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A BASIS THEORY PRIMER*

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CONTENTS

PREFACE

Bases are essential tools in the study of Banach and Hilbert spaces. This manuscript presents a quick and hopefully easy introduction to basis theory for readers with a modest background in real and functional analysis. A short review of the relevant background from analysis is included.

This manuscript grew out of a set of notes originally prepared in 1987 at the instigation of my Ph.D. thesis advisor, Professor John Benedetto of the University of Maryland, College Park. At that time, the now-ubiquitous field of wavelets was in its infancy. An important goal of the new theory was the construction of "good" bases or basis-like systems called frames for function spaces such as $L^2(\mathbf{R})$. A solid understanding of basis theory was therefore needed, and these notes are the offspring of that need. The results presented here were drawn from many sources, but especially from the indispensable books by Lindenstrauss and Tzafriri [LT77], Marti [Mar69], Singer [Sin70], and Young [You80]. Aside from a few minor results in connection with frame theory that are clearly identified, no results presented in this manuscript are original or are claimed as original.

Outline. In the first part of the manuscript, consisting of Chapter 1, we present a review of basic functional-analytic background material. We give the definitions and the statements of the theorems that underlie the material in this manuscript, but we omit the proofs. Most of this material can be found in standard texts on real analysis, functional analysis, or Hilbert space theory, but it is collected here as a single, convenient source of reference.

The second part of the manuscript deals with the meaning of an infinite series $\sum x_n$ in abstract spaces. In Chapter 2, we define what it means for a series to converge, and study several more restrictive forms of convergence, including unconditional convergence in particular. Chapter 3 presents some additional results on unconditional convergence that apply to the specific case of Hilbert spaces.

The third part of the manuscript is devoted to the study of bases and related systems in Banach spaces. In Chapter 4 we present the definitions and essential properties of bases in Banach spaces. Chapter 5 discusses the special case of absolutely convergent bases. In Chapter 6 and 7 we discuss properties of general biorthogonal systems. Chapter 8 considers the duality between bases and their biorthogonal sequences. Chapter 9 presents in detail the important class of unconditional bases. Chapter 10 is devoted to considering some generalizations of bases to the case of the weak or weak* topologies.

The fourth and final part of this manuscript is devoted to the study of bases and basis-like systems in Hilbert spaces. In Chapter 11 we consider unconditional bases in Hilbert space, and characterize the class of bounded unconditional bases. In Chapter 12 we consider frames, which share many of the properties of bounded unconditional bases, yet need not be bases. This final chapter is adapted from my Ph.D. thesis [Hei90] and from my joint research-tutorial with Walnut [HW89].

Acknowledgment. I would like to thank Professor Benedetto and fellow student David Walnut for numerous invaluable discussions and insights on frames, wavelets, harmonic analysis, and all other areas of mathematics, and for their continuing advice and friendship.

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I. PRELIMINARIES

1. NOTATION AND FUNCTIONAL ANALYSIS REVIEW

In this chapter we shall briefly review the basic definitions and theorems that underlie the results presented in this manuscript. Excellent references for this material are [Con85], [GG81], [RS80], [Roy68], [Rud91], [WZ77], and related books.

1.1. General Notation.

$\mathbf{Z} = \{\dots, -1, 0, 1, \dots\}$ is the set of integers, $\mathbf{N} = \{1, 2, 3, \dots\}$ is the natural numbers, $\mathbf{Q}$ is the set of rational numbers, $\mathbf{R}$ is the set of real numbers, and $\mathbf{C}$ is the set of complex numbers. $\mathbf{F}$ will denote the current field of scalars, either $\mathbf{R}$ or $\mathbf{C}$ according to context. The *real part* of a complex number $z = a + ib$ is $\mathrm{Re}(z) = a$, and the *imaginary part* is $\mathrm{Im}(z) = b$. The *complex conjugate* of $z = a + ib$ is $\bar{z} = a - ib$. The *modulus*, or *absolute value*, of $z = a + ib$ is $|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}$. On occasion, we use formally the *extended real numbers* $\mathbf{R} \cup \{-\infty, \infty\}$. For example, the infimum and supremum of a set of real numbers $\{a_n\}$ always exist as extended real numbers, i.e., we always have $-\infty \le \inf a_n \le \sup a_n \le \infty$.

If S is a subset of a set X then $X \backslash S = \{x \in X : x \notin S\}$. The cardinality of a finite set F is denoted by $|F|$. The Lebesgue measure of a subset $S \subset \mathbf{R}$ is denoted by $|S|$. The distinction between these two meanings of $|\cdot|$ is always clear from context.

Sequences or series with unspecified limits are assumed to be over $\mathbf{N}$. That is,

$$(c_n) \;=\; (c_n)_{n=1}^{\infty}, \qquad \{x_n\} \;=\; \{x_n\}_{n=1}^{\infty}, \qquad \sum_n x_n \;=\; \sum_{n=1}^{\infty} x_n.$$

We generally use the notation (c_n) to denote a sequence of scalars and $\{x_n\}$ to denote a sequence of vectors. A series $\sum c_n$ of complex numbers *converges* if $\lim_{N\to\infty} \sum_{n=1}^{N} c_n$ exists as a complex number. If (c_n) is a sequence of *nonnegative* real scalars, we use the notation $\sum c_n < \infty$ to mean that the series $\sum c_n$ converges.

