF L^p

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Notation: When talking about reflexivity, it is convenient to take the action of a functional μ on a vector x to be linear in both μ and x. Usually, we use the ordinary notation $\mu(x)$ to denote an action that is linear in both variables. However, in this note I will use a slightly different notation and write $\langle x, \mu \rangle$ for the action of μ on x. Furthermore, I will take this notation to be *linear* in both μ and x (note that in most of the lecture notes, I take this notation to be linear in x and antilinear in μ).

Theorem 1. If $E \subseteq \mathbb{R}$ is Lebesgue measurable, then $L^p(E)$ is reflexive for each $1 < p < \infty$.

Proof. Let p' be the dual index. Let

$$\begin{aligned} L\colon L^p(E) &\to L^p(E)^{**} \\ g &\mapsto \hat{g} \end{aligned}$$

be the canonical embedding of $L^p(E)$ into $L^p(E)^{**}$. That is, $\hat{g}\colon L^p(E)^* \to \mathbb{F}$ is given by

$$\langle \mu, \hat{g} \rangle \;=\; \langle g, \mu \rangle, \qquad \mu \in L^p(E)^*.$$

Given $g \in L^{p'}(E)$, we know that

$$\langle f, \mu_g \rangle \;=\; \int_E f(x)\, g(x)\, dx$$

defines a bounded linear functional on $L^p(E)$, and furthermore the mapping

$$\begin{aligned} T_p\colon L^{p'}(E) &\to L^p(E)^* \\ g &\mapsto \mu_g \end{aligned}$$

is an isometric isomorphism. Likewise, we have an isometric isomorphism $T_{p'}\colon L^p(E) \to L^{p'}(E)^*$.

As a consequence, we have an isometric isomorphism $U\colon L^{p'}(E)^* \to L^p(E)^{**}$ defined by

$$\langle \nu, U\mu \rangle \;=\; \langle T_p^{-1}\nu, \mu \rangle, \qquad \nu \in L^{p'}(E)^*.$$

Now consider the composition $T_{p'}U\colon L^p(E) \to L^p(E)^{**}$. This is an isometric isomorphism, and we claim it equals the canonical embedding L.

To evaluate $UT_{p'}$, choose any $g \in L^p(E)$. Then $UT_{p'}g \in L^p(E)^*$, i.e., it is a bounded linear functional on $L^p(E)^*$. Given $\nu \in L^p(E)^*$, we know that $\nu = \nu_h = T_p h$ for some $h \in L^{p'}(E)$. Therefore,

$$\begin{aligned}
\langle \nu, UT_{p'}g\rangle &= \langle T_p^{-1}\nu, T_{p'}g\rangle \\
&= \langle h, \mu_g\rangle \\
&= \int h(x)\, g(x)\, dx \\
&= \langle g, \mu_h\rangle \\
&= \langle g, \nu\rangle \\
&= \langle \nu, \hat{g}\rangle \\
&= \langle \nu, Lg\rangle.
\end{aligned}$$

Therefore $L = UT_{p'}$, and since we know the latter is an isometric isomorphism, so is the former. □

Here is another approach. We already know that L is an isometry, so the only thing we really have to do is to show that L is onto. So, suppose that μ is an element of $L^p(E)^{**}$. Then μ is a bounded linear functional on $L^p(E)^*$. But there is an isometric isomorphism $T_p\colon L^{p'}(E) \to L^p(E)^*$ so $\mu T_p\colon L^{p'}(E) \to \mathbb{F}$ is a bounded linear functional on $L^{p'}(E)$, i.e., $\mu T_p \in L^{p'}(E)^*$. Hence $\mu T_p = \mu_g$ for some $g \in L^p(E)$. I claim that $\mu = \hat{g}$.

To see this, choose $\nu \in L^p(E)^*$. Then $\nu = \mu_h = T_p h$ for some $h \in L^{p'}(E)$. We must show that $\langle \nu, \mu\rangle = \langle \nu, \hat{g}\rangle$. Note that with our notation,

$$\langle T_p h, \mu\rangle \;=\; \mu(T_p h) \;=\; (\mu T_p)(h) \;=\; \mu_g(h) \;=\; \langle h, \mu_g\rangle.$$

Therefore

$$\begin{aligned}
\langle \nu, \hat{g}\rangle &= \langle g, \nu\rangle \\
&= \langle g, \mu_h\rangle \\
&= \int g(x)\, h(x)\, dx \\
&= \langle h, \mu_g\rangle \\
&= \langle T_p h, \mu\rangle \\
&= \langle \nu, \mu\rangle.
\end{aligned}$$

Therefore $\mu = \hat{g}$, so L is surjective.