FUNCTIONAL ANALYSIS LECTURE NOTES:

QUOTIENT SPACES

CHRISTOPHER HEIL

1. Cosets and the Quotient Space

Any vector space is an abelian group under the operation of vector addition. So, if you are have studied the basic notions of abstract algebra, the concept of a coset will be familiar to you. However, even if you have not studied abstract algebra, the idea of a coset in a vector space is very natural: it is just a translate of a subspace.

Example 1.1 (Cosets in $\mathbb{R}^2$)**.** Consider the vector space $X = \mathbb{R}^2$. Let M be any one-dimensional subspace of $\mathbb{R}^2$, i.e., M is a line in $\mathbb{R}^2$ through the origin. A coset of M is a rigid translate of M by a vector in $\mathbb{R}^2$. For concreteness, let us consider the case where M is the x_1-axis in $\mathbb{R}^2$, i.e.,

$$M \;=\; \{(x_1, 0) : x_1 \in \mathbb{R}\}.$$

Then given a vector $y = (y_1, y_2) \in \mathbb{R}^2$, the coset $y + M$ is the set

$$y + M \;=\; \{y + m : m \in M\} \;=\; \{(y_1 + x_1, y_2 + 0) : x_1 \in \mathbb{R}\} \;=\; \{(x_1, y_2) : x_1 \in \mathbb{R}\},$$

which is the horizontal line at height y_2. This is not a subspace of $\mathbb{R}^2$, but it is a rigid translate of the x_1-axis. Note that there are infinitely many different choices of y that give the same coset. Furthermore, we have the following facts for this particular setting.

(a) Two cosets of M are either identical or entirely disjoint.

(b) The union of the cosets is all of $\mathbb{R}^2$.

(c) The set of distinct cosets is a partition of $\mathbb{R}^2$.

The preceding example is entirely typical.

Definition 1.2 (Cosets)**.** Let M be a subspace of a vector space X. Then the *cosets* of M are the sets

$$f + M \;=\; \{f + m : m \in M\}, \qquad f \in X.$$

Exercise 1.3. Let X be a vector space, and let M be a subspace of X. Given $f, g \in X$, define $f \sim g$ if $f - g \in M$. Prove the following.

(a) $\sim$ is an equivalence relation on X.

(b) The equivalence class of f under the relation $\sim$ is $[f] = f + M$.

(c) If $f,\, g \in M$ then either $f + M = g + M$ or $(f + M) \cap (g + M) = \emptyset$.

(d) $f + M = g + M$ if and only if $f - g \in M$.

(e) $f + M = M$ if and only if $f \in M$.

(f) If $f \in X$ and $m \in M$ then $f + M = f + m + M$.

(g) The set of distinct cosets of M is a partition of X.

Definition 1.4 (Quotient Space)**.** If M is a subspace of a vector space X, then the quotient space X/M is
$$X/M \;=\; \{f + M : f \in X\}.$$

Since two cosets of M are either identical or disjoint, the quotient space X/M is the set of all the distinct cosets of M.

Example 1.5. Again let $M = \{(x_1, 0) : x_1 \in \mathbb{R}\}$ be the x_1-axis in $\mathbb{R}^2$. Then, by Example 1.1, we have that
$$\mathbb{R}^2/M \;=\; \{y + M : y \in \mathbb{R}^2\} \;=\; \{(x_1, 0) + M : x_1 \in \mathbb{R}\},$$
i.e., $\mathbb{R}^2/M$ is the set of all horizontal lines in $\mathbb{R}^2$. Note that $\mathbb{R}^2/M$ is in 1-1 correspondence with the set of distinct heights, i.e., there is a natural bijection of $\mathbb{R}^2/M$ onto $\mathbb{R}$. This is a special case of a more general fact that we will explore.

Next we define two natural operations on the set of cosets: addition of cosets and multiplication of a coset by a scalar. These are defined formally as follows.

