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Accuracy of Several Multidimensional Refinable Distributions

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ABSTRACT. Compactly supported distributions $f_1, \dots, f_r$ on $\mathbf{R}^d$ are refinable if each f_i is a finite linear combination of the rescaled and translated distributions $f_j(Ax-k)$, where the translates k are taken along a lattice $\Gamma \subset \mathbf{R}^d$ and A is a dilation matrix that expansively maps Γ into itself. Refinable distributions satisfy a refinement equation $f(x) = \sum_{k\in\Lambda} c_k\, f(Ax-k)$, where Λ is a finite subset of Γ, the c_k are $r \times r$ matrices, and $f = (f_1, \dots, f_r)^{\mathrm{T}}$. The accuracy of f is the highest degree p such that all multivariate polynomials q with degree$(q) < p$ are exactly reproduced from linear combinations of translates of $f_1, \dots, f_r$ along the lattice Γ. We determine the accuracy p from the matrices c_k. Moreover, we determine explicitly the coefficients $y_{\alpha,i}(k)$ such that $x^\alpha = \sum_{i=1}^r \sum_{k\in\Gamma} y_{\alpha,i}(k)\, f_i(x+k)$. These coefficients are multivariate polynomials $y_{\alpha,i}(x)$ of degree $|\alpha|$ evaluated at lattice points $k \in \Gamma$.

1. Introduction

A discrete set $\Gamma \subset \mathbf{R}^d$ is a *lattice* if it is the image of $\mathbf{Z}^d$ under some nonsingular linear transformation. A $d \times d$ matrix A is *expansive* if every eigenvalue λ satisfies $|\lambda| > 1$. An expansive matrix A is a *dilation matrix* with respect to a lattice Γ if $A(\Gamma) \subset \Gamma$. Complex-valued functions $f_1, \dots, f_r$ on $\mathbf{R}^d$ are *refinable* with respect to A and Γ if each f_i equals a linear combination of the rescaled and translated functions $f_j(Ax-k)$. We shall only consider the case where the linear combinations are finite. Then the vector-valued function $f\colon \mathbf{R}^d \to \mathbf{C}^r$ defined by $f(x) = (f_1(x), \dots, f_r(x))^{\mathrm{T}}$ satisfies a *refinement equation*, *dilation equation*, or *two-scale difference equation* of the form

$$f(x) \;=\; \sum_{k\in\Lambda} c_k\, f(Ax-k)\,, \tag{1.1}$$

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where Λ is a finite subset of Γ and the c_k are $r \times r$ matrices. By applying a similarity transformation to $\mathbf{R}^d$ we can, if desired, always reduce to the case $\Gamma = \mathbf{Z}^d$. In this case, A will necessarily have integer entries.

Refinable functions play important roles in several areas, including wavelet theory [7] and subdivision schemes in approximation theory [6]. A key goal is the determination of properties of a refinable $f\colon \mathbf{R}^d \to \mathbf{C}^r$ based on the coefficient mask $c = \{c_k\}_{k\in\Lambda}$. One fundamental property is the *accuracy* of f, the largest integer p such that every multivariate polynomial q with $\deg(q) < p$ lies in the shift-invariant space

$$S(f) \;=\; \left\{ \sum_{k\in\Gamma} \sum_{i=1}^{r} w_{k,i}\, f_i(x+k) : w_{k,i} \in \mathbf{C} \right\} \;=\; \left\{ \sum_{k\in\Gamma} w_k\, f(x+k) : w_k \in \mathbf{C}^{1\times r} \right\} \tag{1.2}$$

generated by f, where $\mathbf{C}^r = \mathbf{C}^{r\times 1}$ is the space of column vectors of length r and $\mathbf{C}^{1\times r}$ is the space of row vectors of length r. We shall deal only with compactly supported f, in which case each series in (1.2) is well-defined for all choices of $w_{k,i}$. Accuracy is closely related to order of approximation; we shall not pursue this but instead refer the reader interested in this connection to literature such as [1, 2, 3].

The classical one-dimensional, single-function refinement equation has $d = 1$, $r = 1$, $\Gamma = \mathbf{Z}$, and $A = 2$. It is well-known that, with minor hypotheses, accuracy p holds in this case if and only if

$$\sum_{k=0}^{N} c_k = 2 \qquad \text{and} \qquad \sum_{k=0}^{N} (-1)^k\, k^j\, c_k = 0 \quad \text{for } j = 0, \dots, p-1\,.$$

The extension of these results to multiple functions, higher dimensions, general dilation matrices, and distributional solutions presents several difficulties. For the one-dimensional, multi-function case $d = 1$, $r \geq 1$, $\Gamma = \mathbf{Z}$, $A = 2$, Heil, Strang, and Strela [9] and Plonka [15] independently derived the "matrix sum rules" that characterize accuracy for integrable refinable functions. A previous paper [4] considered the higher-dimensional, multi-function case with an arbitrary dilation matrix A, assuming the existence of an integrable, compactly supported solution to the refinement equation (1.1) and also assuming (for the necessary conditions) that lattice translates of the $f_i(x+k)$ are linearly independent. Some similar results, for the case of diagonalizable A, were also derived in [14].

The determination of whether a refinement equation has an integrable solution is considerably involved and complicated. However, every refinement equation such that, for example, the matrix $\Delta = \frac{1}{|\det(A)|} \sum_{k\in\Lambda} c_k$ has eigenvalues $\lambda_1 = \cdots = \lambda_s = 1$, $|\lambda_{s+1}|, \dots, |\lambda_r| < 1$ with the eigenvalue 1 nondegenerate always has a compactly supported distributional solution (see Proposition 1).

In this paper we consider the general case of compactly supported distributional solutions to higher-dimensional, multi-function refinement equations with arbitrary dilation matrices. We present necessary and sufficient conditions for accuracy, and show that these conditions are equivalent to a finite set of finite linear equations. Our conditions do not require the hypothesis that translates of f be independent. For additional history and motivation of this problem, we refer to [4] and [13] and the references contained therein.

The basic notation for our paper is presented in Section 2. In Section 3 we present some results which apply to arbitrary distributions that are not necessarily refinable, and in Section 4 we present our main results on refinable distributions.

2. Notation

2.1. General Notation.

The Lebesgue measure of a set $E \subset \mathbf{R}^d$ is denoted $|E|$.

We use the standard multi-index notation $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ for $x \in \mathbf{R}^d$ and $\alpha = (\alpha_1, \ldots, \alpha_d)$ a vector of nonnegative integers. The degree of α is $|\alpha| = \alpha_1 + \cdots + \alpha_d$. The number of multi-indices α of a given degree s is $d_s = \binom{s+d-1}{d-1}$. We write $\beta \le \alpha$ if $\beta_i \le \alpha_i$ for $i = 1, \ldots, d$. If $\beta \le \alpha$ then we set $\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_d}{\beta_d}$, otherwise $\binom{\alpha}{\beta} = 0$.

A dilation matrix A necessarily has integer determinant. We set $m = |\det(A)|$. We let $d_1, \ldots, d_m \in \Gamma$ be a *full set of digits*, i.e., a complete set of representatives of the order-m group $\Gamma/A(\Gamma)$. Then Γ is partitioned into the disjoint cosets

$$\Gamma_i \;=\; A(\Gamma) - d_i \;=\; \{Ak - d_i : k \in \Gamma\}\,.$$

Let $u_1, \ldots, u_d \in \mathbf{R}^d$ be a set of generators for the lattice Γ, i.e., $\Gamma = \{m_1 u_1 + \cdots + m_d u_d : m_i \in \mathbf{Z}\}$. Then the rectangular parallelepiped

$$P \;=\; \{x_1 u_1 + \cdots x_d u_d : 0 \le x_i < 1\} \tag{2.1}$$

is a *fundamental domain* for the group $\mathbf{R}^d/\Gamma$, and $\mathbf{R}^d$ is partitioned into the disjoint sets $\{P+k\}_{k\in\Gamma}$. Note that if $\Gamma = \mathbf{Z}^d$ then we can take $P = [0,1)^d$, and for a general lattice P can be taken to be the image of $[0,1)^d$ under the similarity transformation that takes $\mathbf{Z}^d$ to Γ.

We use the following generalized matrix notation. Let J and K be finite or countable index sets. Let $m_{j,k}$ be ordinary $r \times s$ matrices for $j \in J$ and $k \in K$. Then we say that $M = [m_{j,k}]_{j\in J, k\in K} \in (\mathbf{C}^{r\times s})^{J\times K}$ is a "$J \times K$ matrix with $r \times s$ block entries." If $N = [n_{k,\ell}]_{k\in K,\ell\in L} \in (\mathbf{C}^{s\times t})^{K\times L}$, then the product of the $J \times K$ matrix M with the $K \times L$ matrix N is the $J \times L$ matrix

$$MN \;=\; \left[\sum_{k\in K} m_{j,k}\, n_{k,\ell}\right]_{j\in J,\ell\in L}.$$

Most summations encountered in this paper will contain only finitely many nonzero terms. A column vector is a $J \times 1$ matrix, which we denote by $v = [v_j]_{j\in J}$. The entries v_j may be scalars or $r \times s$ blocks. Analogously, a row vector is a $1 \times J$ matrix, which we denote by $u = (u_j)_{j\in J}$.

Integrals of a vector-valued function $f = (f_1, \ldots, f_r)^{\mathrm{T}}$ are computed componentwise. In particular, if the f_i are integrable then we define the Fourier transform of f to be

$$\hat f(\omega) \;=\; \int_{\mathbf{R}^d} f(x)\, e^{-2\pi i x\cdot\omega}\, dx \;=\; \left(\int_{\mathbf{R}^d} f_1(x)\, e^{-2\pi i x\cdot\omega}\, dx, \ldots, \int_{\mathbf{R}^d} f_r(x)\, e^{-2\pi i x\cdot\omega}\, dx\right)^{\mathrm{T}}.$$

2.2 Vector-Valued Distributions.

$C^\infty(\mathbf{R}^d)$ is the space of all infinitely differentiable functions on $\mathbf{R}^d$. The Schwartz class $\mathcal{S}(\mathbf{R}^d)$ contains all infinitely differentiable functions each of whose derivatives decay faster than the reciprocal of any polynomial. $C_c^\infty(\mathbf{R}^d)$ is the space of compactly supported infinitely differentiable functions. The topological dual of $C_c^\infty(\mathbf{R}^d)$ is the space of distributions $\mathcal{D}'(\mathbf{R}^d)$. The space of tempered distributions $\mathcal{S}'(\mathbf{R}^d)$ is the dual of $\mathcal{S}(\mathbf{R}^d)$. The space of compactly supported distributions $\mathcal{E}'(\mathbf{R}^d)$ is the dual of $C^\infty(\mathbf{R}^d)$. We write $\langle \varphi, g\rangle$ or $\langle \varphi(x), g(x)\rangle$ to denote the evaluation of a distribution g on a test function φ. The Fourier transform maps $\mathcal{S}(\mathbf{R}^d)$ into itself, and extends to $\mathcal{S}'(\mathbf{R}^d)$ by duality. The Paley–Wiener theorem for distributions implies that the Fourier transform of a compactly supported distribution is a continuous function on $\mathbf{R}^d$ [16, p. 198].

