REAL ANALYSIS LECTURE NOTES:

SHORT REVIEW OF METRICS, NORMS, AND CONVERGENCE

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In these notes we will give a brief review of basic notions and terminology for metrics, norms, and convergence.

1. Metrics and Convergence

1.1. **The Definition of a Metric.** A metric defines a notion of distance between points in a set.

Definition 1.1 (Metric Space)**.** Let X be a set. A *metric* on X is a function $\mathrm{d}\colon X \times X \to \mathbb{R}$ such that:

(a) $0 \le \mathrm{d}(f,g) < \infty$ for all $f,\, g \in X$,

(b) $\mathrm{d}(f,g) = 0$ if and only if $f = g$,

(c) $\mathrm{d}(f,g) = \mathrm{d}(g,f)$ for all $f,\, g \in X$,

(d) The Triangle Inequality: for all $f,\, g,\, h \in X$ we have

$$\mathrm{d}(f,h) \;\le\; \mathrm{d}(f,g) + \mathrm{d}(g,h).$$

In this case, X is a called a *metric space*. We refer to the number $\mathrm{d}(f,g)$ as the *distance* from f to g. $\diamondsuit$

A metric space need not be a vector space, although this will be true of many of the metric spaces that we will encounter.

1.2. **An Example.** In a finite measure space, the definition of convergence in measure can be reformulated in terms of a metric.

Exercise 1.2. Suppose that (X, Σ, μ) is a finite measure space, and let $\mathcal{M}$ be the set of all measurable complex-valued functions on X:

$$\mathcal{M} \;=\; \{f\colon X \to \mathbb{C} \,:\, f \text{ is measurable}\}.$$

For $f,\, g \in \mathcal{M}$, define

$$\mathrm{d}(f,g) \;=\; \int \frac{|f-g|}{1+|f-g|}\, d\mu.$$

Show that d is a metric on $\mathcal{M}$ if we identify functions that are equal a.e., and show that

$$f_n \overset{\mathrm{m}}{\to} f \qquad \Longleftrightarrow \qquad \mathrm{d}(f_n, f) \to 0.$$

Hint: The function $\frac{x}{x+1}$ is an increasing function of x. $\diamondsuit$

1.3. **Convergence.** Once we have a notion of distance, we have a corresponding notion of convergence.

Definition 1.3 (Convergent and Cauchy sequences)**.** Let X be a metric space with metric d, and let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of elements of X.

(a) We say that $\{f_n\}_{n\in\mathbb{N}}$ *converges* to $f \in X$ if $\lim\limits_{n\to\infty} \mathrm{d}(f_n, f) = 0$, i.e., if

$$\forall\, \varepsilon > 0, \quad \exists\, N > 0, \quad \forall\, n \geq N, \quad \mathrm{d}(f_n, f) < \varepsilon.$$

In this case, we write $\lim\limits_{n\to\infty} f_n = f$ or $f_n \to f$.

(b) We say that $\{f_n\}_{n\in\mathbb{N}}$ is *Cauchy* if

$$\forall\, \varepsilon > 0, \quad \exists\, N > 0, \quad \forall\, m, n \geq N, \quad \mathrm{d}(f_m, f_n) < \varepsilon. \qquad \diamondsuit$$

1.4. **Open and Closed Sets.**

Definition 1.4 (Open and Closed Sets)**.** Let X be a metric space.

(a) The *open ball of radius r centered at f* is

$$B_r(f) \;=\; \bigl\{g \in X : \mathrm{d}(f, g) < r\bigr\}.$$

(b) A subset U of X is *open* if for each $f \in U$ there exists some $r > 0$ such that $B_r(f) \subseteq U$.

(c) A subset F of X is *closed* if its complement $X \setminus F$ is open. $\diamondsuit$

We can equivalently characterize closed sets as being those sets that contain all of their limits.

Lemma 1.5. If E is a subset of a metric space X then the following two statements are equivalent.

(a) E is closed.

(b) E contains every limit of points of E, i.e.,

$$f_n \in E \text{ and } f_n \to f \in X \implies f \in E. \qquad \diamondsuit$$

1.5. **Complete Metric Spaces.** Here is an easy but important exercise.

Exercise 1.6. Prove that if X is a metric space, then every convergent sequence in X is Cauchy. $\diamondsuit$

In general, however, a Cauchy sequence need not be convergent (see Exercise 3.2 for an example). We give a name to spaces in which every Cauchy sequence does converge.

Definition 1.7 (Complete Metric Space)**.** If every Cauchy sequence in a metric space X has the property that it converges to an element of X, then X is said to be *complete*. $\diamondsuit$

2. Norms and Banach Spaces

A norm provides us with a notion of the length of a vector in a vector space. In these notes, we will take our vector spaces to be over the complex field $\mathbb{C}$, but only minor changes are needed if we instead assume that they are over the real field $\mathbb{R}$.