Let X and Y be sets. We write $f\colon X \to Y$ to denote a function with domain X and range Y. The image or range of f is $\mathrm{Range}(f) = f(X) = \{f(x) : x \in X\}$. A function $f\colon X \to Y$ is *injective*, or *1-1*, if $f(x) = f(y)$ implies $x = y$. It is *surjective*, or *onto*, if $f(X) = Y$. It is *bijective* if it is both injective and surjective.

Let E be a subset of the real line $\mathbf{R}$ and let $f\colon E \to \mathbf{C}$ be a complex-valued function defined on E. f is *bounded* if there exists a number M such that $|f(x)| \le M$ for every $x \in E$. f is *essentially bounded* if there exists a number M such that $|f(x)| \le M$ almost everywhere, i.e., if the set $Z = \{x \in E : |f(x)| > M\}$ has Lebesgue measure zero. In general, a property is said to hold *almost everywhere* (a.e.) if the Lebesgue measure of the set on which the property fails is zero.

The *Kronecker delta* is

$$\delta_{mn} \;=\; \begin{cases} 1, & \text{if } m = n, \\ 0, & \text{if } m \ne n. \end{cases}$$

We use the symbol $\square$ to denote the end of a proof, and the symbol $\diamondsuit$ to denote the end of a definition or the end of the statement of a theorem whose proof will be omitted.

1.2. Banach Spaces. We assume that the reader is familiar with vector spaces. The scalar field associated with the vector spaces in this manuscript will always be either the real line $\mathbf{R}$ or the complex plane $\mathbf{C}$. We use the symbol $\mathbf{F}$ to denote the generic choice of one of these fields. For simplicity, some definitions and examples are stated specifically for complex scalars, but the required changes for the case of real scalars are always obvious.

Our first step is to define what we mean by the "size" or "norm" of a vector.

Definition 1.1. A vector space X is called a *normed linear space* if for each $x \in X$ there is a real number $\|x\|$, called the *norm* of x, such that:

(a) $\|x\| \geq 0$,

(b) $\|x\| = 0$ if and only if $x = 0$,

(c) $\|cx\| = |c|\,\|x\|$ for every scalar c, and

(d) $\|x + y\| \leq \|x\| + \|y\|$.

If only properties (a), (c), and (d) hold then $\|\cdot\|$ is called a *seminorm.* $\diamondsuit$

It is usually clear from context which normed linear space X and which norm $\|\cdot\|$ is being referred to. However, when there is the possibility of confusion, we write $\|\cdot\|_X$ to specify which norm we mean.

Definition 1.2. Let X be a normed linear space.

(a) A sequence of vectors $\{x_n\}$ in X *converges* to $x \in X$ if $\lim_{n\to\infty} \|x - x_n\| = 0$, i.e., if

$$\forall\, \varepsilon > 0, \quad \exists\, N > 0, \quad \forall\, n \geq N, \quad \|x - x_n\| < \varepsilon.$$

In this case, we write $x_n \to x$, or $\lim_{n\to\infty} x_n = x$.

(b) A sequence of vectors $\{x_n\}$ in X is *Cauchy* if $\lim_{m,n\to\infty} \|x_m - x_n\| = 0$, i.e., if

$$\forall\, \varepsilon > 0, \quad \exists\, N > 0, \quad \forall\, m, n \geq N, \quad \|x_m - x_n\| < \varepsilon.$$

(c) It is easy to show that every convergent sequence in a normed space is a Cauchy sequence. However, the converse is not true in general. We say that X is *complete* if it is the case that every Cauchy sequence in X is a convergent sequence. A complete normed linear space is called a *Banach space.* $\diamondsuit$

Definition 1.3. A sequence $\{x_n\}$ in a Banach space X is:

(a) *bounded below* if $\inf \|x_n\| > 0$,

(b) *bounded above* if $\sup \|x_n\| < \infty$,

(c) *normalized* if $\|x_n\| = 1$ for all n. $\diamondsuit$

Sometimes, to emphasize that the boundedness discussed in Definition 1.3 refers to the norm of the elements of the sequence, we will say that $\{x_n\}$ is *norm*-bounded below, etc. For example, it is easy to show that if $x_n \to x$ then $\|x_n\| \to \|x\|$. Hence all convergent sequences are norm-bounded above.

We sometimes use the term "bounded" without the qualification "above" or "below." In most cases, we mean only that the sequence is bounded above. However, in certain contexts we may require that the sequence be bounded both above and below. For example, this is what we mean when we refer to a "bounded basis" (see Definition 4.2). This more restricted meaning for "bounded" is always stated explicitly in a definition, and in general the exact meaning should always be clear from context.