Definition 1.6. Let M be a subspace of a vector space X. Given $f,\, g \in X$, define addition of cosets by
$$(f + M) + (g + M) \;=\; (f + g) + M.$$
Given $f \in X$ and $c \in \mathbb{F}$, define scalar multiplication by
$$c(f + M) \;=\; cf + M.$$

Remark 1.7. Before proceeding, we must show that these operations are actually well-defined. After all, there need not be just one f that determines the coset $f + M$—how do we know that if we choose different vectors that determine the same cosets, we will get the same result when we compute $(f + g) + M$? We must show that $f_1 + M = f_2 + M$ and $g_1 + M = g_2 + M$ then $(f_1 + g_1) + M = (f_2 + g_2) + M$ in order to know that Definition 1.6 makes sense.

Proposition 1.8. If M is a subspace of a vector space X, then the addition of cosets of M given in Definition 1.6 is well-defined.

Proof. Suppose that $f_1 + M = f_2 + M$ and $g_1 + M = g_2 + M$. Then by Exercise 1.3(d) we know that $f_1 - f_2 = k \in M$ and $g_1 - g_2 = l \in M$. If $h \in (f_1 + g_1) + M$ then we have $h = f_1 + g_1 + m$ for some $m \in M$. Hence

$$h \;=\; (f_2 + k) + (g_2 + l) + m \;=\; (f_2 + g_2) + (k + l + m) \;\in\; (f_2 + g_2) + M.$$

Thus $(f_1 + g_1) + M \subseteq (f_2 + g_2) + M$, and the converse inclusion is symmetric. □

Exercise 1.9. Show that scalar multiplication is likewise well-defined.

Now we can show that the quotient space is actually a vector space under the operations just defined.

Proposition 1.10. If M is a subspace of a vector space X, then X/M is a vector space with respect to the operations given in Definition 1.6.

Proof. Addition of cosets is commutative because

$$(f + M) + (g + M) \;=\; (f + g) + M \;=\; (g + f) + M \;=\; (g + M) + (f + M).$$

The zero vector in X/M is the coset $0+M = M$, because $(f+M)+(0+M) = (f+0)+M = f + M$.

Exercise: Show that the remaining axioms of a vector space are satisfied. □

Definition 1.11 (Codimension)**.** If M is a subspace of a vector space X, then the codimension of M is the vector space dimension of X/M, i.e.,

$$\operatorname{codim}(M) \;=\; \dim(X/M).$$

Example 1.12. Let $C(\mathbb{R})$ be space of continuous functions on $\mathbb{R}$, and let $\mathcal{P}$ be the subspace containing the polynomials. Given $f \in C(\mathbb{R})$, the coset determined by f is

$$f + \mathcal{P} \;=\; \{f + p : p \text{ is a polynomial}\}.$$

Further, $f + \mathcal{P} = g + \mathcal{P}$ if and only if $f - g$ is a polynomial. Thus, $f + \mathcal{P}$ can be thought of as "f modulo the polynomials," i.e., it is the equivalence class obtained by identifying functions which differ by a polynomial.

In the same way, a coset $f + M$ can be thought of as the equivalence class obtained by identifying vectors which differ by an element of M. We can imagine the mapping that takes f to $f + M$ as "collapsing information modulo M."

2. The Canonical Projection

Definition 2.1. If M is a subspace of a vector space X, then the *canonical projection* or the *canonical mapping* of X onto X/M is $\pi\colon X \to X/M$ defined by

$$\pi(f) \;=\; f + M, \quad f \in X.$$

Exercise 2.2. Let M be a subspace of a vector space X.

(a) Prove that the canonical projection π is linear.

(b) Prove that π is surjective and $\ker(\pi) = M$.

(c) Prove that if $E \subseteq X$, then the inverse image of $\pi(E)$ is

$$\pi^{-1}\big(\pi(E)\big) \;=\; E + M \;=\; \{u + m : u \in E, m \in M\}.$$

We will mostly be interested in the case where X is a normed space. The following result shows that X/M is a semi-normed space in general, and is a normed space if M is closed.