We let $C^\infty(\mathbf{R}^d, \mathbf{C}^r) = C^\infty(\mathbf{R}^d) \times \cdots \times C^\infty(\mathbf{R}^d)$ denote the space of vector-valued functions $\varphi = (\varphi_1, \ldots, \varphi_r)^{\mathrm{T}}$ with each component $\varphi_i \in C^\infty(\mathbf{R}^d)$. We use analogous notations for other cross products of spaces of test functions or distributions. If $\varphi = (\varphi_1, \ldots, \varphi_r)^{\mathrm{T}}$ is a vector-valued test function and $f = (f_1, \ldots, f_r)^{\mathrm{T}}$ is a vector-valued distribution, then we write

$$\langle \varphi, f\rangle \;=\; \begin{bmatrix} \langle \varphi_1, f_1\rangle \\ \vdots \\ \langle \varphi_r, f_r\rangle \end{bmatrix}.$$

We allow a vector-valued distribution f to act on a scalar-valued test function φ by defining $\langle \varphi, f\rangle = (\langle \varphi, f_1\rangle, \dots, \langle \varphi, f_r\rangle)^{\mathrm{T}}$ in this case.

Note that if $v = (v_1, \dots, v_r) \in \mathbf{C}^{1\times r}$ is an ordinary row vector and $f = (f_1, \dots, f_r)^{\mathrm{T}} \in \mathcal{E}'(\mathbf{R}^d, \mathbf{C}^r)$ is a compactly supported vector-valued distribution, then $vf = \sum_{i=1}^r v_i f_i \in \mathcal{E}'(\mathbf{R}^d)$ is the scalar-valued distribution defined by

$$\langle \varphi, vf\rangle \;=\; \sum_{i=1}^r \langle \varphi, v_i f_i\rangle, \qquad \varphi \in C^\infty(\mathbf{R}^d).$$

In particular,

$$v\hat{f}(0) \;=\; \sum_{i=1}^r v_i\, \hat{f}_i(0) \;=\; \sum_{i=1}^r v_i\, \langle 1, f_i\rangle \;=\; \Big\langle 1, \sum_{i=1}^r v_i f_i\Big\rangle \;=\; \langle 1, vf\rangle.$$

Given a compactly supported vector-valued distribution $f \in \mathcal{E}'(\mathbf{R}^d, \mathbf{C}^r)$ and given arbitrary row vectors $w_k \in \mathbf{C}^{1\times r}$, the series $\sum_{k\in\Gamma} w_k\, f(x+k)$ defines a distribution in $\mathcal{D}'(\mathbf{R}^d)$ by the formula

$$\Big\langle \varphi(x), \sum_{k\in\Gamma} w_k\, f(x+k)\Big\rangle \;=\; \sum_{k\in\Gamma} w_k\, \langle \varphi(x-k), f(x)\rangle, \qquad \varphi \in C_c^\infty(\mathbf{R}^d)\,. \tag{2.2}$$

Since both f and φ are compactly supported, the right-hand side of (2.2) contains only finitely many nonzero terms. Letting $W = (w_k)_{k\in\Gamma} \in (\mathbf{C}^{1\times r})^{1\times\Gamma}$ denote the infinite row vector with block entries w_k, and writing formally the infinite column vector of distributions

$$F(x) \;=\; [f(x+k)]_{k\in\Gamma}\,,$$

we define WF to be the distribution in $\mathcal{D}'(\mathbf{R}^d)$ given by

$$WF(x) \;=\; \sum_{k\in\Gamma} w_k\, f(x+k)\,.$$

Note that if the w_k grow in size at most polynomially, then $WF \in \mathcal{S}'(\mathbf{R}^d)$.

As a consequence of the preceding remarks, if $f \in \mathcal{E}'(\mathbf{R}^d, \mathbf{C}^r)$ then we can define the shift-invariant space $S(f)$ generated by f to be

$$S(f) \;=\; \Big\{\sum_{k\in\Gamma} w_k\, f(x+k) : w_k \in \mathbf{C}^{1\times r}\Big\} \;=\; \Big\{WF : W \in (\mathbf{C}^{1\times r})^{1\times\Gamma}\Big\}\,.$$

Note that $S(f)$ is a subspace of $\mathcal{D}'(\mathbf{R}^d)$. Since all polynomials are distributions, the definition of accuracy extends to distributions, i.e., we say that $f \in \mathcal{E}'(\mathbf{R}^d, \mathbf{C}^r)$ has accuracy p if each multivariate polynomial q with $\deg(q) < p$ lies in $S(f)$.

Given $f \in \mathcal{E}'(\mathbf{R}^d, \mathbf{C}^r)$, we set

$$K(f) \;=\; \Big\{W = (w_k)_{k\in\Gamma} \in (\mathbf{C}^{1\times r})^{1\times\Gamma} : WF = 0\Big\}\,.$$

We say that translates of f along Γ are *independent* if $K(f) = \{0\}$.

Finally, we say that a compactly supported vector-valued distribution $f \in \mathcal{E}'(\mathbf{R}^d, \mathbf{C}^r)$ is *refinable* if it is a solution of the refinement equation (1.1) in the sense of distributions, i.e., if

$$\langle \varphi, f\rangle \;=\; \sum_{k\in\Lambda} c_k\, \langle \varphi(x), f(Ax-k)\rangle \;=\; \frac{1}{m}\sum_{k\in\Lambda} c_k\, \Big\langle \varphi\big(A^{-1}(x+k)\big),\, f(x)\Big\rangle$$

for all $\varphi \in C^\infty(\mathbf{R}^d, \mathbf{C}^r)$. If we set $c_k = 0$ for $k \notin \Lambda$ and define L to be the $\Gamma \times \Gamma$ matrix

$$L \;=\; [c_{Ai-j}]_{i,j\in\Gamma}\,,$$

then we can recast the refinement equation in the following form:

$$f(x) \;=\; \sum_{k\in\Lambda} c_k\, f(Ax-k) \qquad \Longleftrightarrow \qquad F(x) \;=\; L\,F(Ax)\,.$$

Additionally, if we define $B = (A^{-1})^{\mathrm{T}}$ and let $M(\omega) = \frac{1}{m}\sum_{k\in\Lambda} c_k\, e^{-2\pi i k\cdot\omega}$ be the matrix-valued symbol of the refinement equation, then the refinement equation can also be recast into the following form by applying the Fourier transform:

$$f(x) \;=\; \sum_{k\in\Lambda} c_k\, f(Ax-k) \qquad \Longleftrightarrow \qquad \hat f(\omega) \;=\; M(B\omega)\,\hat f(B\omega).$$

In this case, if $\Delta = M(0) = \frac{1}{m}\sum_{k\in\Lambda} c_k$, then $\hat f(0) = M(0)\,\hat f(0) = \Delta\,\hat f(0)$, so $\hat f(0)$ is a right 1-eigenvector of Δ if $\hat f(0) \neq 0$.

2.3. Translation and Dilation of Multidimensional Polynomials.

We shall often deal with matrix-valued functions $u = [u_{j,k}]_{j\in J,k\in K}\colon \mathbf{R}^d \to \mathbf{C}^{J\times K}$ each of whose entries $u_{j,k}\colon \mathbf{R}^d \to \mathbf{C}$ is a polynomial. In this case, we refer to u as a *matrix of polynomials*. The *degree* of u is $\deg(u) = \max\{\deg(u_{j,k})\}_{j\in J,k\in K}$.

Recall that the number of monomials x^α of degree s is $d_s = \binom{s+d-1}{d-1}$. For a given degree $s \geq 0$, we collect the monomials of this degree together to form the vector of monomials $X_{[s]}\colon \mathbf{R}^d \to \mathbf{C}^{d_s}$ defined by

$$X_{[s]}(x) \;=\; [\,x^\alpha\,]_{|\alpha|=s}\,, \qquad x \in \mathbf{R}^d\,.$$

For each integer $0 \leq t \leq s$, we define a matrix of polynomials $Q_{[s,t]}\colon \mathbf{R}^d \to \mathbf{C}^{d_s\times d_t}$ by

$$Q_{[s,t]}(y) \;=\; (-1)^{s-t}\left[\binom{\alpha}{\beta} y^{\alpha-\beta}\right]_{|\alpha|=s,|\beta|=t}.$$

Then, by [4, Section 2.4], translation of $X_{[s]}(x)$ obeys the rule

$$X_{[s]}(x-y) \;=\; \sum_{t=0}^{s} Q_{[s,t]}(y)\, X_{[t]}(x)\,.$$

Given any $d \times d$ matrix $Z = [z_{i,j}]_{i,j=1,\ldots,d}$ and given $s \geq 0$, we let $Z_{[s]} = [z^s_{\alpha,\beta}]_{|\alpha|=s,|\beta|=s}$ be the $d_s \times d_s$ matrix whose scalar entries $z^s_{\alpha,\beta}$ are defined by the equation

$$\sum_{|\beta|=s} z^s_{\alpha,\beta}\, x^\beta \;=\; (Zx)^\alpha \;=\; \prod_{i=1}^{d} (z_{i,1}x_1 + \cdots + z_{i,d}x_d)^{\alpha_i}\,.$$

By [4, Section 2.4], dilation of $X_{[s]}(x)$ by Z obeys the rule

$$X_{[s]}(Zx) \;=\; Z_{[s]}\, X_{[s]}(x)\,.$$

The matrix $Z_{[s]}$ has a number of interesting properties (see [4, Section 4.1] for a discussion). In particular, if $\lambda = (\lambda_1, \ldots, \lambda_d)^{\mathrm{T}}$ is the vector consisting of all eigenvalues of Z, then $[\lambda^\alpha]_{|\alpha|=s}$ is the vector consisting of all eigenvalues of $Z_{[s]}$.

2.4. Some Special Matrices and Polynomial Functions.

Given a collection

$$\{v_\alpha = (v_{\alpha,1}, \dots, v_{\alpha,r}) \in \mathbf{C}^{1\times r} : 0 \le |\alpha| < p\}$$

of row vectors of length r, we shall associate a number of special matrices and functions.

First, we group the v_α by degree to form $d_s \times 1$ column vectors $v_{[s]} \in (\mathbf{C}^{1\times r})^{d_s\times 1}$ with block entries that are the $1 \times r$ row vectors v_α, i.e.,

$$v_{[s]} \;=\; [v_\alpha]_{|\alpha|=s} \;=\; \begin{bmatrix} v_{\alpha_1,1} & \cdots & v_{\alpha_1,r} \\ \vdots & \ddots & \vdots \\ v_{\alpha_{d_s},1} & \cdots & v_{\alpha_{d_s},r} \end{bmatrix}.$$

Note that $v_{[0]} = [v_0] = v_0$.

Next, for each α, we define a row vector of polynomials $y_\alpha\colon \mathbf{R}^d \to \mathbf{C}^{1\times r}$ by

$$y_\alpha(x) \;=\; \sum_{0\le\beta\le\alpha} (-1)^{|\alpha|-|\beta|} \binom{\alpha}{\beta} v_\beta\, x^{\alpha-\beta}\,.$$

If we write $y_\alpha(x) = (y_{\alpha,1}(x), \dots, y_{\alpha,r}(x))$, then the coefficients of the polynomial $y_{\alpha,i}$ are determined by the scalars $v_{\beta,i}$ for those β with $0 \le \beta \le \alpha$. Further, $\deg(y_\alpha) \le |\alpha|$, and $\deg(y_\alpha) = |\alpha|$ if and only if $v_0 \ne 0$. In particular, y_0 is the constant polynomial $y_0(x) \equiv v_0$.