The simplest examples of Banach spaces are $\mathbf{R}^n$ (using real scalars) or $\mathbf{C}^n$ (using complex scalars). There are many choices of norm for these finite-dimensional Banach spaces. In particular, we can use any of the following norms:

$$|v|_p = \begin{cases} \big(|v_1|^p + \cdots + |v_n|^p\big)^{1/p}, & 1 \le p < \infty, \\ \max\{|v_1|, \dots, |v_n|\}, & p = \infty, \end{cases}$$

where $v = (v_1, \dots, v_n)$. The *Euclidean norm* of a vector v is the norm corresponding to the choice $p = 2$, i.e.,

$$|v| \;=\; |v|_2 \;=\; \sqrt{|v_1|^2 + \cdots + |v_n|^2}.$$

This particular norm has some extra algebraic properties that will we discuss further in Section 1.3.

Definition 1.4. Suppose that X is a normed linear space with respect to a norm $\|\cdot\|$ and also with respect to another norm $|\!|\!|\cdot|\!|\!|$. These norms are *equivalent* if there exist constants $C_1, C_2 > 0$ such that $C_1\,\|x\| \le |\!|\!|x|\!|\!| \le C_2\,\|x\|$ for every $x \in X$. If $\|\cdot\|$ and $|\!|\!|\cdot|\!|\!|$ are equivalent then they define the same convergence criterion, i.e.,

$$\lim_{n\to\infty} \|x - x_n\| = 0 \quad\iff\quad \lim_{n\to\infty} |\!|\!|x - x_n|\!|\!| = 0. \quad \diamondsuit$$

Any two of the norms $|\cdot|_p$ on $\mathbf{C}^n$ are equivalent. In fact, it can be shown that *any* two norms on a *finite*-dimensional vector space are equivalent.

Example 1.5. The following are Banach spaces whose elements are complex-valued functions with domain $E \subset \mathbf{R}$.

(a) Fix $1 \le p < \infty$, and define

$$L^p(E) \;=\; \Big\{f\colon E \to \mathbf{C} \;:\; \int_E |f(x)|^p\,dx < \infty\Big\}.$$

This is a Banach space with norm

$$\|f\|_{L^p} \;=\; \left(\int_E |f(x)|^p\,dx\right)^{1/p}.$$

(b) For $p = \infty$, define

$$L^\infty(E) \;=\; \bigl\{f \colon E \to \mathbf{C} \;:\; f \text{ is essentially bounded on } E\bigr\}.$$

This is a Banach space under the "sup-norm" or "uniform norm"

$$\|f\|_{L^\infty} \;=\; \operatorname*{ess\,sup}_{x\in E} |f(x)| \;=\; \inf\bigl\{M \geq 0 : |f(x)| \leq M \text{ a.e.}\bigr\}.$$

(c) Define

$$C(E) \;=\; \bigl\{f \colon E \to \mathbf{C} \;:\; f \text{ is continuous on } E\bigr\}.$$

If E is a *compact* subset of $\mathbf{R}$ then any continuous function on E must be bounded. It can be shown that, in this case, $C(E)$ is a Banach space using the sup-norm

$$\|f\|_{L^\infty} \;=\; \sup_{x\in E} |f(x)|.$$

Note that for a continuous function, the supremum of $|f(x)|$ coincides with the essential supremum of $|f(x)|$. Therefore, $C(E)$ is a subspace of $L^\infty(E)$ that is itself a Banach space using the norm of $L^\infty(E)$. $\diamondsuit$

Example 1.6. The following are Banach spaces whose elements are sequences $c = (c_n) = (c_n)_{n=1}^\infty$ of scalars.

(a) Fix $1 \leq p < \infty$, and define

$$\ell^p \;=\; \Bigl\{c = (c_n) : \sum_{n\in\mathbf{Z}} |c_n|^p < \infty\Bigr\}.$$

This is a Banach space with norm

$$\|c\|_{\ell^p} \;=\; \|(c_n)\|_{\ell^p} \;=\; \biggl(\sum_{n\in\mathbf{Z}} |c_n|^p\biggr)^{1/p}.$$

(b) For $p = \infty$, define

$$\ell^\infty \;=\; \bigl\{c = (c_n) : (c_n) \text{ is a bounded sequence}\bigr\}.$$

This is a Banach space under the "sup-norm"

$$\|c\|_{\ell^\infty} \;=\; \|(c_n)\|_{\ell^\infty} \;=\; \Bigl(\sup_{n\in\mathbf{Z}} |c_n|\Bigr).$$

(c) Define

$$c_0 \;=\; \Bigl\{c = (c_n) : \lim_{|n|\to\infty} c_n = 0\Bigr\}.$$

This is a subspace of ℓ^∞ that is itself a Banach space under the norm $\|\cdot\|_{\ell^\infty}$. $\diamondsuit$

We have the following important inequalities on the norm of a product of two functions or sequences.