Proposition 2.3. Let M be a subspace of a normed linear space X. Given $f \in X$, define

$$\|f + M\| \;=\; \operatorname{dist}(f, M) \;=\; \inf_{m \in M} \|f - m\|.$$

Then the following statements hold.

(a) $\|\cdot\|$ is well-defined.

(b) $\|\cdot\|$ is a semi-norm on X/M.

(c) If M is closed, then $\|\cdot\|$ is a norm on X/M.

Proof. (a), (b) Exercises.

(c) Suppose that M is closed, and that $\|f + M\| = 0$. Then $\inf_{m\in M} \|f - m\| = 0$. Hence there exist vectors $g_n \in M$ such that $\|f - g_n\| \to 0$ as $n \to \infty$. But M is closed, so this implies $f \in M$. By Exercise 1.3(e), we therefore have $f + M = M = 0 + M$, which is the zero vector in X/M. □

In the Hilbert space case, there is a very close relationship between π and the orthogonal projection of H onto $M^\perp$.

Exercise 2.4. Let M be a closed subspace of a Hilbert space H, and let π be the canonical projection of H onto H/M. Prove that the restriction of π to $M^\perp$ is an isometric isomorphism of $M^\perp$ onto H/M.

Remark 2.5. There is no analog of the preceding result for arbitrary Banach spaces. If X is a Banach space and M is a closed subspace then we say that M is *complemented* in X if there exists another closed subspace N such that $M \cap N = \{0\}$ and $M + N = X$.

It is not true that every closed subspace of every Banach space is complemented. In particular, c_0 is not complemented in ℓ^∞. Also, if $1 < p \leq \infty$ and $p \neq 2$, then ℓ^p has uncomplemented subspaces.

Now we derive some basic properties of the canonical projection π of X onto X/M.

Proposition 2.6. Let M be a closed subspace of a normed linear space X. Then the following statements hold.

(a) π is continuous, with $\|\pi(f)\| = \|f + M\| \le \|f\|$ for each $f \in X$.

(b) Let $B_r^X(f)$ denote the open ball of radius r in X centered at f, and let $B_r^{X/M}(f+M)$ denote the open ball of radius r in X/M centered at $f + M$. Then for any $f \in X$ and $r > 0$ we have

$$\pi\big(B_r^X(f)\big) \;=\; B_r^{X/M}(f+M).$$

(c) $W \subseteq X/M$ is open in X/M if and only if $\pi^{-1}(W) = \{f \in X : f + M \in W\}$ is open in X.

(d) π is an *open mapping*, i.e., if U is open in X then $\pi(U)$ is open in X/M.

Proof. (a) Choose any $f \in X$. Since 0 is one of the elements of M, we have

$$\|\pi(f)\| \;=\; \|f+M\| \;=\; \inf_{m\in M} \|f-m\| \;\le\; \|f-0\| \;=\; \|f\|.$$

(b) First consider the case $f = 0$ and $r > 0$. Suppose that $g + M \in \pi\big(B_r^X(0)\big)$. Then $g + M = h + M$ for some $h \in B_r^X(0)$, i.e., $\|h\| < r$. Hence $\|g + M\| = \|h + M\| \le \|h\| < r$, so $g + M \in B_r^{X/M}(0 + M)$.

Now suppose that $g + M \in B_r^{X/M}(0 + M)$. Then $\inf_{m\in M} \|g - m\| = \|g + M\| < r$. Hence there exists $m \in M$ such that $\|g - m\| < r$. Thus $g - m \in B_r^X(0)$, so

$$g+M \;=\; g-m+M \;=\; \pi(g-m) \;\in\; \pi\big(B_r^X(0)\big).$$

Exercise: Show that statement (b) holds for an arbitrary $f \in X$.