As with the vectors v_α, we collect the vectors of polynomials y_α by degree and arrange them as block entries in a column vector to form the matrix of polynomials $y_{[s]}\colon \mathbf{R}^d \to (\mathbf{C}^{1\times r})^{d_s\times 1}$ defined by

$$y_{[s]}(x) \;=\; [\,y_\alpha(x)\,]_{|\alpha|=s} \;=\; \left[\sum_{t=0}^{s}\sum_{|\beta|=t} (-1)^{s-t}\binom{\alpha}{\beta} x^{\alpha-\beta}\, v_\beta\right]_{|\alpha|=s} = \sum_{t=0}^{s} Q_{[s,t]}(x)\, v_{[t]}.$$

We have $\deg(y_{[s]}) \le s$, and $\deg(y_{[s]}) = s$ if and only if $v_0 \ne 0$.

Finally, for each x we collect the blocks $y_{[s]}(x+k)$ into an infinite row vector to form a function $Y_{[s]}\colon \mathbf{R}^d \to \big((\mathbf{C}^{1\times r})^{d_s\times 1}\big)^{1\times\Gamma}$. Specifically,

$$Y_{[s]}(x) \;=\; \big(y_{[s]}(x+k)\big)_{k\in\Gamma}\,.$$

We adopt the convention that

$$Y_{[s]} \;=\; Y_{[s]}(0) \;=\; \big(y_{[s]}(k)\big)_{k\in\Gamma}\,.$$

Thus $Y_{[s]}$ is the row vector of evaluations of the matrix of polynomials $y_{[s]}$ at lattice points.

The following fact on the behavior of the matrix of polynomials $y_{[s]}$ under translation will be useful.

Lemma 1.

Given a collection $\{v_\alpha \in \mathbf{C}^{1\times r} : 0 \le |\alpha| < p\}$ of row vectors, define the matrix of polynomials $y_{[s]}(x) = \sum_{t=0}^{s} Q_{[s,t]}(x)\, v_{[t]}$ as above. Then

$$y_{[s]}(x+y) \;=\; \sum_{t=0}^{s} Q_{[s,t]}(y)\, y_{[t]}(x) \qquad \textit{and} \qquad Y_{[s]}(x+y) \;=\; \sum_{t=0}^{s} Q_{[s,t]}(y)\, Y_{[t]}(x)\,.$$

3. Results for Arbitrary Distributions

We first show that the coefficients used to reproduce any given polynomial from translates of f must have a considerable amount of structure; specifically, the kth coefficient is completely determined from the 0th coefficient modulo choices of coefficients lying in $K(f)$. Moreover, we show that with the additional assumption of independence of translates, the coefficients used to reproduce polynomials must have a specific polynomial structure.

Because we consider all monomials of a given degree simultaneously, we introduce a grouping notation for arbitrary coefficients similar to the grouping notation introduced in Section 2.4 for coefficients with polynomial structure. Specifically, given row vectors $w_{\alpha,k} \in C^{1\times r}$ for $0 \le |\alpha| < p$ and $k \in \Gamma$, we first group by degree to form the column vectors

$$w_{[s]}(k) \;=\; [w_{\alpha,k}]_{|\alpha|=s} \;\in\; (C^{1\times r})^{d_s\times 1}\,, \qquad k\in\Gamma. \tag{3.1}$$

Then we collect these vectors to form the row vectors

$$W_{[s]}(\ell) \;=\; \big(w_{[s]}(\ell+k)\big)_{k\in\Gamma} \;\in\; \big((\mathbf{C}^{1\times r})^{d_s\times 1}\big)^{1\times\Gamma}\,, \qquad \ell\in\Gamma. \tag{3.2}$$

We adopt the convention that

$$W_{[s]} \;=\; W_{[s]}(0) \;=\; \big(w_{[s]}(k)\big)_{k\in\Gamma}\,. \tag{3.3}$$

Although a slight abuse of notation, we define

$$K(f)^{d_s} \;=\; \left\{W_{[s]} = \big([w_{\alpha,k}]_{|\alpha|=s}\big)_{k\in\Gamma} \in \big((\mathbf{C}^{1\times r})^{d_s\times 1}\big)^{1\times\Gamma} : W_\alpha = (w_{\alpha,k})_{k\in\Gamma} \in K(f) \text{ for all } |\alpha| = s\right\}.$$

In particular, if $W_{[s]} \in K(f)^{d_s}$, then $W_{[s]}F = [0]_{|\alpha|=s}$, the zero vector in $\mathcal{E}'(\mathbf{R}^d)^{d_s}$.

Using the above notation, we can formulate the main result of this section as follows.

Theorem 1.

Assume that $f \in \mathcal{E}'(\mathbf{R}^d, \mathbf{C}^r)$ has accuracy p. Let $w_{\alpha,k} \in \mathbf{C}^{1\times r}$ be any vectors such that

$$x^\alpha \;=\; \sum_{k\in\Gamma} w_{\alpha,k}\, f(x+k) \;\; a.e., \qquad 0 \le |\alpha| < p\,. \tag{3.4}$$

Then

$$X_{[s]}(x) \;=\; \sum_{k\in\Gamma} w_{[s]}(k)\, f(x+k) \;=\; W_{[s]}\, F(x), \qquad 0 \le s < p\,,$$

and

$$W_{[s]}(\ell) - \sum_{t=0}^{s} Q_{[s,t]}(\ell)\, W_{[t]} \;\in\; K(f)^{d_s}, \qquad 0 \le s < p, \quad \ell\in\Gamma\,. \tag{3.5}$$

Furthermore, if translates of $f \in \mathcal{E}'(\mathbf{R}^d, \mathbf{C}^r)$ along Γ are independent and f has accuracy p, then there exists a collection $\{v_\alpha \in \mathbf{C}^{1\times r} : 0 \le |\alpha| < p\}$ of row vectors such that

(i) *$v_0 \ne 0$, and*

(ii) $X_{[s]}(x) \;=\; \sum_{k\in\Gamma} y_{[s]}(k)\, f(x+k) \;=\; Y_{[s]}\, F(x)$ *for* $0 \le s < p$,

where $Y_{[s]} = \big(y_{[s]}(k)\big)_{k\in\Gamma}$ *is the row vector of evaluations at lattice points of the matrix of polynomials* $y_{[s]}(x) = \sum_{t=0}^{s} Q_{[s,t]}(x)\, v_{[t]}$. *Consequently, if* q *is any polynomial with* $\deg(q) < p$, *then there exists a unique row vector of polynomials* $u_q\colon \mathbf{R}^d \to \mathbf{C}^{1\times r}$, *with* $\deg(u_q) = \deg(q)$, *such that* $q(x) = \sum_{k\in\Gamma} u_q(k)\, f(x+k)$.

Proof. Assume that f has accuracy p and that (3.4) holds. Then given $0 \le s < p$, we have

$$X_{[s]}(x) \;=\; [\,x^\alpha\,]_{|\alpha|=s} \;=\; \left[\sum_{k\in\Gamma} w_{\alpha,k}\, f(x+k)\right]_{|\alpha|=s} \;=\; \sum_{k\in\Gamma} w_{[s]}(k)\, f(x+k) \;=\; W_{[s]}\, F(x)\,.$$

Therefore, for each $\ell \in \Gamma$,

$$W_{[s]}(\ell)\, F(x) \;=\; W_{[s]}\, F(x-\ell) \;=\; X_{[s]}(x-\ell) \;=\; \sum_{t=0}^{s} Q_{[s,t]}(\ell)\, X_{[t]}(x) \;=\; \left(\sum_{t=0}^{s} Q_{[s,t]}(\ell)\, W_{[t]}\right) F(x)\,.$$

This implies that (3.5) holds.

Now assume that translates of f are independent. Then (3.5) reduces to the statement that $W_{[s]}(\ell) = \sum_{t=0}^{s} Q_{[s,t]}(\ell)\, W_{[t]}$. Therefore,

$$\big(w_{[s]}(k+\ell)\big)_{k\in\Gamma} \;=\; W_{[s]}(\ell) \;=\; \sum_{t=0}^{s} Q_{[s,t]}(\ell)\, W_{[t]} \;=\; \left(\sum_{t=0}^{s} Q_{[s,t]}(\ell)\, w_{[t]}(k)\right)_{k\in\Gamma}.$$

In particular, taking $k = 0$ we obtain

$$w_{[s]}(\ell) \;=\; \sum_{t=0}^{s} Q_{[s,t]}(\ell)\, w_{[t]}(0)\,.$$

If we define $v_\alpha = w_{\alpha,0}$, then, following the notation of Section 2.4, we have

$$v_{[t]} \;=\; [v_\alpha]_{|\alpha|=t} \;=\; [w_{\alpha,0}]_{|\alpha|=t} \;=\; w_{[t]}(0)\,,$$

and therefore

$$w_{[s]}(\ell) \;=\; \sum_{t=0}^{s} Q_{[s,t]}(\ell)\, w_{[t]}(0) \;=\; \sum_{t=0}^{s} Q_{[s,t]}(x)\, v_{[t]} \;=\; y_{[s]}(\ell)\,.$$

Hence $W_{[s]} = Y_{[s]}$ for each s, and therefore $X_{[s]}(x) = W_{[s]}\, F(x) = Y_{[s]}\, F(x)$. Further, when $s = 0$ we have $y_{[0]}(k) = v_0$ for every k, so

$$1 \;=\; x^0 \;=\; X_{[0]}(x) \;=\; \sum_{k\in\Gamma} y_{[0]}(k)\, f(x+k) \;=\; v_0 \sum_{k\in\Gamma} f(x+k)\,.$$

Therefore $v_0 \neq 0$.

Finally, suppose that $q(x) = \sum_{0\le|\alpha|\le s} t_\alpha\, x^\alpha$ is any polynomial with $\deg(q) = s < p$. Since $y_{[s]} = [y_\alpha]_{|\alpha|=s}$, we have that $x^\alpha = \sum_{k\in\Gamma} y_\alpha(k)\, f(x+k)$. Therefore,

$$q(x) \;=\; \sum_{k\in\Gamma} \left(\sum_{0\le|\alpha|\le s} t_\alpha\, y_\alpha(k)\right) f(x+k) \;=\; \sum_{k\in\Gamma} u_q(k)\, f(x+k)\,. \tag{3.6}$$

Since translates of f are independent, the coefficients $u_q(k)$ in (3.6) are unique. However, $u_q(k)$ is the evaluation at lattice points of the row vector of polynomials $u_q(x) = \sum_{0\le|\alpha|\le s} t_\alpha\, y_\alpha(x)$. Since such evaluations uniquely determine a polynomial, we conclude that u_q is unique. It remains only to show

that $\deg(u_q) = s$. Since $y_\alpha(x) = \sum_{0 \le \beta \le \alpha} (-1)^{|\alpha|-|\beta|} \binom{\alpha}{\beta} x^{\alpha-\beta} v_\beta$ and $v_0 \neq 0$, we have $\deg(y_\alpha) = |\alpha|$. Moreover, y_α contains only a single term of degree $|\alpha|$, namely, $(-1)^{|\alpha|} x^\alpha v_0$. Therefore, $\deg(u_q) = \max\{|\alpha| : t_\alpha \neq 0\} = s$. $\square$

The following result states that, regardless of whether f has accuracy p or not, if any monomial x^α can be reproduced from lattice translates of f using coefficients that are themselves polynomials evaluated at lattice points, then for each $0 \le \beta \le \alpha$ the monomial x^β can also be reproduced from translates of f. Moreover, the coefficients used to obtain x^β are the evaluations at lattice points of a constant times the $(\alpha - \beta)$ derivative of the coefficients used to obtain x^α.