2.1. **The Definition of a Norm.**

Definition 2.1 (Seminorms and Norms)**.** Let X be a vector space over the field $\mathbb{C}$ of complex scalars. A *seminorm* on X is a function $\|\cdot\|\colon X \to \mathbb{R}$ such that for all $f,\, g \in X$ and all scalars $c \in \mathbb{C}$ we have:

(a) $0 \le \|f\| < \infty$,

(b) $\|cf\| = |c|\,\|f\|$, and

(c) The Triangle Inequality: $\|f+g\| \le \|f\| + \|g\|$.

A seminorm is a *norm* if we also have:

(d) $\|f\| = 0$ if and only if $f = 0$.

A vector space X together with a norm $\|\cdot\|$ is called a *normed linear space* or simply a *normed space*. $\diamondsuit$

Note that if S is a subspace of a normed space X, then S is itself a normed space with respect to the norm on X (restricted to S).

2.2. **The Induced Metric.**

The following exercise shows that all normed spaces are metric spaces. In particular, the notions of convergent and Cauchy sequences apply in any normed space.

Exercise 2.2. If X is a normed space, then

$$\mathrm{d}(f,g) \;=\; \|f-g\|$$

defines a metric on X, called the *induced metric*. $\diamondsuit$

Not every metric is induced from a norm; the metric in Exercise 1.2 corresponding to convergence in measure is an example.

2.3. **Some Properties of Norms.**

Exercise 2.3. Show that if X is a normed linear space, then the following statements hold.

(a) Reverse Triangle Inequality: $\big|\|f\| - \|g\|\big| \le \|f-g\|$.

(b) Continuity of the norm: $f_n \to f \implies \|f_n\| \to \|f\|$.

(c) Continuity of vector addition: $f_n \to f$ and $g_n \to g \implies f_n + g_n \to f + g$.

(d) Continuity of scalar multiplication: $f_n \to f$ and $c_n \to c \implies c_n f_n \to cf$.

(e) Boundedness of convergent sequences: if $\{f_n\}_{n\in\mathbb{N}}$ is convergent then $\sup \|f_n\| < \infty$.

(f) Boundedness of Cauchy sequences: if $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy then $\sup \|f_n\| < \infty$. $\diamondsuit$

2.4. **Banach Spaces.** We give a special name to normed spaces that are complete.

Definition 2.4 (Banach Space)**.** A normed linear space X is called a *Banach space* if it is complete with respect to the induced metric, i.e., if every Cauchy sequence in X converges to an element of X. $\diamondsuit$

Thus, the terms "Banach space" and "complete normed space" are interchangeable.

Example 2.5. The standard norm on the complex plane $\mathbb{C}$ is absolute value $|\cdot|$. An important fact is that $\mathbb{C}$ is a Banach space with respect to absolute value. $\diamondsuit$

The following exercise states that a subspace of a Banach space is itself a Banach space if and only if it is closed.

Exercise 2.6. Let M be a subspace of a Banach space X. Show that M is a normed space using the norm of X restricted to M, and show that

$$M \text{ is a Banach space} \quad \Longleftrightarrow \quad M \text{ is closed.} \qquad \diamondsuit$$

All subspaces of a finite-dimensional normed space are closed. However, subspaces of an infinite-dimensional normed space need not be closed. We will see some examples in Section 5.

3. Examples of Normed Spaces: ℓ^p and c_{00}

We will give a few examples of Banach spaces and complete metric spaces. We begin with the ℓ^p spaces on countable index sets.

Definition 3.1. Let I be a finite or countably infinite index sequence.

(a) If $1 \le p < \infty$, then $\ell^p(I)$ consists of all sequences of scalars $x = (x_k)_{k\in I}$ such that

$$\|x\|_p \;=\; \|(x_k)_{k\in I}\|_p \;=\; \biggl(\sum_{k\in I} |x_k|^p\biggr)^{1/p} \;<\; \infty.$$

(b) For $p = \infty$, the space $\ell^\infty(I)$ consists of all sequences of scalars $x = (x_k)_{k\in I}$ such that

$$\|x\|_\infty \;=\; \|(x_k)_{k\in I}\|_\infty \;=\; \sup_{k\in I} |x_k| \;<\; \infty.$$

If $I = \mathbb{N}$, then we write simply ℓ^p instead of $\ell^p(\mathbb{N})$.