Theorem 1.7 (Hölder's Inequality). *Fix $1 \le p \le \infty$, and define p' by $\frac{1}{p} + \frac{1}{p'} = 1$, where we set $1/0 = \infty$ and $1/\infty = 0$.*

(a) *If $f \in L^p(E)$ and $g \in L^{p'}(E)$ then $fg \in L^1(E)$, and*

$$\|fg\|_{L^1} \le \|f\|_{L^p} \, \|g\|_{L^{p'}}.$$

For $1 < p < \infty$ this is equivalent to the statement

$$\int_E |f(x)\, g(x)|\, dx \le \left(\int_E |f(x)|^p \, dx\right)^{1/p} \left(\int_E |g(x)|^{p'} \, dx\right)^{1/p'}.$$

(b) *If $(a_n) \in \ell^p$ and $(b_n) \in \ell^{p'}$ then $(a_n b_n) \in \ell^1$, and*

$$\|(a_n b_n)\|_{\ell^1} \le \|(a_n)\|_{\ell^p} \, \|(b_n)\|_{\ell^{p'}}.$$

For $1 < p < \infty$ this is equivalent to the statement

$$\sum_n |a_n\, b_n| \le \Bigl(\sum_n |a_n|\Bigr)^{1/p} \Bigl(\sum_n |b_n|^{p'}\Bigr)^{1/p'}. \quad \diamondsuit$$

Note that if $p = 2$ then we also have $p' = 2$. Therefore, we have the following special cases of Hölder's inequality, which are usually referred to as the *Schwarz* or *Cauchy–Schwarz* inequalities:

$$\|fg\|_{L^1} \le \|f\|_{L^2} \, \|g\|_{L^2} \qquad \text{and} \qquad \|(a_n b_n)\|_{\ell^1} \le \|(a_n)\|_{\ell^2} \, \|(b_n)\|_{\ell^2}. \tag{1.1}$$

$L^2(E)$ and ℓ^2 are specific examples of *Hilbert spaces*, which are discussed more fully in Section 1.3. In particular, Theorem 1.16 gives a generalization of (1.1) that is valid in any Hilbert space.

Next, we present some results related to the topology of X induced by the norm $\|\cdot\|$.

Definition 1.8. Let X be a Banach space.

(a) If $x \in X$ and $\varepsilon > 0$, then the *open ball in X centered at x with radius ε* is

$$B_\varepsilon(x) = \{y \in X : \|x - y\| < \varepsilon\}.$$

(b) A subset $U \subset X$ is *open* if for each $x \in U$ there exists an $\varepsilon > 0$ such that $B_\varepsilon(x) \subset U$.

(c) Let $E \subset X$. Then $x \in X$ is a *limit point* of E if there exist $x_n \in E$ such that $x_n \to x$.

(d) A subset $E \subset X$ is *closed* if $X \backslash E$ is open. Equivalently, E is closed if it contains all its limit points, i.e., if $\{x_n\} \subset E$ and $x_n \to x$ always implies $x \in E$.

(e) The *closure* of a subset $E \subset X$ is the smallest closed set $\bar{E}$ that contains E. Equivalently, $\bar{E}$ is equal to E plus all the limit points of E.

(f) A subset $E \subset X$ is *dense* in X if $\bar{E} = X$. $\diamondsuit$

Lemma 1.9. *Let S be a subspace of a Banach space X. Then S is itself a Banach space under the norm of X if and only if S is a closed subset of X.* $\diamondsuit$

Example 1.10. If E is a compact subset of the real line $\mathbf{R}$, then $C(E)$ is a closed subspace of $L^\infty(E)$. If $E = \mathbf{R}$, then the following are both closed subspaces of $L^\infty(\mathbf{R})$:

$$\begin{aligned} C_b(\mathbf{R}) &= C(\mathbf{R}) \cap L^\infty(\mathbf{R}), \\ C_0(\mathbf{R}) &= \bigl\{ f\colon \mathbf{R} \to \mathbf{C} \;:\; f \text{ is continuous and } \lim_{|x|\to\infty} f(x) = 0 \bigr\}. \quad \diamondsuit \end{aligned}$$

We end this section with some important definitions.

Definition 1.11. A normed linear space X is *separable* if it contains a countable dense subset. $\diamondsuit$

Example 1.12. $L^p(E)$ and ℓ^p are separable for $1 \le p < \infty$, but not for $p = \infty$. $\diamondsuit$

Definition 1.13. Let $\{x_n\}$ be a sequence in a normed linear space X.

(a) The *finite linear span*, or simply the *span*, of $\{x_n\}$ is the set of all finite linear combinations of elements of $\{x_n\}$, i.e.,

$$\operatorname{span}\{x_n\} \;=\; \biggl\{ \sum_{n=1}^{N} c_n\, x_n : \text{all } N > 0 \text{ and all } c_1, \dots, c_N \in \mathbf{F} \biggr\}.$$

(b) The *closed linear span* of $\{x_n\}$ is the closure in X of the finite linear span, and is denoted $\overline{\operatorname{span}}\{x_n\}$.

(c) $\{x_n\}$ is *complete* (or *total* or *fundamental*) in X if $\overline{\operatorname{span}}\{x_n\} = X$, i.e., if $\operatorname{span}\{x_n\}$ is dense in X. $\diamondsuit$

Corollary 1.41 gives an equivalent characterization of complete sequences.

Note that the term "complete" has two distinct uses: (a) a normed linear space X is complete if every Cauchy sequence in X is convergent, and (b) a sequence $\{x_n\}$ in a normed linear space X is complete if $\operatorname{span}\{x_n\}$ is dense in X. These two distinct uses should always be clear from context.