(c) $\Rightarrow$. Part (a) implies that π is continuous. Hence $\pi^{-1}(W)$ must be open in X if W is open in X/M.

$\Leftarrow$. Suppose that W is a subset of X/M such that $\pi^{-1}(W)$ is open in X. We must show that W is open in X/M. Choose any point $f + M \in W$. Then $f \in \pi^{-1}(W)$, which is open in X. Hence, there exists an $r > 0$ such that $B_r^X(f) \subseteq \pi^{-1}(W)$. By part (b) we therefore have

$$B_r^{X/M}(f+M) \;=\; \pi\big(B_r^X(f)\big) \;\subseteq\; \pi\big(\pi^{-1}(W)\big) \;=\; W.$$

Therefore W is open.

(d) Suppose that U is an open subset of X. Then by Exercise 2.2(c), we have

$$\pi^{-1}\big(\pi(U)\big) \;=\; U+M \;=\; \{u+m : u\in U, m\in M\} \;=\; \bigcup_{m\in M} (U+m).$$

But each set $U + m$, being the translate of the open set U, is itself open. Hence $\pi^{-1}\big(\pi(U)\big)$ is open, since it is a union of open sets. Part (c) therefore implies that $\pi(U)$ is open in X/M. ◻

Exercise 2.7. Let M be a closed subspace of a normed space X, and let π be the canonical projection π of X onto X/M. Prove that $\|\pi\| = 1$.

Hint: F. Riesz's Lemma.

Lemma 2.8 (F. Riesz's Lemma)**.** Let M be a proper, closed subspace of a normed space X. Then for each $\varepsilon > 0$, there exists $g \in X$ with $\|g\| = 1$ such that

$$\operatorname{dist}(g, M) \;=\; \inf_{f \in M} \|g - f\| \;>\; 1 - \varepsilon.$$

Proof. Choose any $u \in X \setminus M$. Since M is closed, we have

$$a \;=\; \operatorname{dist}(u, M) \;=\; \inf_{f \in M} \|u - f\| \;>\; 0.$$

Fix $\delta > 0$ small enough that $\frac{a}{a+\delta} > 1 - \varepsilon$. By definition of infimum, there exists $v \in M$ such that $a \le \|u - v\| < a + \delta$. Set

$$g \;=\; \frac{u - v}{\|u - v\|},$$

and note that $\|g\| = 1$. Given $f \in M$ we have $h = v + \|u - v\|\, f \in M$, so

$$\|g - f\| \;=\; \left\| \frac{u - v - \|u - v\|\, f}{\|u - v\|} \right\| \;=\; \frac{\|u - h\|}{\|u - v\|} \;>\; \frac{a}{a + \delta} \;>\; 1 - \varepsilon. \qquad \square$$

Exercise 2.9. Let X be a normed linear space. Let $B = \{x \in X : \|x\| \le 1\}$ be the closed unit ball in X. Prove that if B is compact, then X is finite-dimensional.

Hints: Suppose that X is infinite-dimensional. Given any nonzero e_1 with $\|e_1\| \le 1$, by Lemma 2.8 there exists $e_2 \in X \setminus \operatorname{span}\{e_1\}$ with $\|e_2\| \le 1$ such that $\|e_2 - e_1\| > \frac{1}{2}$. Continue in this way to construct vectors e_k such that $\{e_1, \dots, e_n\}$ are independent for any n. Conclude that X is infinite-dimensional.

3. The Banach Space X/M

Now we can prove that if X is a Banach space, then X/M inherits a Banach space structure from X.

Theorem 3.1. If M is a closed subspace of a Banach space X, then X/M is a Banach space.

Proof. We have already shown that X/M is a normed space, so we must show that it is complete in that norm.