If $I = \{1, \dots, d\}$, then $\ell^p(I) = \mathbb{C}^d$, and in this case we refer to $\ell^p(I)$ as "$\mathbb{C}^d$ under the ℓ^p norm." The ℓ^2 norm on $\mathbb{C}^d$ is called the *Euclidean norm.* $\diamondsuit$

It is a fact that each ℓ^p space for $1 \le p \le \infty$ is a normed space. This is easy to prove for $p = 1$ and $p = \infty$. However, it is not trivial to prove the Triangle Inequality when $1 < p < \infty$ — for this you need a result called *Hölder's Inequality.* For these notes we will assume that $\|\cdot\|_p$ is a norm on ℓ^p when $1 \le p \le \infty$. The following exercise addresses the issue of completeness.

Exercise 3.2. Given $1 \le p \le \infty$, prove that ℓ^p is complete.

Hints: Suppose that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in ℓ^p, and write $x_n = (x_n(1), x_n(2), \dots)$. Then show that for each *fixed* k we have that $\{x_n(k)\}_{n\in\mathbb{N}}$ is a Cauchy sequence of scalars. Since $\mathbb{C}$ is complete, this sequence of scalars converges, say $y_k = \lim_{k\to\infty} x_n(k)$. Thus, for each k, the kth component of x_n converges to the kth component of y; this is called *componentwise convergence.*

Now we have a candidate sequence $y = (y_1, y_2, \dots)$ for the limit of $\{x_n\}_{n\in\mathbb{N}}$. Use the fact that $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy in ℓ^p together with the componentwise convergence to show that $\|x - x_n\|_p \to 0$. $\diamondsuit$

3.1. **An Incomplete Space.** Let c_{00} denote the set of all sequences that contain only finitely many nonzero components:

$$c_{00} \;=\; \bigl\{x = (x_1, \dots, x_N, 0, 0, \dots) : N > 0,\, x_1, \dots, x_N \in \mathbb{C}\bigr\}.$$

The vectors in c_{00} are sometimes called *finite sequences* because they contain at most finitely many nonzero components. If we fix a particular value of p, then we have $c_{00} \subsetneq \ell^p$, so c_{00} is a normed space with respect to $\|\cdot\|_p$. However, according to the following exercise, c_{00} is not a closed subspace of ℓ^p and consequently is not complete.

Exercise 3.3. Fix $1 \le p \le \infty$, and let $x = \bigl(2^{-k}\bigr)_{k\in\mathbb{N}}$. Observe that $x \in \ell^p$, and find a sequence of vectors $\{x_n\}_{n\in\mathbb{N}}$ in c_{00} such that $\|x - x_n\|_p \to 0$ as $n \to \infty$. Use this to show that c_{00} is not a closed subset of ℓ^p, and that c_{00} is not a Banach space with respect to the norm $\|\cdot\|_p$. $\diamondsuit$

4. Function Space Examples of Normed Spaces

We will give some examples of Banach spaces of functions on $\mathbb{R}$. The *support* of a function is the closure of the set where it is nonzero, i.e.,

$$\operatorname{supp}(f) \;=\; \overline{\{x \in \mathbb{R} : f(x) \ne 0\}}.$$

Since $\operatorname{supp}(f)$ is closed, it is compact if and only if it is bounded. Thus, a function has *compact support* if and only if it is zero outside of some finite interval.

Exercise 4.1. (a) Let $\mathcal{F}_b(\mathbb{R})$ denote the space of bounded functions $f\colon \mathbb{R} \to \mathbb{C}$. Show that $\mathcal{F}_b(\mathbb{R})$ is a Banach space with respect to the *uniform norm*

$$\|f\|_{\mathrm{u}} \;=\; \sup_{t\in\mathbb{R}} |f(t)|.$$

(b) Let $C_b(\mathbb{R})$ denote the space of continuous, bounded functions $f\colon \mathbb{R} \to \mathbb{C}$. Show that $C_b(\mathbb{R})$ is a closed subspace of $\mathcal{F}_b(\mathbb{R})$ with respect to the *uniform norm*, and hence is itself a Banach space.

(c) Let $C_0(\mathbb{R})$ be the subspace of $C_b(\mathbb{R})$ consisting of functions that "decay to zero at infinity." Specifically,

$$C_0(\mathbb{R}) \;=\; \bigl\{f \in C_b(\mathbb{R}) : \lim_{|t|\to\infty} f(t) = 0\bigr\}.$$

Prove that $C_0(\mathbb{R})$ is a closed subspace of $C_b(\mathbb{R})$, and hence is a Banach space with respect to the uniform norm. $\diamondsuit$

Convergence with respect to the uniform norm is called *uniform convergence*, i.e., we say that $f_n \to f$ uniformly if $\|f - f_n\|_{\mathrm{u}} \to 0$ as $n \to \infty$.