Theorem 2.

Assume that $f \in \mathcal{E}'(\mathbf{R}^d, \mathbf{C}^r)$, and let α be any multi-index. If $u: \mathbf{R}^d \to \mathbf{C}^{1\times r}$ is a row vector of polynomials such that

$$x^\alpha \;=\; \sum_{k\in\Gamma} u(k)\, f(x+k)\,,$$

then for each $0 \le \beta \le \alpha$,

$$x^\beta \;=\; C_\beta \sum_{k\in\Gamma} (D^{\alpha-\beta}u)(k)\, f(x+k)\,,$$

where

$$D^\gamma u \;=\; \left(\frac{\partial^{|\gamma|}}{\partial x^\gamma} u_1, \ldots, \frac{\partial^{|\gamma|}}{\partial x^\gamma} u_r\right) \qquad \textit{and} \qquad C_\gamma \;=\; (-1)^{|\alpha-\gamma|}\,\frac{\gamma!}{\alpha!} \;=\; (-1)^{|\alpha-\gamma|}\,\frac{\gamma_1!}{\alpha_1!}\cdots\frac{\gamma_d!}{\alpha_d!}\,.$$

Proof. Note that since $u(k)$ has polynomial growth, the series $\sum_{k\in\Gamma} u(k)\, f(x+k)$ defines a tempered distribution. Fix any $\varphi \in \mathcal{S}(\mathbf{R}^d)$. Then

$$\begin{aligned}
\langle \varphi(x), (x+\ell)^\alpha \rangle \;&=\; \langle \varphi(x-\ell), x^\alpha \rangle \\
&=\; \Big\langle \varphi(x-\ell), \sum_{k\in\Gamma} u(k)\, f(x+k) \Big\rangle \\
&=\; \Big\langle \varphi(x), \sum_{k\in\Gamma} u(k)\, f(x+\ell+k) \Big\rangle \\
&=\; \Big\langle \varphi(x), \sum_{k\in\Gamma} u(k-\ell)\, f(x+k) \Big\rangle\,. \qquad (3.7)
\end{aligned}$$

For each $y \in \mathbf{R}^d$, define g_y, $h_y \in \mathcal{S}'(\mathbf{R}^d)$ by

$$g_y(x) \;=\; (x+y)^\alpha \qquad \text{and} \qquad h_y(x) \;=\; \sum_{k\in\Gamma} u(k-y)\, f(x+k)\,.$$

Given $\varphi \in \mathcal{S}(\mathbf{R}^d)$, the quantities

$$\begin{aligned}
P(y) \;&=\; \langle \varphi, g_y \rangle \;=\; \langle \varphi(x), (x+y)^\alpha \rangle\,, \\
Q(y) \;&=\; \langle \varphi, h_y \rangle \;=\; \Big\langle \varphi(x), \sum_{k\in\Gamma} u(k-y)\, f(x+k) \Big\rangle
\end{aligned}$$

are both polynomials in the unknown y. From (3.7), we have $P(\ell) = Q(\ell)$ for every lattice point $\ell \in \Gamma$, and therefore $P(y) = Q(y)$ for every $y \in \mathbf{R}^d$.

Let e_j be the multi-index of degree 1 with a 1 in the jth coordinate and 0's elsewhere. Then for every $y \in \mathbf{R}^d$,

$$\langle \varphi(x), \alpha_j \, (x+y)^{\alpha - e_j} \rangle \;=\; \frac{\partial P}{\partial y_j}(y) \;=\; \frac{\partial Q}{\partial y_j}(y) \;=\; \sum_{k \in \Gamma} (-1) \, (D^{e_j} u)(k-y) \, \langle \varphi(x), f(x+k) \rangle \, .$$

Evaluating at $y = 0$, we have

$$\langle \varphi(x), \alpha_j \, x^{\alpha - e_j} \rangle \;=\; \sum_{k \in \Gamma} (-1) \, (D^{e_j} u)(k) \, \langle \varphi(x), f(x+k) \rangle \, .$$

Hence, distributionally,

$$\alpha_j \, x^{\alpha - e_j} \;=\; \sum_{k \in \Gamma} (-1) \, (D^{e_j} u)(k) \, f(x+k) \, .$$

The general result then follows by iteration. $\square$

4. Results for Refinable Distributions

For the remainder of this paper we shall concentrate on compactly supported distributions which satisfy the refinement equation (1.1). We assume throughout that A is a dilation matrix with respect to a lattice Γ in $\mathbf{R}^d$, and that the coefficient mask $c = \{c_k\}_{k \in \Lambda}$ is a finite collection of $r \times r$ matrices.

We first note a mild condition on the coefficient mask which ensures that a compactly supported distributional solution of the refinement equation exists.

Proposition 1.

Define $M(\omega) = \frac{1}{m} \sum_{k \in \Lambda} c_k \, e^{-2\pi i k \cdot \omega}$. If the matrix $\Delta = M(0) = \frac{1}{m} \sum_{k \in \Lambda} c_k$ has eigenvalues $\lambda_1 = \cdots = \lambda_s = 1$, $|\lambda_{s+1}|, \ldots, |\lambda_r| < 1$, with the eigenvalue 1 nondegenerate, then the following statements hold.

- **(a)** *The infinite matrix product $P(\omega) = \prod_{j=1}^{\infty} M(B^j \omega)$, where $B = (A^{-1})^{\mathrm{T}}$, converges uniformly on compact sets to a continuous function with at most polynomial growth at infinity.*
- **(b)** *If v is any right 1-eigenvector for Δ, then $\hat{f}(\omega) = P(\omega) v$ defines a distribution $f \in \mathcal{E}'(\mathbf{R}^d, \mathbf{C}^r)$ that satisfies the refinement equation (1.1), and $\hat{f}(0) = v$.*
- **(c)** *If $f \in \mathcal{E}'(\mathbf{R}^d, \mathbf{C}^r)$ is any distributional solution to the refinement equation (1.1), then $\hat{f}(\omega) = P(\omega) \, \hat{f}(0)$. In particular, if f is nontrivial then $\hat{f}(0) \neq 0$ and $\hat{f}(0)$ is a right 1-eigenvector for Δ.*

Proof. (a) It was shown in [4, Theorem A.3] that the infinite matrix product $P(\omega)$ converges uniformly on compact sets to a continuous function. Since A is expansive, the matrix B is contractive. Hence, there exists a vector norm $|\cdot|$ on $\mathbf{R}^d$ such that the corresponding matrix norm of B satisfies $|B| < 1$. Let $\|\cdot\|$ be any norm on $\mathbf{C}^r$, and define $K = \sup_{|\omega| \leq 1} \|P(\omega)\|$ and $R = \sup_{\omega \in \mathbf{R}^d} \|M(\omega)\|$. Set $\theta = 1/|B|$. Then

$$\sup_{|\omega| \leq \theta^n} \|P(\omega)\| \;\leq\; \sup_{|\omega| \leq \theta^n} \|P(B^n \omega)\| \, \|M(B^n \omega)\| \cdots \|M(B\omega)\| \;\leq\; K R^n \, .$$

Hence $\|P(\omega)\| \leq K R \, (1 + |\omega|)^{\log_\theta R}$, so P has at most polynomial growth at infinity.

(b) Let v be any right 1-eigenvector for Δ. By part (a), $\hat{f}(\omega) = P(\omega) v$ then defines a tempered distribution $f \in \mathcal{S}'(\mathbf{R}^d, \mathbf{C}^r)$. This distribution clearly satisfies $\hat{f}(\omega) = M(B\omega) \, \hat{f}(B\omega)$, so f

is refinable. It therefore remains only to show that f is compactly supported. For each n, define $\mu_n \in \mathcal{S}'(\mathbf{R}^d, \mathbf{C}^r)$ by $\hat{\mu}_n(\omega) = \big(\prod_{j=1}^n M(B^j\omega)\big)v$. Since $M(B^j\omega) = \frac{1}{m}\sum_{k\in\Lambda} c_k\, e^{-2\pi i k\cdot B^j\omega} = \frac{1}{m}\sum_{k\in\Lambda} c_k\, e^{-2\pi i A^{-j}k\cdot\omega}$, the entries of $\hat{\mu}_n(\omega)$ are finite linear combinations of the exponentials $e^{-2\pi i\ell\cdot\omega}$ with ℓ restricted to the discrete set $\Lambda_n = \sum_{j=1}^n A^{-j}(\Lambda)$. Since A^{-1} is contractive, there exists a compact set $\Omega \subset \mathbf{R}^d$ such that $\Lambda_n \subset \Omega$ for each n. Thus the entries of μ_n are finite linear combinations of point masses δ_ℓ with $\ell \in \Omega$. Hence $\mathrm{supp}(\mu_n) \subset \Omega$. However, $\hat{\mu}_n(\omega) \to P(\omega)v = \hat{f}(\omega)$ uniformly on compact sets, so $\mu_n \to f$ weakly. Hence $\mathrm{supp}(f) \subset \Omega$ as well.

(c) Note that $\hat{f}(\omega) = \big(\prod_{j=1}^n M(B^j\omega)\big)\,\hat{f}(B^j\omega)$ for each n, and that $\hat{f}(B^j\omega) \to \hat{f}(0)$ since B is contractive and $\hat{f}$ is a continuous function. □

In order to prove a result giving necessary and sufficient conditions for a refinable distribution to have accuracy p, we require the following ergodic-type lemma for tempered distributions.

Lemma 2.
Let $\mu \in \mathcal{S}'(\mathbf{R}^d)$. If there exists a $\lambda \in \mathbf{C}$ such that

$$\mu(Ax) \;=\; \lambda\mu(x) \qquad \textit{and} \qquad \mu(x-\ell) \;=\; \mu(x), \quad \ell \in \Gamma\,,$$

then μ is a constant. Moreover, if $\lambda \neq 1$ then $\mu = 0$.