1.3. Hilbert Spaces. A *Hilbert space* is a Banach space with additional geometric properties. In particular, the norm of a Hilbert space is obtained from an *inner product* that mimics the properties of the dot product of vectors in $\mathbf{R}^n$ or $\mathbf{C}^n$. Recall that the dot product of u, $v \in \mathbf{C}^n$ is defined by

$$u \cdot v \;=\; u_1 \bar{v}_1 + \cdots + u_n \bar{v}_n.$$

Therefore, if we use the Euclidean norm $|v| = (|v_1|^2 + \cdots + |v_n|^2)^{1/2}$, then this norm is related to the dot product by the equation $|v| = (v \cdot v)^{1/2}$. On the other hand, when $p \neq 2$ the norm

$|v|_p = (|v_1|^p + \cdots + |v_n|^p)^{1/p}$ is not obtainable from the dot product of v with itself. In fact, there is *no* way to define a "generalized dot product" $u \cdot v$ which has the same algebraic properties as the usual dot product and which also satisfies $|v|_p = (v \cdot v)^{1/2}$. The essential algebraic properties of the dot product are given in the following definition.

Definition 1.14. A vector space H is an *inner product space* if for each $x, y \in H$ we can define a complex number $\langle x, y\rangle$, called the *inner product* of x and y, so that:

(a) $\langle x, x\rangle$ is real and $\langle x, x\rangle \geq 0$ for each x,

(b) $\langle x, x\rangle = 0$ if and only if $x = 0$,

(c) $\langle y, x\rangle = \overline{\langle x, y\rangle}$, and

(d) $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$.

If $\langle x, y\rangle = 0$, then x and y are said to be *orthogonal*. In this case, we write $x \perp y$.

If H is an inner product space, then it can be shown that $\|x\| = \langle x, x\rangle^{1/2}$ defines a norm for H, called the *induced norm*. Hence all inner product spaces are normed linear spaces. If H is complete in this norm then H is called a *Hilbert space*. Thus Hilbert spaces are those Banach spaces whose norms can be derived from an inner product. ◇

A given Hilbert space may have many possible inner products. We say that two inner products for H are equivalent if the two corresponding induced norms are equivalent (compare Definition 1.4).

Example 1.15. The following are examples of Hilbert spaces.

(a) $L^p(E)$ is a Hilbert space when $p = 2$, but not for $p \neq 2$. For $p = 2$ the inner product is defined by

$$\langle f, g\rangle \;=\; \int_E f(x)\,\overline{g(x)}\,dx.$$

The fact that this integral converges is a consequence of the Cauchy–Schwarz inequality (1.1).

(b) Similarly, ℓ^p is a Hilbert space when $p = 2$, but not for $p \neq 2$. For $p = 2$ the inner product is defined by

$$\big\langle (a_n),\ (b_n)\big\rangle \;=\; \sum_{n=1}^{\infty} a_n\,\overline{b_n}.$$

Again, the convergence of this series is a consequence of the Cauchy–Schwarz inequality (1.1). ◇

The following result generalizes the Cauchy–Schwarz inequality to any Hilbert space, and gives some basic properties of the inner product.

Theorem 1.16. *Let H be a Hilbert space, and let $x, y \in H$.*

(a) (Cauchy–Schwarz Inequality) $|\langle x, y\rangle| \leq \|x\|\,\|y\|$.

(b) $\|x\| = \sup_{\|y\|=1} |\langle x, y\rangle|$.

(c) (Parallelogram Law) $\|x+y\|^2 + \|x-y\|^2 = 2\big(\|x\|^2 + \|y\|^2\big)$.

(d) (Pythagorean Theorem) *If* $\langle x, y\rangle = 0$ *then* $\|x+y\|^2 = \|x\|^2 + \|y\|^2$. ◇

Sequences in a Hilbert space which possess the property that any two distinct elements are orthogonal have a number of mathematically appealing and useful features, which the following result describes. The precise meaning of the infinite series used in Definition 1.17(c) is explained in Definition 2.1.

Definition 1.17. Let $\{x_n\}$ be a sequence in a Hilbert space H.

(a) $\{x_n\}$ is *orthogonal* if $\langle x_m, x_n\rangle = 0$ whenever $m \neq n$.

(b) $\{x_n\}$ is *orthonormal* if $\langle x_m, x_n\rangle = \delta_{mn}$, i.e., if $\{x_n\}$ is orthogonal and $\|x_n\| = 1$ for every n.

(c) $\{x_n\}$ is a *basis* for H if every $x \in H$ can be written $x = \sum_{n=1}^{\infty} c_n x_n$ for a unique choice of scalars c_n.

(d) An orthonormal sequence $\{x_n\}$ is an *orthonormal basis* if it is both orthonormal and a basis. In this case, the unique representation of $x \in H$ in this basis is $x = \sum \langle x, x_n\rangle\, x_n$ (see Theorem 1.20). ◇

Example 1.18. Here are some examples of orthonormal bases.

(a) Consider $H = \ell^2$, and define sequences $e_n = (\delta_{mn})_{m=1}^{\infty} = (0, \dots, 0, 1, 0, \dots)$, where the 1 is in the nth position. Then $\{e_n\}$ is an orthonormal basis for ℓ^2, often called the *standard basis* for ℓ^2.