Suppose that $\{f_n + M\}_{n \in \mathbb{N}}$ is a Cauchy sequence in X/M. It would be convenient if this implies that $\{f_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in X, but this need not be the case. For, the vectors f_n are not unique in general: if we replace f_n by any vector $f_n + m$ with $m \in M$, then we obtain the same coset. We will show that by choosing an appropriate subsequence $\{f_{n_k}\}_{k \in \mathbb{N}}$ and replacing the f_{n_k} by appropriate vectors that determine the same cosets $f_{n_k} + M$, we can create a sequence in X that is Cauchy and hence converges, and then use this to show that the original sequence of cosets $\{f_n + M\}_{n \in \mathbb{N}}$ converges in X/M.

We begin by applying an earlier exercise about Cauchy sequences: there exists a subsequence $\{f_{n_k} + M\}_{k\in\mathbb{N}}$ such that

$$\forall\, k \in \mathbb{N}, \quad \|(f_{n_{k+1}} - f_{n_k}) + M\| \;=\; \|(f_{n_{k+1}} + M) - (f_{n_k} + M)\| \;<\; 2^{-k}.$$

Now we seek to create vectors $g_k \in M$ so that $\{f_{n_k} - g_k\}_{k\in\mathbb{N}}$ will converge in X. Note that the cosets determined by f_{n_k} and by $f_{n_k} - g_k$ are identical.

Set $g_1 = 0$. Then

$$\inf_{g\in M} \|(f_{n_1} - g_1) - (f_{n_2} - g)\| \;=\; \inf_{g\in M} \|(f_{n_1} - f_{n_2}) + g\| \;=\; \|(f_{n_1} - f_{n_2}) + M\| \;<\; \frac{1}{2}.$$

Therefore, there exists a $g_2 \in M$ such that

$$\|(f_{n_1} - g_1) - (f_{n_2} - g_2)\| \;<\; \frac{1}{2}.$$

Then, since $g_2 \in M$,

$$\inf_{g\in M} \|(f_{n_2} - g_2) - (f_{n_3} - g)\| \;=\; \inf_{g\in M} \|(f_{n_2} - f_{n_3}) + g\| \;=\; \|(f_{n_2} - f_{n_3}) + M\| \;<\; \frac{1}{2^2}.$$

Therefore, there exists a $g_3 \in M$ such that

$$\|(f_{n_2} - g_2) - (f_{n_3} - g_3)\| \;<\; \frac{1}{2^2}.$$

Continuing in this way, by induction we construct $h_k = f_{n_k} - g_k$ such that

$$\|h_k - h_{k+1}\| \;<\; \frac{1}{2^k}.$$

An earlier exercise therefore implies that $\{h_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in X. Since X is complete, this sequence converges, say $h_k \to h$.

Since

$$\begin{aligned} \|(f_{n_k} + M) - (h + M)\| \;&=\; \|f_{n_k} - g_k - h + M\| \qquad (\text{since } g_k \in M) \\ &=\; \|h_k - h + M\| \\ &\leq\; \|h_k - h\| \;\to\; 0, \end{aligned}$$

we see that $\{f_{n_k} + M\}_{k\in\mathbb{N}}$ is a convergent subsequence of $\{f_n + M\}_{n\in\mathbb{N}}$. Thus $\{f_n + M\}_{n\in\mathbb{N}}$ is a Cauchy sequence that has a convergent subsequence, and hence it must converge. Therefore X/M is complete. □

The following exercise shows that the converse of the preceding theorem is true as well.

Exercise 3.2. Let M be a closed subspace of a normed space X. Prove that if M and X/M are both complete, then X must be complete.

Note: Since X is not assumed to be a Banach space, the fact that M is closed does not imply that M is complete. We need that as a separate assumption.

Hints: Suppose that $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Show that $\{f_n + M\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X/M, hence converges to some coset $f + M$. Thus $\|f - f_n + M\| \to 0$. Does this imply that there exists a $g \in M$ such that $\|f - f_n + g\| \to 0$? Or perhaps vectors $g_n \in M$ such that $\|f - f_n + g_n\| \to 0$?

The quotient space and canonical map will be useful tools for proving many later results. The following proof illustrates their utility.