Proof. For each integer $j \in \mathbf{Z}$ and each lattice point $k \in \Gamma$, define $\mu_{j,k}(x) = \mu(A^j x - k)$. Then, by hypothesis, $\mu_{j,k}(x) = \lambda^j\mu(x)$. Let $B = (A^{-1})^{\mathrm{T}}$. With $m = |\det(A)|$, we then have distributionally that

$$\lambda^j\hat{\mu}(\omega) \;=\; \hat{\mu}_{j,k}(\omega) \;=\; m^{-j}\, e^{-2\pi i k\cdot B^j\omega}\,\hat{\mu}(B^j\omega).$$

Choose now any $\varphi \in \mathcal{S}(\mathbf{R}^d)$ such that $\hat{\varphi}$ is compactly supported. Then $\nu_j(\omega) = \hat{\varphi}((A^{\mathrm{T}})^j\omega)\,\hat{\mu}(\omega)$ is a compactly supported tempered distribution. Moreover, $\mathrm{supp}(\nu_j) \subset \mathrm{supp}(\hat{\varphi}((A^{\mathrm{T}})^j\omega)$. Since A is expansive, by choosing j large enough we will have $\mathrm{supp}(\nu_j)$ contained in the rectangular parallelepiped $R = \{x_1u_1 + \cdots + x_du_d : -1/2 \le x_i < 1/2\}$, which is a fundamental domain for Γ. In this case, ν_j is completely determined by the values $\hat{\nu}_j(k)$ for $k \in \Gamma$. Since ν_j is compactly supported, we can compute these values as follows.

$$\begin{aligned}
\hat{\nu}_j(k) \;&=\; \big\langle e^{-2\pi i k\cdot\omega},\, \nu_j(\omega)\big\rangle \\
&=\; \Big\langle e^{-2\pi i k\cdot\omega},\, \hat{\varphi}((A^{\mathrm{T}})^j\omega)\,\hat{\mu}(\omega)\Big\rangle \\
&=\; \Big\langle \hat{\varphi}((A^{\mathrm{T}})^j\omega),\, e^{-2\pi i k\cdot\omega}\,\hat{\mu}(\omega)\Big\rangle \\
&=\; \Big\langle \hat{\varphi}(\omega),\, m^{-j}\, e^{-2\pi i k\cdot(A^{\mathrm{T}})^{-j}\omega}\,\hat{\mu}((A^{\mathrm{T}})^{-j}\omega)\Big\rangle \\
&=\; \langle\hat{\varphi}, \hat{\mu}_{j,k}\rangle \\
&=\; \langle\hat{\varphi}, \lambda^j\hat{\mu}\rangle \;=\; C.
\end{aligned}$$

Thus $\hat{\nu}_j(k)$ is a constant independent of k, so $\nu_j = C\,\delta$. Since $\nu_j(\omega) = \hat{\varphi}((A^{\mathrm{T}})^j\omega)\,\hat{\mu}(\omega)$, it follows that $\mathrm{supp}(\hat{\mu}) = \{0\}$. As a consequence, μ must be a polynomial [16, p. 194]. However, μ is Γ-periodic by hypothesis, so this implies that μ must be a constant. Finally, since $\mu(Ax) = \lambda\,\mu(x)$, this constant must be zero if $\lambda \neq 1$. □

The following result gives necessary and sufficient conditions for a refinable distribution to have accuracy p.

Theorem 3.

Assume that $f \in \mathcal{E}'(\mathbf{R}^d, \mathbf{C}^r)$ is a distributional solution of the refinement equation (1.1)*. Then the following statements are equivalent.*

(I) *f has accuracy p.*

(II) *There exists a collection of row vectors $\{w_{\alpha,k} \in \mathbf{C}^{1\times r} : k \in \Gamma, 0 \le |\alpha| < p\}$ such that if $w_{[s]}(k)$, $W_{[s]}(\ell)$, and $W_{[s]}$ are defined as in* (3.1) *through* (3.3)*, then:*

(i) *$W_{[0]}\, F(x) = \sum_k w_{0,k}\, f(x) \neq 0$,*

(ii) *$W_{[s]}(\ell) - \sum_{t=0}^{s} Q_{[s,t]}(\ell)\, W_{[t]}(0) \in K(f)^{d_s}$ for $0 \le s < p$ and $\ell \in \Gamma$, and*

(iii) *$W_{[s]} - A_{[s]}\, W_{[s]}\, L \in K(f)^{d_s}$ for $0 \le s < p$.*

Moreover, in case these hold, we have $W_{[0]}\, F(x) = C \neq 0$, and $W_{[s]}\, F(x) = C\, X_{[s]}(x)$ for $0 \le s < p$.

Proof. (I) $\Rightarrow$ (II). Assume that f has accuracy p. Then there exist vectors $w_{\alpha,k} \in \mathbf{C}^{1\times r}$ such that

$$x^\alpha \;=\; \sum_{k\in\Gamma} w_{\alpha,k}\, f(x+k) \text{ a.e.}, \qquad 0 \le |\alpha| < p.$$

It therefore follows from Theorem 1 that

$$X_{[s]}(x) \;=\; \sum_{k\in\Gamma} w_{[s]}(k)\, f(x+k) \;=\; W_{[s]}\, F(x), \qquad 0 \le s < p\,. \tag{4.1}$$

In particular, $1 = X_{[0]}(x) = \sum_k w_{[0]}(k)\, f(x+k) = \sum_k w_{0,k}\, f(x+k)$, so (i) holds. It also follows from Theorem 1 that property (ii) holds.

Combining (4.1) with the refinement equation $F(x) = L\, F(Ax)$ and with the definition of $A_{[s]}$, we have

$$W_{[s]}\, F(Ax) \;=\; X_{[s]}(Ax) \;=\; A_{[s]}\, X_{[s]}(x) \;=\; A_{[s]}\, W_{[s]}\, F(x) \;=\; A_{[s]}\, W_{[s]}\, L\, F(Ax)\,.$$

Consequently, $W_{[s]} - A_{[s]}\, W_{[s]}\, L \in K(f)^{d_s}$ for $0 \le s < p$, which proves (iii).

(II) $\Rightarrow$ (I) Assume that statement (II) holds. For each $0 \le s < p$, define a vector-valued distribution $G_{[s]} \in \mathcal{S}'(\mathbf{R}^d, \mathbf{C}^{d_s})$ by

$$G_{[s]}(x) \;=\; \sum_{k\in\Gamma} w_{[s]}(k)\, f(x+k) \;=\; W_{[s]}\, F(x)\,.$$

Using the assumption $W_{[s]} - A_{[s]}\, W_{[s]}\, L \in K(f)^{d_s}$ and the refinement equation $L\, F(Ax) = F(x)$, we have

$$G_{[s]}(Ax) \;=\; W_{[s]}\, F(Ax) \;=\; A_{[s]}\, W_{[s]}\, L\, F(Ax) \;=\; A_{[s]}\, W_{[s]}\, F(x) \;=\; A_{[s]}\, G_{[s]}(x)\,. \tag{4.2}$$

We will show by induction that there is a nonzero constant C independent of s such that $G_{[s]}(x) = C\, X_{[s]}(x)$ for $0 \le s < p$.

Consider the case $s = 0$. We have $d_0 = 1$, so $G_{[0]} \in \mathcal{S}'(\mathbf{R}^d)$. Since $A_{[0]} = 1$, we have by (4.2) that $G_{[0]}(Ax) = G_{[0]}(x)$. By hypothesis (ii), we have $W_{[0]}(\ell) - W_{[0]} \in K(f)$ for every $\ell \in \Gamma$, so

$$G_{[0]}(x-\ell) \;=\; W_{[0]}\, F(x-\ell) \;=\; W_{[0]}(\ell)\, F(x) \;=\; W_{[0]}\, F(x) \;=\; G_{[0]}(x), \qquad \ell \in \Gamma\,.$$

Lemma 2 therefore implies that $G_{[0]}$ is a constant C, and by hypothesis (i), this constant is not zero.

Suppose now, inductively, that $G_{[t]}(x) = C\,X_{[t]}(x)$ for $0 \le t < s$. Then,

$$\begin{aligned}
G_{[s]}(x-\ell) &= W_{[s]}\,F(x-\ell)\\
&= W_{[s]}(\ell)\,F(x)\\
&= \sum_{t=0}^{s} Q_{[s,t]}(\ell)\,W_{[t]}\,F(x) && \text{by hypothesis (ii)}\\
&= \sum_{t=0}^{s} Q_{[s,t]}(\ell)\,G_{[t]}(x) && \text{by definition of } G_{[t]}\\
&= Q_{[s,s]}(\ell)\,G_{[s]}(x) \;+\; \sum_{t=0}^{s-1} Q_{[s,t]}(\ell)\,G_{[t]}(x)\\
&= Q_{[s,s]}(\ell)\,G_{[s]}(x) \;+\; C\sum_{t=0}^{s-1} Q_{[s,t]}(\ell)\,X_{[t]}(x) && \text{inductive hypothesis}\\
&= Q_{[s,s]}(\ell)\,G_{[s]}(x) \;+\; C\sum_{t=0}^{s} Q_{[s,t]}(\ell)\,X_{[t]}(x) \;-\; C\,Q_{[s,s]}(\ell)\,X_{[s]}(x)\\
&= G_{[s]}(x) \;+\; C\,X_{[s]}(x-\ell) \;-\; C\,X_{[s]}(x) && \text{by definition of } Q_{[s,t]}\,.
\end{aligned}$$

Defining $H_{[s]}(x) = G_{[s]}(x) - C\,X_{[s]}(x)$, it therefore follows from the preceding calculation that

$$H_{[s]}(x-\ell) \;=\; H_{[s]}(x), \quad \ell \in \Gamma\,. \tag{4.3}$$

In addition, it follows from (4.2) that $H_{[s]}$ also satisfies

$$H_{[s]}(Ax) \;=\; A_{[s]}\,H_{[s]}(x)\,.$$

We will now invoke Lemma 2 to show that each of the entries of $H_{[s]}$ is zero. From [4, Lemma 4.2], by choosing an appropriate basis for $\mathbf{C}^{d_s}$ and appropriate ordering of the monomials x^α of degree s, we may assume that the $d_s \times d_s$ matrix $A_{[s]}$ is lower-triangular. Let $\lambda_1, \ldots, \lambda_{d_s}$ be the eigenvalues of $A_{[s]}$. Let $h_i \in \mathcal{S}'(\mathbf{R}^d)$ be the ith component of $H_{[s]}$, and let h_n be the first nonzero component, so that $H_{[s]} = (0, \ldots, 0, h_n, \ldots, h_{d_s})^{\mathrm{T}}$. By (4.3), we have $h_n(x-\ell) = h_n(x)$ for $\ell \in \Gamma$. Moreover, since $A_{[s]}$ is lower-triangular with $\lambda_1, \ldots, \lambda_{d_s}$ on its diagonal and since $H_{[s]}(Ax) = A_{[s]}\,H_{[s]}(x)$, we also have $h_n(Ax) = \lambda_n\,h_n(x)$. Since A is expansive and since $s > 0$, it follows from [4, Lemma 4.2] that $A_{[s]}$ is also expansive. Therefore $|\lambda_n| > 1$, so Lemma 2 implies that $h_n = 0$. Hence $H_{[s]} = 0$, whence $G_{[s]}(x) = C\,X_{[s]}(x)$. This completes the proof. □

With the assumption of independent translates, we have the following simplifications.

Corollary 1.

Assume that $f \in \mathcal{E}'(\mathbf{R}^d, \mathbf{C}^r)$ is a distributional solution of the refinement equation (1.1) *and that translates of f along Γ are independent. Then f has accuracy p if and only if there exists a collection of row vectors $\{v_\alpha \in \mathbf{C}^{1\times r} : 0 \le |\alpha| < p\}$ such that*

(i) *$v_0 \hat{f}(0) \ne 0$, and*

(ii) *$Y_{[s]} = A_{[s]}\,Y_{[s]}\,L$ for $0 \le s < p$,*

where $Y_{[s]} = \big(y_{[s]}(k)\big)_{k\in\Gamma}$ is the row vector of evaluations at lattice points of the matrix of polynomials $y_{[s]}(x) = \sum_{t=0}^{s} Q_{[s,t]}(x)\,v_{[t]}$. Moreover, in case these hold, after scaling the vectors v_α by the nonzero constant $C = (v_0\hat{f}(0))^{-1}\,|P|$, we have

$$X_{[s]}(x) \;=\; \sum_{k\in\Gamma} y_{[s]}(k)\,f(x+k) \;=\; Y_{[s]}\,F(x), \qquad 0 \le s < p.$$

Proof. Assume that f has accuracy p. Then by Theorem 1, there exist row vectors $\{v_\alpha \in \mathbf{C}^{1\times r} : 0 \le |\alpha| < p\}$ such that $v_0 \neq 0$ and $X_{[s]}(x) = Y_{[s]}\, F(x)$ for $0 \le s < p$. By Theorem 3 combined with the fact that translates of f are independent, we have that $Y_{[s]} = A_{[s]}\, Y_{[s]}\, L$ for $0 \le s < p$. It therefore remains only to show that $v_0 \hat{f}(0) \neq 0$.