(b) Consider $H = L^2[0,1]$, the space of functions that are square-integrable on the interval $[0,1]$. Define functions $e_n(x) = e^{2\pi i n x}$, with n ranging through the set $\mathbf{Z}$ of all integers. Then $\{e_n\}_{n\in\mathbf{Z}}$ is an orthonormal basis for H. If $f \in L^2[0,1]$ then the expansion $f = \sum_{n\in\mathbf{Z}} \langle f, e_n\rangle\, e_n$ is called the *Fourier series* of f, and $(\langle f, e_n\rangle)_{n\in\mathbf{Z}}$ is the sequence of *Fourier coefficients* of f. The Fourier coefficients are often denoted by $\hat{f}(n) = \langle f, e_n\rangle = \int_0^1 f(x)\, e^{-2\pi i n x}\, dx$. Note that we are only guaranteed that the Fourier series of f will converge in L^2-norm. There is no guarantee that it will converge pointwise, and indeed, there exist continuous functions whose Fourier series do not converge at every point [Kat68].

We can also regard the functions $e_n(x) = e^{2\pi i n x}$ as being 1-periodic functions defined on the entire real line. In this case, we can again show that $\{e_n\}_{n\in\mathbf{Z}}$ is an orthonormal basis for the Hilbert space $L^2(\mathbf{T})$ consisting of all 1-periodic functions that are square-integrable over a single period, i.e.,

$$L^2(\mathbf{T}) \;=\; \Big\{f\colon \mathbf{R} \to \mathbf{R} \;:\; f(x+1) = f(x) \text{ for all } x, \text{ and } \int_0^1 |f(x)|^2 < \infty\Big\}. \quad ◇$$

In light of Example 1.18(b), if $\{e_n\}$ is an orthonormal basis for an arbitrary Hilbert space H, then the representation $x = \sum \langle x, e_n \rangle\, e_n$ is sometimes called the *generalized Fourier series* of $x \in H$, and $(\langle x, e_n \rangle)$ is called the sequence of *generalized Fourier coefficients.*

Theorem 1.19. *Let $\{x_n\}$ be an orthonormal sequence in a Hilbert space H.*

(a) *The series $x = \sum c_n x_n$ converges if and only if $(c_n) \in \ell^2$. In this case we have the* Plancherel Formula $\|x\|^2 = \sum |c_n|^2$.

(b) *If $x = \sum c_n x_n$ converges then $c_n = \langle x, x_n \rangle$. In particular, $(c_n) = (\langle x, x_n \rangle)$ is the unique choice of coefficients such that $x = \sum c_n x_n$.*

(c) (Bessel Inequality) *If $x \in H$ then $\sum |\langle x, x_n \rangle|^2 \le \|x\|^2$.* $\diamondsuit$

It is tempting to conclude from Theorem 1.19 that if $\{x_n\}$ is any orthonormal sequence in a Hilbert space H, then *every* $x \in H$ can be written $x = \sum \langle x, x_n \rangle\, x_n$. This, however, is not always the case, for there may not be "enough" vectors in the sequence to span all of H. In particular, if $\{x_n\}$ is not complete then its closed span is only a proper closed subspace of H and not all of H. For example, a finite sequence of orthonormal vectors $\{x_1, \dots, x_N\}$ can only span a finite-dimensional subspace of an infinite-dimensional Hilbert space, and therefore cannot be complete in an infinite-dimensional space. As another example, if $\{x_n\}$ is an orthonormal sequence in H then $\{x_{2n}\}$ is also an orthonormal sequence in H. However, x_1 is orthogonal to every x_{2n}, so it follows from Corollary 1.41 that $\{x_{2n}\}$ is incomplete.

The next theorem presents several equivalent conditions which imply that an orthonormal sequence is complete in H.

Theorem 1.20. *Let $\{x_n\}$ be an orthonormal sequence in a Hilbert space H. Then the following statements are equivalent.*

(a) *$\{x_n\}$ is complete in H.*

(b) *$\{x_n\}$ is an orthonormal basis for H.*

(c) (Plancherel Formula) *$\sum |\langle x, x_n \rangle|^2 = \|x\|^2$ for every $x \in H$.*

(d) *$x = \sum \langle x, x_n \rangle\, x_n$ for every $x \in H$.* $\diamondsuit$

Note that this theorem implies that every *complete* orthonormal sequence in a Hilbert space is actually a *basis* for H. This need not be true for nonorthogonal sequences. Indeed, it is easy to construct complete sequences that are not bases. Moreover, we show in Chapter 6 that there exist complete sequences that are finitely linearly independent (i.e., such that no finite linear combination is zero except the trivial combination), yet are not bases.

Suppose that H does have an orthonormal basis $\{x_n\}$. Then

$$E = \left\{ \sum_{n=1}^{N} r_n c_n : N > 0,\ \mathrm{Re}(r_n),\ \mathrm{Im}(r_n) \in \mathbf{Q} \right\}$$

is a countable, dense subset of H, so H is separable. The converse is also true, i.e., every separable Hilbert space does possess an orthonormal basis. Moreover, by mapping one orthonormal basis for one separable Hilbert space onto an orthonormal basis for another separable Hilbert space, it follows that all separable Hilbert spaces are isomorphic. We state this explicitly in the following theorem (see Definitions 1.22 and 1.45 for an explanation of the terms *isomorphic* and *norm-preserving*).