Proposition 3.3. Let X be a normed linear space. If M is a closed subspace of X and N is finite-dimensional, then $M+N$ is a closed subspace of X.

Proof. Let π be the canonical projection of X onto X/M. Since N is finite-dimensional, it has a finite basis, say $\{e_1,\dots,e_n\}$. Then since π is linear,

$$\pi(N) \;=\; \pi\big(\mathrm{span}\{e_1,\dots,e_n\}\big) \;=\; \mathrm{span}\{\pi(e_1),\dots,\pi(e_n)\} \;=\; \mathrm{span}\{e_1+M,\dots,e_n+M\}.$$

Thus $\pi(N)$ is a finite-dimensional subspace of X/M, and therefore is closed. Since π is continuous, it follows that $\pi^{-1}(\pi(N))$ is closed in X. However, by Exercise 2.2(c), we have $\pi^{-1}(\pi(N)) = M+N$. ☐

4. Additional Exercises on Quotient Spaces

Exercise 4.1. Let M be a closed subspace of a normed linear space X.

(a) Prove that if X is separable, then X/M is separable.

(b) Prove that if X/M and M are both separable, then X is separable.
Hint: Let $\{f_n+M\}_{n\in\mathbb{N}}$ be a countable dense subset of X/M, and let $\{g_n\}_{n\in\mathbb{N}}$ be a countable dense subset of M. Then $S=\{f_m+g_n\}_{m,n\in\mathbb{N}}$ is a countable subset of X. Show that it is dense in X.

(c) Give an example of X, M such that X/M is separable, but X is not separable.

Exercise 4.2. Fix $1\le p\le\infty$.

(a) Show that

$$M \;=\; \big\{x=(x_k)_{k\in\mathbb{N}}\in\ell^p : x_1=0\big\}$$

is a closed subspace of ℓ^p. Further, $\mathrm{codim}(M)=1$ and $\ell^p\cong M$.

(b) Show that

$$M \;=\; \big\{x=(x_k)_{k\in\mathbb{N}}\in\ell^p : x_{2k}=0\ \forall\, k\in\mathbb{N}\big\}$$

is a closed subspace of ℓ^p. Further, $\mathrm{codim}(M)=\infty$, $\ell^p\cong M$, and $\ell^p/M\cong\ell^p$.

Exercise 4.3. Define

$$c_0 \;=\; \big\{x=(x_k)_{k\in\mathbb{N}} : \lim_{k\to\infty} x_k = 0\big\},$$

$$c \;=\; \big\{x=(x_k)_{k\in\mathbb{N}} : \lim_{k\to\infty} x_k \text{ exists}\big\}.$$

(a) Show that c and c_0 are closed subspaces of ℓ^∞.

(b) Show that $\dim(c/c_0)=1$.

(c) Show directly that ${c_0}^*\cong\ell^1$ and $c^*\cong\ell^1$.

(d) Is $c_0\cong c$?

Exercise 4.4. Recall that c_0 is a closed subspace of the Banach space ℓ^∞.

(a) Prove that ℓ^∞ is not separable.
Hint: Consider

$$S = \{(x_1, x_2, \dots) : x_k = 0 \text{ or } 1 \text{ for every } k\}.$$

What is the distance between two distinct elements of S?

(b) Let x, y be vectors in S. Prove that if $x_k \neq y_k$ for at most finitely many k, then $x + c_0 = y + c_0$. Prove that if $x_k \neq y_k$ for infinitely many k, then $\|x - y + c_0\| = 1$.

(c) Use part (b) to prove directly that ℓ^∞/c_0 is not separable.

(e) Prove that the standard basis $\{e_n\}_{n\in\mathbb{N}}$ is a Schauder basis for c_0 (with respect to the ℓ^∞-norm).

(f) Use part (e) to show that c_0 is separable.

(g) Use part (f) and Exercise 4.1 to show that ℓ^∞/c_0 is not separable.