Since $y_{[0]}(k) = v_0$ for every k, we have

$$1 \;=\; x^0 \;=\; X_{[0]}(x) \;=\; \sum_{k\in\Gamma} v_0 f(x+k)\,.$$

Recall that the rectangular parallelepiped P defined by (2.1) is a fundamental domain for Γ. Let $\varphi \in C_c^\infty(\mathbf{R}^d)$ be a nonnegative, compactly supported function such that $\sum_{k\in\Gamma} \varphi(x-k) = 1$. Then, necessarily, $\int \varphi(x)\,dx = |P|$. Therefore,

$$\begin{aligned} v_0\hat{f}(0) \;=\; \langle 1, v_0 f\rangle \;&=\; \left\langle \sum_{k\in\Gamma} \varphi(x-k),\, v_0 f(x)\right\rangle \\ &=\; \left\langle \varphi(x),\, \sum_{k\in\Gamma} v_0 f(x+k)\right\rangle \;=\; \langle \varphi, 1\rangle \;=\; \int_{\mathbf{R}^d} \varphi(x)\,dx \;=\; |P| \;\neq\; 0. \end{aligned}$$

For the converse, assume that **(i)** and **(ii)** hold, and define $w_{\alpha,k}$ by setting $[w_{\alpha,k}]_{|\alpha|=s} = y_{[s]}(k)$. Then $W_{[s]}(\ell) = Y_{[s]}(\ell)$, so $W_{[s]} = A_{[s]}\, W_{[s]}\, L$ by hypothesis (ii). Further, $W_{[s]}(\ell) = \sum_{t=0}^{s} Q_{[s,t]}(\ell)\, W_{[t]}(0)$ by Lemma 1. Therefore, if we can show that $W_{[0]}\, F(x) = \sum_k w_{0,k}\, f(x) \neq 0$, then it follows from Theorem 3 that f has accuracy p.

As in the proof of Theorem 3, the function $G_{[0]}(x) = W_{[0]}\, F(x) = Y_{[0]}\, F(x)$ is a constant C. To evaluate this constant explicitly, fix a nonnegative, compactly supported function $\varphi \in \mathcal{S}(\mathbf{R}^d)$ with $\operatorname{supp}(\varphi) \subset P$ such that $\sum_{k\in\Gamma} \varphi(x-k) = 1$. Then $\int \varphi(x)\,dx = |P|$, so

$$\begin{aligned} C\,|P| \;=\; \langle \varphi, C\rangle \;=\; \langle \varphi, G_{[0]}\rangle \;&=\; \left\langle \varphi(x),\, \sum_{k\in\Gamma} v_0 f(x+k)\right\rangle \\ &=\; \left\langle \sum_{k\in\Gamma} \varphi(x-k),\, v_0 f(x)\right\rangle \;=\; \langle 1, v_0 f\rangle \;=\; v_0\hat{f}(0) \;\neq\; 0. \end{aligned}$$

Theorem 3 therefore implies that f has accuracy p and that $Y_{[s]}\, F(x) = C\, X_{[s]}(x)$ for $0 \le s < p$. □

Note that in the proof of Corollary 1, the hypothesis of independence of translates is in fact only required for the necessary conditions for f to have accuracy p, not for the sufficient conditions.

Note that the condition $Y_{[s]} = A_{[s]}\, Y_{[s]}\, L$ in Corollary 1 involves only the coefficients c_k and the structure of the matrix L. In particular, it does not depend on whether the solution of the refinement equation is a function or a distribution. It was shown in [4] that the seemingly infinite set of conditions on the vectors v_α given by the statement $Y_{[s]} = A_{[s]}\, Y_{[s]}\, L$ is in fact equivalent to a *finite* system of *finite* linear equations. This equivalence is quoted here for completeness, since it will be used for further results below.

Theorem 4. *Let $m = |\det(A)|$, and let $d_1, \dots, d_m \in \Gamma$ be a full set of digits. Set $\Gamma_i = A(\Gamma) - d_i$. Given a collection $\{v_\alpha \in \mathbf{C}^{1\times r} : 0 \le |\alpha| < p\}$ of row vectors, let $y_{[s]}(x) = \sum_{t=0}^{s} Q_{[s,t]}(x)\, v_{[t]}$ be the associated matrix of polynomials and let $Y_{[s]} = \big(y_{[s]}(k)\big)_{k\in\Gamma}$ be the row vector of evaluations of these polynomials at lattice points. If $v_0 \neq 0$, then the following statements are equivalent.*

(a) $Y_{[p-1]} = A_{[p-1]}\, Y_{[p-1]}\, L$.

(b) $Y_{[s]} = A_{[s]}\, Y_{[s]}\, L$ *for* $0 \le s < p$.

(c) $v_{[s]} \;=\; \sum_{k\in\Gamma_i} \sum_{t=0}^{s} Q_{[s,t]}(k)\, A_{[t]}\, v_{[t]}\, c_k$ *for* $0 \le s < p$ *and* $i = 1, \ldots, m$.

The test for accuracy in statement (II) of Theorem 3 includes the condition that $v_0 \hat{f}(0) \neq 0$. We can formulate the following sufficient conditions so that $v_0 \neq 0$ implies $v_0 \hat{f}(0) \neq 0$.

Theorem 5.

Assume that $f \in \mathcal{E}'(\mathbf{R}^d, \mathbf{C}^r)$ *is a distributional solution of the refinement equation* (1.1). *Let* $m = |\det(A)|$, *and let* $d_1, \ldots, d_m \in \Gamma$ *be a full set of digits. Assume that* $v_0 \in \mathbf{C}^{1\times r}$ *satisfies statement* (c) *in Theorem 4 for the case* $s = 0$, *i.e.,*

$$v_0 \;=\; v_0 \sum_{k\in\Gamma_i} c_k, \qquad i = 1, \ldots, m\,.$$

If $v_0 \neq 0$, *then either of the following two conditions is sufficient to imply that* $v_0 \hat{f}(0) \neq 0$, *and therefore that* f *has accuracy at least* $p = 1$:

(a) *translates of* f *along* Γ *are independent, or*

(b) *the matrix* $\Delta = \frac{1}{m} \sum_{k\in\Lambda} c_k$ *has eigenvalues* $\lambda_1 = 1$ *and* $|\lambda_2|, \ldots, |\lambda_r| < 1$.

Proof. (a) Define $G_{[0]}(x) = v_0 \sum_{k\in\Gamma} f(x+k)$. Then the argument of the proof of Corollary 1(b) shows that $G_{[0]}(x) = C$ a.e., with $C = (v_0 \hat{f}(0))|P|^{-1}$. Hence $v_0 \hat{f}(0) \neq 0$ if and only if $C \neq 0$. However, if translates of f along Γ are independent, then we must have $C \neq 0$ since $C = \sum_{k\in\Gamma} v_0\, f(x+k)$ and $v_0 \neq 0$.

(b) Assume that $\Delta = \frac{1}{m} \sum_{k\in\Lambda} c_k$ has eigenvalues $\lambda_1 = 1$ and $|\lambda_2|, \ldots, |\lambda_r| < 1$. Then, by Proposition 1(b), $\hat{f}(0)$ is the right 1-eigenvector for Δ. On the other hand, since $v_0 = v_0 \sum_{k\in\Gamma_i} c_k$ and since Γ is the disjoint union of the Γ_i, we have

$$v_0 \;=\; v_0\, \frac{1}{m} \sum_{i=1}^{m} \sum_{k\in\Gamma_i} c_k \;=\; v_0 \Delta\,.$$

Hence v_0 is the left 1-eigenvector for Δ. Since the dot product of the left and right 1-eigenvectors must be nonzero when 1 is a simple eigenvalue, we have $v_0 \hat{f}(0) \neq 0$. □

Thus, if either of these two conditions are satisfied, in order to determine the accuracy of f we can use any of the three equivalent statements of Theorem 4. Moreover, the following result implies that test (b) of Theorem 4 imposes a necessary condition on the left eigenvalues of L.

Proposition 2.

Let $\lambda = (\lambda_1, \ldots, \lambda_d)^{\mathrm{T}}$ *be the vector of all eigenvalues of* A. *If there exist row vectors* $Y_{[s]} \in \left((\mathbf{C}^{1\times r})^{d_s \times 1}\right)^{1\times\Gamma}$ *such that* $Y_{[s]} = A_{[s]}\, Y_{[s]}\, L$ *for* $0 \le s < p$, *then* $\lambda^{-\alpha}$ *is a left eigenvalue for* L *for each multi-index* α *with* $0 \le |\alpha| < p$.

Proof. By [4, Lemma 4.2], the eigenvalues of $A_{[s]}$ are $[\lambda^\alpha]_{|\alpha|=s}$. Let S be such that $J = S^{-1} A_{[s]}\, S$ is in Jordan form. If J is diagonal, then its diagonal entries are λ^α. Therefore, by thinking of $Z_{[s]} = S^{-1} Y_{[s]} = [z_\alpha]_{|\alpha|=s}$ as having "rows" $z_\alpha = (z_\alpha(k))_{k\in\Gamma} \in (\mathbf{C}^{1\times r})^{1\times\Gamma}$, we can compute

$$[z_\alpha]_{|\alpha|=s} \;=\; Z_{[s]} \;=\; S^{-1} Y_{[s]} \;=\; S^{-1} A_{[s]}\, S\, S^{-1} Y_{[s]}\, L \;=\; J\, Z_{[s]}\, L \;=\; [\lambda^\alpha z_\alpha L]_{|\alpha|=s}\,.$$

Thus z_α is a left $\lambda^{-\alpha}$-eigenvector for L for each $|\alpha| = s$. If J is not diagonal, then for each distinct value of λ^α there is still at least one z_α such that $z_\alpha = \lambda^\alpha z_\alpha L$, so each of the distinct values of $\lambda^{-\alpha}$ is still a left eigenvalue for L. □

Considering Theorem 3, Theorem 4, and Proposition 2 together, we see that if f is to have accuracy p, then $\lambda^{-\alpha}$ must be a left eigenvalue for L for each $0 \le |\alpha| < p$. An example from [13] shows that even in the case $d = 1$, $r = 1$, the existence of such eigenvalues alone is not sufficient to imply accuracy for f; the corresponding left eigenvectors must have the polynomial structure specified in Theorem 3.

Since L is an infinite matrix, it is conceivable that the determination of its eigenvalues could be a difficult task. However, the eigenvalues and eigenvectors of L are in fact completely determined by a particular finite submatrix of L. This was shown by Jia for the one-dimensional, single-function case in [11]. The higher-dimensional, single-function case was considered in [12], and the one-dimensional, multi-function case was discussed in [13]. Because the characterization of the eigenvalues of L leads to an alternative test for the accuracy of a refinable distribution, we briefly sketch the extension of these ideas to the general higher-dimensional, multi-function setting of this paper.