Theorem 1.21. *If H is a Hilbert space, then there exists an orthonormal basis $\{x_n\}$ for H if and only if H is separable. As a consequence, all separable Hilbert spaces are isometrically isomorphic, and in fact are isomorphic to ℓ^2. That is, if H is a separable Hilbert space, then there exists a bijective, norm-preserving mapping S of H onto ℓ^2.* $\diamondsuit$

1.4. Operators. Let X and Y be normed linear spaces. An *operator* is a simply a function $T\colon X \to Y$. If $Y = \mathbf{F}$ is the field of scalars, then an operator $T\colon X \to \mathbf{F}$ is called a *functional* on X. For simplicity, we will write either Tx or $T(x)$ to denote the operator T applied to an element x.

Definition 1.22. Let X and Y be normed linear spaces, and let $T\colon X \to Y$ be an operator.

(a) T is *linear* if $T(ax + by) = aTx + bTy$ for all $x, y \in X$ and all scalars $a, b \in \mathbf{F}$.

(b) T is *injective*, or $1-1$, if $Tx = Ty$ if and only if $x = y$.

(c) The *range*, or *image*, of T is $\mathrm{Range}(T) = T(X) = \{Tx : x \in X\}$.

(d) T is *surjective*, or *onto*, if $\mathrm{Range}(T) = Y$.

(e) T is *bijective* if it is both injective and surjective.

(f) T is *continuous* if $x_n \to x$ in X implies $T(x_n) \to Tx$ in Y.

(g) The *operator norm*, or simply the *norm*, of a linear operator T is

$$\|T\| \;=\; \sup_{\|x\|_X=1} \|Tx\|_Y.$$

T is *bounded* if $\|T\| < \infty$.

(h) T is *norm-preserving*, or *isometric*, if $\|Tx\|_Y = \|x\|_X$ for every $x \in X$.

(i) T is a *functional* if $Y = \mathbf{F}$. $\diamondsuit$

A critical property of linear operators on normed linear spaces is that boundedness and continuity are equivalent!

Theorem 1.23. *Let $T\colon X \to Y$ be a linear operator mapping a normed linear space X into another normed linear space Y. Then:*

$$T \text{ is continuous} \quad\Longleftrightarrow\quad T \text{ is bounded.} \quad \diamondsuit$$

As a consequence of this result, we use the terms *continuous* and *bounded* interchangeably when speaking of linear operators.

1.5. Dual Spaces.

Not all linear functionals on a Banach space X are continuous if X is infinite-dimensional (see Example 4.1). The class of all *continuous* linear functionals on X is especially important. We consider this *dual space* in this section.

Notation 1.24. We often use the symbol x^* to denote a typical continuous linear functional on X. It is important to note that x^* is simply a functional on X, and is not somehow determined from some specific element $x \in X$. That is, x^* is a mapping from X to $\mathbf{F}$, and the value of x^* at an arbitrary $x \in X$ is $x^*(x)$. For continuous linear functionals, we often denote the action of x^* on $x \in X$ by the notation

$$\langle x, x^* \rangle \;=\; x^*(x).$$

With this notation, the linearity of x^* is expressed by the statement

$$\forall\, x, y \in X, \quad \langle ax + by, x^* \rangle \;=\; a\langle x, x^* \rangle + b\langle y, x^* \rangle.$$

Similarly, the continuity of x^* is expressed in this notation by the statement

$$\lim_{n\to\infty} x_n = x \quad \Longrightarrow \quad \lim_{n\to\infty} \langle x_n, x^* \rangle = \langle x, x^* \rangle.$$

Additionally, since since the norm on the scalar field $\mathbf{F}$ is simply absolute value, the operator norm of a linear functional x^* is given in this notation by the formula

$$\|x^*\| \;=\; \sup_{\|x\|_X = 1} |\langle x, x^* \rangle|. \quad \diamondsuit$$

The collection of all continuous linear functionals on X is a key space in Banach space theory.

Definition 1.25. Let X be a normed linear space. Then the *dual space* of X is

$$X^* \;=\; \{x^*\colon X \to F \;:\; x^* \text{ is a continuous linear functional on } X\}. \quad \diamondsuit$$

The dual space of a Banach space is itself Banach space.

Theorem 1.26. *If X is a normed linear space, then its dual space X^* is a Banach space when equipped with the operator norm*

$$\|x^*\|_{X^*} \;=\; \sup_{\|x\|_X = 1} |\langle x, x^* \rangle|. \quad \diamondsuit$$

By definition, the norm of a functional $x^* \in X$ is determined by its evaluations $\langle x, x^* \rangle$ on elements of X. Conversely, the following result states that the norm of an element $x \in X$, can be recovered from the evaluations $\langle x, x^* \rangle$ over functionals in X^*.

Theorem 1.27. *Let X be a Banach space, and let $x \in X$. Then*

$$\|x\|_X \;=\; \sup_{\|x^*\|_{X^*}=1} |\langle x, x^* \rangle|. \quad \diamondsuit$$

In is often difficult to explicitly characterize the dual space X^* of a general Banach space. However, we can characterize the dual spaces of some particular Banach spaces.