4.1. **An Incomplete Space.** Consider the space $C_c(\mathbb{R})$ that consists of all continuous functions with compact support:

$$C_c(\mathbb{R}) \;=\; \bigl\{f \in C_b(\mathbb{R}) : \mathrm{supp}(f) \text{ is compact}\bigr\} \tag{4.1}$$

This is a subspace of $C_0(\mathbb{R})$, so it is a normed space with respect to the uniform norm. However, the following exercise shows that it is not a Banach space.

Exercise 4.2. Let $g(x) = e^{-|x|}$, and observe that g belongs to $C_0(\mathbb{R})$ but not $C_c(\mathbb{R})$. For each integer $n > 0$, define a compactly supported approximation to g by setting $g_n(x) = g(x)$ for $|x| \le n$ and $g_n(x) = 0$ for $|x| > n+1$, and let g_n be linear on $[n, n+1]$ and $[-n-1, -n]$ (see Figure 1). Prove that g_n converges uniformly to g, and conclude that $C_c(\mathbb{R})$ is not a closed subspace of $C_0(\mathbb{R})$. Consequently $C_c(\mathbb{R})$ is not complete with respect to the uniform norm. In fact, show directly that $\{g_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C_c(\mathbb{R})$, but also show that this sequence cannot converge to *any* element of $C_c(\mathbb{R})$. $\diamondsuit$

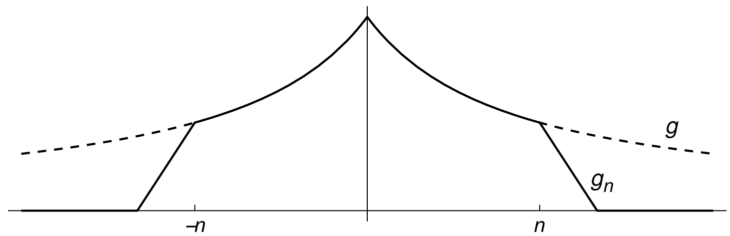

FIGURE 1. A function g and a compactly supported approximation g_n.

4.2. **Spaces of Differentiable Functions.** We define some spaces of functions with higher-order smoothness.

Exercise 4.3. Let $C_b^m(\mathbb{R})$ be the space of all m-times differentiable functions on $\mathbb{R}$ each of whose derivatives is bounded and continuous, i.e.,

$$C_b^m(\mathbb{R}) \;=\; \bigl\{f \in C_b(\mathbb{R}) : f, f', \dots, f^{(m)} \in C_b(\mathbb{R})\bigr\}.$$

Show that $C_b^m(\mathbb{R})$ is a Banach space with respect to the norm

$$\|f\|_{C_b^m} \;=\; \|f\|_{\mathrm{u}} + \|f'\|_{\mathrm{u}} + \cdots + \|f^{(m)}\|_{\mathrm{u}}.$$

5. Dense Subspaces

Definition 5.1. Let X be a normed linear space. We say that a subset S of X is *dense* in X if every element of X is a limit of points of S, i.e., if for each $f \in X$ there exist $f_n \in S$ such that $f_n \to f$. $\diamondsuit$

For example, the set $\mathbb{Q} + i\mathbb{Q}$ of all complex numbers whose real and imaginary parts are both rational is dense in $\mathbb{C}$. However, $\mathbb{Q} + i\mathbb{Q}$ is not a *subspace* of the vector space $\mathbb{C}$ (implicitly taken with the scalar field to be $\mathbb{C}$) because it is not closed under multiplication by elements of $\mathbb{C}$.

The only dense subspace of $\mathbb{C}$ is $\mathbb{C}$ itself. In finite-dimensional spaces, there are no proper dense subspaces. There do exist examples in infinite-dimensional spaces, some are listed in the next exercise.

Exercise 5.2. (a) Fix $1 \le p \le \infty$. Prove that the space c_{00} introduced in Exercise 3.2 is a subspace of ℓ^p that is not closed (with respect to the ℓ^p-norm). Prove that c_{00} is dense in $\ell^p(\mathbb{N})$ if $p < \infty$, but that it is not dense in ℓ^∞.

(b) Define

$$c_0 = \Bigl\{x = (x_k)_{k=1}^\infty : \lim_{k\to\infty} x_k = 0\Bigr\}.$$

Prove c_0 is a closed subspace of $\ell^\infty(\mathbb{N})$, and that c_{00} is dense in c_0 with respect to the ℓ^∞-norm. $\diamondsuit$

Exercise 5.3. Show that the space $C_c(\mathbb{R})$ introduced in equation (4.1) is a proper dense subspace of $C_0(\mathbb{R})$ (with respect to the uniform norm).

Hint: Modify the idea of Exercise 4.2. $\diamondsuit$