Define the support of a column vector $a = [a_k]_{k\in\Gamma}$ to be $\operatorname{supp}(a) = \{k \in \Gamma : a_k \neq 0\}$, and set

$$\begin{aligned} \ell(\Gamma) &= \{a = [a_k]_{k\in\Gamma} : a_k \in \mathbf{C}^{r\times 1}\} = (\mathbf{C}^{r\times 1})^{\Gamma\times 1}, \\ \ell_c(\Gamma) &= \{a \in \ell(\Gamma) : \operatorname{supp}(a) \text{ is finite}\}. \end{aligned}$$

For each nonempty $\Omega \subset \Gamma$ define

$$\ell(\Omega) = \{a \in \ell(\Gamma) : \operatorname{supp}(a) \subset \Omega\},$$

and define $\ell(\emptyset) = \{0\}$. We will study a class of finite sets Ω for which $\ell(\Omega)$ is right-invariant under L. In particular, we will show in Lemma 3 that $\ell(\Omega)$ is invariant if

$$A^{-1}(\Omega + \Lambda) \cap \Gamma \subset \Omega. \tag{4.4}$$

We will say that a finite nonempty set $\Omega \subset \Gamma$ is *admissible* if (4.4) holds.

The language of Iterated Function Systems (IFS) is convenient for discussing the properties of admissible sets. Let $\mathcal{H}(\mathbf{R}^d)$ be the metric space of all nonempty compact subsets of $\mathbf{R}^d$ under the Hausdorff metric. For each $k \in \Lambda$, define $w_k : \mathbf{R}^d \to \mathbf{R}^d$ by $w_k(x) = A^{-1}(x + k)$. Then define $w_\Lambda : \mathcal{H}(\mathbf{R}^d) \to \mathcal{H}(\mathbf{R}^d)$ by

$$w_\Lambda(K) = \bigcup_{k\in\Lambda} w_k(K) = A^{-1}(K + \Lambda), \qquad K \in \mathcal{H}(\mathbf{R}^d).$$

Note that:

$$\Omega \subset \Gamma \text{ is admissible} \quad \Longleftrightarrow \quad w_\Lambda(\Omega) \cap \Gamma \subset \Omega.$$

Since A is expansive, there exists a vector norm $\|\cdot\|$ on $\mathbf{R}^d$ such that $\|A^{-1}\| < 1$. Therefore each w_k is a contractive mapping on $\mathbf{R}^d$, and as a consequence it can be shown that w_Λ is a contractive mapping on $\mathcal{H}(\mathbf{R}^d)$. By the Contraction Mapping Theorem, there must therefore exist a unique nonempty compact set $K_\Lambda \subset \mathbf{R}^d$ such that $w_\Lambda(K_\Lambda) = K_\Lambda$, i.e., such that $A^{-1}(K_\Lambda + \Lambda) = K_\Lambda$. In fact,

$$K_\Lambda = \sum_{n=1}^{\infty} A^{-n}(\Lambda) = \left\{\sum_{n=1}^{\infty} A^{-n}\lambda_n : \lambda_n \in \Lambda\right\}. \tag{4.5}$$

The set K_Λ is called the *attractor* of the iterated function system generated by $\{w_k\}_{k\in H}$ [10]. It can be shown that if f is compactly supported and satisfies the refinement equation (1.1), then $\operatorname{supp}(f) \subset K_\Lambda$ [5].

We now derive some basic properties of admissible sets. In particular, we show that the set

$$\Omega_\Lambda = K_\Lambda \cap \Gamma$$

is admissible, and possesses some important special properties among the class of all admissible sets. For the remainder of this article, we will let $\|\cdot\|$ denote any vector norm on $\mathbf{R}^d$ such that $\|A^{-1}\| < 1$, and we let

$$B(\rho) \;=\; \{x \in \mathbf{R}^d : \|x\| \le \rho\}$$

denote the corresponding closed ball of radius ρ centered at the origin. We fix R so that $\Lambda \subset B(R)$, and we define

$$\rho_0 \;=\; \frac{R}{\|A^{-1}\|^{-1} - 1}\,.$$

Lemma 3.

(a) *Let Ω be any finite subset of Γ. Then L maps $\ell(\Omega)$ into $\ell(w_\Lambda(\Omega) \cap \Gamma)$.*

(b) *If $\Omega \subset \Gamma$ is admissible, then $\ell(\Omega)$ is right-invariant under L.*

(c) *$\Omega = B(\rho) \cap \Gamma$ is admissible for all $\rho \ge \rho_0$. In particular, every finite subset of Γ is contained in an admissible set.*

(d) *$\Omega_\Lambda = K_\Lambda \cap \Gamma$ is admissible, and satisfies $\Omega_\Lambda = w_\Lambda(\Omega_\Lambda) \cap \Gamma$.*

(e) *Let Ω be an arbitrary finite subset of Γ. If $\Omega \subset w_\Lambda(\Omega) \cap \Gamma$, then $\Omega \subset \Omega_\Lambda$.*

(f) *If $\Omega \subset \Gamma$ is admissible, then $\Omega' = w_\Lambda(\Omega) \cap \Gamma \subset \Omega$ is also admissible. Further, if $\Omega_\Lambda \subset \Omega$, then $\Omega_\Lambda \subset \Omega'$.*

(g) *Assume $\rho \ge \rho_0$ is such that $\Omega_\Lambda \subset B(\rho)$. Then there exist admissible sets*

$$\Omega_\Lambda = \Omega_0 \;\subsetneq\; \Omega_1 \;\subsetneq\; \cdots \;\subsetneq\; \Omega_{N-1} \;\subsetneq\; \Omega_N = B(\rho) \cap \Gamma$$

such that

$$w_\Lambda(\Omega_{n+1}) \cap \Gamma \;\subset\; \Omega_n, \qquad n = 0, \ldots, N-1\,. \tag{4.6}$$

Proof. (a) Let $a = [a_k]_{k\in\Gamma} \in \ell(\Omega)$. Since $La = \big[\sum_{j\in\Gamma} c_{Ai-j}\, a_j\big]_{i\in\Gamma}$ and $a_j \ne 0$ only when $j \in \Omega$, we can only have $(La)_i \ne 0$ if there is a $j \in \Omega$ such that $Ai - j \in \Lambda$. In this case, $Ai \in j + \Lambda \subset \Omega + \Lambda$, and therefore $i \in A^{-1}(\Omega + \Lambda) \cap \Gamma = w_\Lambda(\Omega) \cap \Gamma$.

(b) Follows immediately from (a).

(c) Let $\Omega = B(\rho) \cap \Gamma$. If $\rho \ge \rho_0$, then $\|A^{-1}\|\,(\rho + R) \le \rho$, so

$$A^{-1}(\Omega + \Lambda) \cap \Gamma \;\subset\; A^{-1}\big(B(\rho + R)\big) \cap \Gamma \;\subset\; B\big(\|A^{-1}\|\,(\rho + R)\big) \cap \Gamma \;\subset\; B(\rho) \cap \Gamma \;=\; \Omega\,.$$

(d) Since $\Omega_\Lambda \subset K_\Lambda$, we have $w_\Lambda(\Omega_\Lambda) \cap \Gamma \subset w_\Lambda(K_\Lambda) \cap \Gamma = K_\Lambda \cap \Gamma = \Omega_\Lambda$, and therefore Ω_Λ is admissible. To show that this inclusion is an equality, suppose that $k \in \Omega_\Lambda = K_\Lambda \cap \Gamma$. By (4.5), there exist $\lambda_n \in \Lambda$ so that $k = \sum_{n=1}^\infty A^{-n}\lambda_n$. Therefore $w = Ak - \lambda_1 = \sum_{n=1}^\infty A^{-n}\lambda_{n+1} \in K_\Lambda$, and furthermore $w \in \Gamma$ since Ak and λ_1 lie in Γ. Hence $w \in \Omega_\Lambda$, and therefore $k = A^{-1}(w + \lambda_1) \in A^{-1}(\Omega_\Lambda + \Lambda) \cap \Gamma = w_\Lambda(\Omega_\Lambda) \cap \Gamma$.

(e) If $\Omega \subset w_\Lambda(\Omega) \cap \Gamma = A^{-1}(\Lambda + \Omega) \cap \Gamma$, then

$$\Omega \;\subset\; A^{-1}(\Lambda) + A^{-1}(\Omega) \;\subset\; A^{-1}(\Lambda) + A^{-2}(\Lambda) + A^{-2}(\Omega) \;\subset\; \cdots\,.$$

Since A^{-1} is a contraction, it follows that $\Omega \subset \sum_{n=1}^\infty A^{-n}(\Lambda) = K_\Lambda$. Since we also have $\Omega \subset \Gamma$, we conclude that $\Omega \subset \Omega_\Lambda$.

(f) If Ω is admissible, then $\Omega' = w_\Lambda(\Omega) \cap \Gamma \subset \Omega$. Hence, $w_\Lambda(\Omega') \cap \Gamma \subset w_\Lambda(\Omega) \cap \Gamma = \Omega'$, so Ω' is admissible. Further, if $\Omega_\Lambda \subset \Omega$, then $\Omega_\Lambda = w_\Lambda(\Omega_\Lambda) \cap \Gamma \subset w_\Lambda(\Omega) \cap \Gamma = \Omega'$.

(g) Define $E_0 = B(\rho)$. Since $\rho \ge \rho_0$, we have $\rho \ge \|A^{-1}\|\,(\rho + R)$. Therefore

$$w_\Lambda(E_0) \;=\; A^{-1}\big(B(\rho) + \Lambda\big) \;\subset\; A^{-1}\big(B(\rho + R)\big) \;\subset\; B\big(\|A^{-1}\|\,(\rho + R)\big) \;\subset\; B(\rho) \;=\; E_0\,.$$

Recursively define $E_{\nu+1} = w_\Lambda(E_\nu)$ for $\nu \geq 0$. An easy induction establishes that $E_{\nu+1} \subset E_\nu$ for every ν. Further, since E_0 is compact, we have $\cap E_\nu = K_\Lambda$ by the Contraction Mapping Theorem. Therefore, we must have $E_\nu \cap \Gamma = \Omega_\Lambda$ for all ν large enough, so $\{E_\nu \cap \Gamma\}_{\nu \geq 0}$ is a finite collection of sets. Let $\Omega_\Lambda = \Omega_0 \subsetneq \Omega_1 \subsetneq \cdots \subsetneq \Omega_N = E_0 \cap \Gamma$ be the distinct elements of this collection. Fix $0 \leq n < N$. Then there exists a ν such that $\Omega_n = E_\nu \cap \Gamma \subsetneq E_{\nu-1} \cap \Gamma = \Omega_{n+1}$. Therefore

$$w_\Lambda(\Omega_{n+1}) \cap \Gamma \;\subset\; w_\Lambda(E_{\nu-1}) \cap \Gamma \;=\; E_\nu \cap \Gamma \;=\; \Omega_n\,, \tag{4.7}$$

so (4.6) holds. Moreover, since $\Omega_n \subset \Omega_{n+1}$, it also follows from (4.7) that Ω_{n+1} is admissible. Since we know that $\Omega_0 = \Omega_\Lambda$ is admissible, the proof is complete. $\square$

We now show that the nonzero eigenvalues of L acting on $\ell_c(\Gamma)$ on the right coincide with the nonzero eigenvalues of L acting on $\ell(\Gamma)$ on the left, and that these further coincide with the eigenvalues of the finite submatrix

$$L_{\Omega_\Lambda} \;=\; [c_{Ai-j}]_{i,j\in\Omega_\Lambda}\,.$$

In particular, since L can have only finitely many nonzero eigenvalues, the accuracy of a refinable f is necessarily finite. The use of $\ell_c(\Gamma)$ on one side of L and $\ell(\Gamma)$ on the other is in fact natural, because $\ell_c(\Gamma)$ and $\ell(\Gamma)$ are algebraic adjoints of each other.

Let $P\colon \ell(\Gamma) \to (\mathbf{C}^{r\times 1})^{\Omega_\Lambda \times 1}$ denote the restriction mapping defined by

$$P\big([\,a_k\,]_{k\in\Gamma}\big) \;=\; [\,a_k\,]_{k\in\Omega_\Lambda}\,,$$

and let $E\colon (\mathbf{C}^{r\times 1})^{\Omega_\Lambda\times 1} \to \ell(\Gamma)$ denote the zero extension mapping defined by

$$E\big([\,a_k\,]_{k\in\Omega_\Lambda}\big) \;=\; [\,a_k\,]_{k\in\Gamma}, \quad \text{where } a_k = 0 \text{ if } k \notin \Omega_\Lambda\,.$$

Theorem 6.
Fix $\lambda \neq 0$.

- **(a)** *If $a \in \ell_c(\Gamma)$ is a λ-eigenvector for L, then $\operatorname{supp}(a) \subset \Omega_\Lambda$, so Pa is a λ-eigenvector for L_{Ω_Λ}. Conversely, if $a \in (\mathbf{C}^{r\times 1})^{\Omega_\Lambda\times 1}$ is a λ-eigenvector for L_{Ω_Λ}, then Ea is a λ-eigenvector for L.*
- **(b)** *If $a \in \ell(\Gamma)$ is a λ-eigenvector for L^{T}, then Pa is a λ-eigenvector for $L^{\mathrm{T}}_{\Omega_\Lambda}$. Conversely, if $a \in (\mathbf{C}^{r\times 1})^{\Omega_\Lambda\times 1}$ is a λ-eigenvector for $L^{\mathrm{T}}_{\Omega_\Lambda}$, then there exists a λ-eigenvector $b \in \ell(\Gamma)$ for L^{T} such that $Pb = a$.*

Proof. (a) Suppose $a \in \ell_c(\Gamma)$ satisfies $La = \lambda a$ with $a \neq 0$. Set $\Omega = \operatorname{supp}(a)$. Then $\Omega = \operatorname{supp}(\lambda a) = \operatorname{supp}(La) \subset w_\Lambda(\Omega) \cap \Gamma$ by Lemma 3(a), and therefore $\Omega \subset \Omega_\Lambda$ by Lemma 3(e). In particular, $Pa \neq 0$. Moreover, since $\operatorname{supp}(a) \subset \Omega_\Lambda$, it follows that $c_{Ai-j}\,a_j \neq 0$ can only hold when $j \in \Omega_\Lambda$, and therefore

$$L_{\Omega_\Lambda}(Pa) \;=\; \left[\sum_{j\in\Omega_\Lambda} c_{Ai-j}\,a_j\right]_{i\in\Omega_\Lambda} \;=\; \left[\sum_{j\in\Gamma} c_{Ai-j}\,a_j\right]_{i\in\Omega_\Lambda} \;=\; P(La) \;=\; \lambda Pa\,.$$

Thus Pa is a λ-eigenvector for L_{Ω_Λ}.

For the converse statement, suppose that $L_{\Omega_\Lambda}a = \lambda a$ for some nonzero $a \in (\mathbf{C}^{r\times 1})^{\Omega_\Lambda\times 1}$. Note that if $j \in \Omega_\Lambda$, then $c_{Ai-j}\,a_j \neq 0$ can hold only if $Ai-j \in \Lambda$, which implies $i \in A^{-1}(\Omega_\Lambda+\Lambda)\cap\Gamma = \Omega_\Lambda$. Therefore,

$$L(Ea) \;=\; \left[\sum_{j\in\Omega_\Lambda} c_{Ai-j}\,a_j\right]_{i\in\Gamma} \;=\; E\left(\left[\sum_{j\in\Omega_\Lambda} c_{Ai-j}\,a_j\right]_{i\in\Omega_\Lambda}\right) \;=\; E(L_{\Omega_\Lambda}a) \;=\; \lambda Ea\,.$$

Therefore Ea is a λ-eigenvector for L.

(b) Choose any $a = [a_k]_{k\in\Gamma} \in \ell(\Gamma)$. Note that if $j \in \Omega_\Lambda$, then $Ai - j \in \Lambda$ can hold only if $i \in \Omega_\Lambda$ as well. Therefore,

$$PL^{\mathrm{T}}a \;=\; P\biggl(\Bigl[\sum_{i\in\Gamma} c^{\mathrm{T}}_{Ai-j}\, a_i\Bigr]_{j\in\Gamma}\biggr) \;=\; \Bigl[\sum_{i\in\Gamma} c^{\mathrm{T}}_{Ai-j}\, a_i\Bigr]_{j\in\Omega_\Lambda} \;=\; \Bigl[\sum_{i\in\Omega_\Lambda} c^{\mathrm{T}}_{Ai-j}\, a_i\Bigr]_{j\in\Omega_\Lambda} \;=\; L^{\mathrm{T}}_{\Omega_\Lambda} Pa\,,$$

and hence $PL^{\mathrm{T}} = L^{\mathrm{T}}_{\Omega_\Lambda} P$. Therefore, if $a \neq 0$ is a λ-eigenvector for L^{T}, then $\lambda Pa = PL^{\mathrm{T}}a = L^{\mathrm{T}}_{\Omega_\Lambda} Pa$. Therefore, if we can show that $Pa \neq 0$, then we can conclude that Pa is a λ-eigenvector for L^{T}.

Suppose that we had $Pa = 0$, i.e., that $a_k = 0$ for $k \in \Omega_\Lambda$. Since $a \neq 0$, there must be some $k_0 \in \Gamma$ such that $a_{k_0} \neq 0$. Choose $\rho \geq \rho_0$ such that $B(\rho)$ contains both Ω_Λ and k_0, and then let $\Omega_\Lambda = \Omega_0 \subsetneq \cdots \subsetneq \Omega_N = B(\rho) \cap \Gamma$ be the admissible sets constructed in Lemma 3(g). Recall that Ω_N contains a point k_0 such that $a_{k_0} \neq 0$. We claim that the strictly smaller set Ω_{N-1} must also contain a point k_1 such that $a_{k_1} \neq 0$. Suppose on the contrary that $a_k = 0$ for all $k \in \Omega_{N-1}$. Note that $\lambda a_{k_0} = (L^{\mathrm{T}}a)_{k_0} = \sum_{i\in\Gamma} c^{\mathrm{T}}_{Ai-k_0}\, a_i$. However, if $Ai - k_0 \in \Lambda$, then $i \in A^{-1}(\Omega_N + \Lambda) \cap \Gamma = w_\Lambda(\Omega_N) \subset \Omega_{N-1}$, and therefore $a_i = 0$. Hence $\lambda a_{k_0} = 0$, which is a contradiction. Therefore there must be some $k_1 \in \Omega_{N-1}$ such that $a_{k_1} \neq 0$. Repeating this argument with Ω_{N-1} in place of Ω_N, we see that each Ω_n must contain a point k_n such that $a_{k_n} \neq 0$. However, since $\Omega_0 = \Omega_\Lambda$ contains no such points, this is impossible. Therefore we must have $Pa \neq 0$, so Pa is indeed a λ-eigenvector for L^{T}.

For the converse statement, let $a \in (\mathbf{C}^{r\times 1})^{\Omega_\Lambda \times 1}$ be a λ-eigenvector for $L^{\mathrm{T}}_{\Omega_\Lambda}$. Choose $\rho \geq \rho_0$ large enough that $K_\Lambda \subset B(\rho)$, and let $\Omega_\Lambda = \Omega_0 \subsetneq \cdots \subsetneq \Omega_N = B(\rho) \cap \Gamma$ be the admissible sets constructed in Lemma 3(g). Set $\rho_N = \rho$, and recursively define $\rho_{n+1} = (\rho_n / \|A^{-1}\|) - R$ for $n \geq N$. If ρ is chosen large enough, then $\rho_N < \rho_{N+1} < \cdots$ is an increasing sequence of numbers. Define $\Omega_n = B(\rho_n) \cap \Gamma$ for $n > N$. By Lemma 3(c), these Ω_n are admissible. Combining with the sets $\Omega_0, \ldots, \Omega_N$ constructed previously, we see that $\{\Omega_n\}_{n\geq 0}$ is a strictly increasing sequence of admissible sets whose union is Γ and such that $\Omega_0 = \Omega_\Lambda$. Further, by (4.6), we have $w_\Lambda(\Omega_{n+1}) \cap \Gamma \subset \Omega_n$ for $n = 0, \ldots, N-1$. Moreover, this inclusion holds for $n \geq N$ since for these n we have

$$w_\Lambda(\Omega_{n+1}) \;=\; A^{-1}(\Omega_{n+1} + \Lambda) \;\subset\; B\bigl(\|A^{-1}\|(\rho_{n+1} + R)\bigr) \;=\; B(\rho_n)\,.$$

Now define a_j for $j \notin \Omega_\Lambda$ recursively by the formula

$$a_j \;=\; \frac{1}{\lambda} \sum_{i\in\Omega_n} c^{\mathrm{T}}_{Ai-j}\, a_i, \qquad j \in \Omega_{n+1} \setminus \Omega_n\,.$$

Note that if $Ai - j \in \Lambda$ and $j \in \Omega_{n+1}$, then $i \in A^{-1}(\Omega_{n+1} + \Lambda) \cap \Gamma = w_\Lambda(\Omega_{n+1}) \subset \Omega_n$. It follows that the vector $b = [a_j]_{j\in\Gamma}$ satisfies $L^{\mathrm{T}}b = \lambda b$. Since $Pb = a \neq 0$, the vector b is a λ-eigenvector for L^{T}. $\square$

We can now compare two possible finite tests for accuracy. The first test is provided by Theorem 4(c). To determine the accuracy corresponding to a given refinement equation, we find the largest value of p such that the finite system of linear equations in Theorem 4(c) has a solution. This is an iterative process: the system is solved for $s = 0, 1, \ldots$ in turn until the maximum number of solvable equations is found. This test only involves solving linear equations. An upper bound on the accuracy can be found in advance by computing the right eigenvalues of $L|_{\ell(\Omega_\Lambda)}$, since accuracy p requires that $\lambda^{-\alpha}$ be an eigenvalue of L for each eigenvalue λ of A and each $0 \leq |\alpha| < p$.

An alternative test for accuracy, in the spirit of the one-dimensional results of [13], can be based on Theorem 4(a). Once an upper bound for p has been computed by checking the eigenvalues

of $L|_{\ell(\Omega_\Lambda)}$, the left eigenvectors for L lead to the vectors $Y_{[p-1]}$ such that $Y_{[p-1]} = A_{[p-1]}\, Y_{[p-1]}\, L$. If these vectors have a polynomial structure, then the accuracy is p. If they do not have a polynomial structure, then the test must be repeated replacing p by $p-1$. This test requires the computation of the eigenvalues of a (possibly large) finite matrix, which cannot be done using only systems of linear equations.

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