Example 1.28. Fix $1 \le p \le \infty$, and define p' by $\frac{1}{p} + \frac{1}{p'} = 1$, where we set $1/0 = \infty$ and $1/\infty = 0$. For each $g \in L^{p'}(E)$, define $\mu_g \colon L^p(E) \to \mathbf{C}$ by

$$\mu_g(f) \;=\; \int_E f(x)\, g(x)\, dx, \qquad f \in L^p(E).$$

Then, by Hölder's Inequality (Theorem 1.7), $|\mu_g(f)| \le \|f\|_{L^p}\, \|g\|_{L^{p'}}$. Therefore, $\|\mu_g\| \le \|g\|_{L^{p'}} < \infty$, so μ_g is a continuous linear functional on $L^p(E)$. In fact, it is easy to show that $\|\mu_g\| = \|g\|_{L^{p'}}$.

Thus, each element $g \in L^{p'}(E)$ determines a continuous linear functional $\mu_g \in (L^p(E))^*$. Further, it can be shown that if $1 \le p < \infty$, then for each continuous linear functional $\mu \in (L^p(E))^*$ there exists a unique function $g \in L^{p'}(E)$ such that $\mu = \mu_g$. Thus, if $1 \le p < \infty$ then every function $g \in L^{p'}(E)$ is associated with a unique continuous linear functional $\mu_g \in (L^p(E))^*$, and conversely. We therefore "identify" the functional μ_g with the function g, and write simply "$\mu_g = g$." The fact that μ_g is a functional on $L^p(E)$ while g is a function in $L^{p'}(E)$ usually causes no confusion, as the meaning is clear from context. In the same way, we write $(L^p(E))^* = L^{p'}(E)$, when we actually mean that $g \mapsto \mu_g$ is an isomorphism between $L^{p'}(E)$ and $(L^p(E))^*$. For the case $p = \infty$, we have $L^1(E) \subset (L^\infty(E))^*$, but we do not have equality.

Similar statements apply to the sequence spaces ℓ^p. In particular, each $y = (y_n) \in \ell^{p'}$ determines a continuous linear functional $\mu_y \in (\ell^p)^*$ by the formula

$$\mu_y(x) \;=\; \sum_n x_n y_n, \qquad x = (x_n) \in \ell^p.$$

If $1 \le p < \infty$ then $(\ell^p)^* = \ell^{p'}$, while $\ell^1 \subset (\ell^\infty)^*$. Moreover, it can be shown that $(c_0)^* = \ell^1$, and therefore $(c_0)^{**} = (\ell^1)^* = \ell^\infty$. $\diamondsuit$

Remark 1.29. Consider again the situation of Example 1.28. If we identify the function $g \in L^{p'}(E)$ with the functional $\mu_g \in (L^p(E))^*$ and use the notation of Notation 1.24, we would write

$$\langle f, g \rangle \;=\; \langle f, \mu_g \rangle \;=\; \mu_g(f) \;=\; \int_E f(x)\, g(x)\, dx. \tag{1.2}$$

Consider in particular the case $p = 2$. We then have $p' = 2$ as well, so in this case, f and g are both elements of $L^2(E)$ in (1.2). Moreover, $L^2(E)$ is a Hilbert space, and therefore has an inner product that is defined by the formula

$$\langle f, g \rangle \;=\; \int_E f(x)\, \overline{g(x)}\, dx, \qquad f, g \in L^2(E). \tag{1.3}$$

Hence there is a conflict of notation between the inner product of f and g and the action of g as a linear functional on f. Therefore, in the case $p = 2$ we usually identify the function g with the functional $\mu_{\bar{g}}$ instead of μ_g, thus preserving the meaning of $\langle \cdot, \cdot \rangle$ as an inner product. As a consequence, if we are dealing with an arbitrary value of p then there is the possibility of confusion in the meaning of $\langle f, g \rangle$, since it might refer to either (1.2) or (1.3). However, the meaning is usually clear from context. An additional problem in the case $p = 2$ is that the identification $I \colon g \mapsto \mu_{\bar{g}}$ is anti-linear, because $I(cg) = \mu_{\overline{cg}} = \bar{c}\mu_{\bar{g}} = \bar{c}I(g)$. Again, this does not cause confusion in practice, and we continue to write $(L^2(E))^* = L^2(E)$. $\diamondsuit$

We have seen that $(L^2(E))^*$ can be identified with $L^2(E)$. The following result states that if H is any Hilbert space, then H^* can be identified with H. In particular, any continuous linear functional on H is formed by taking the inner product with some unique element of H.

Theorem 1.30 (Riesz Representation Theorem). *Let H be a Hilbert space. For each $y \in H$ let μ_y be the functional on H defined by $\mu_y(x) = \langle x, y \rangle$.*

(a) *If $y \in H$ then $\mu_y \in H^*$, i.e., μ_y is a continuous linear functional on H, and $\|\mu_y\| = \|y\|$.*

(b) *If $\mu \in H^*$, i.e., μ is a continuous linear functional on H, then there exists a unique $y \in H$ such that $\mu = \mu_y$.* $\diamondsuit$

Remark 1.31. Thus, there is a $1-1$ correspondence between elements of H and elements of H^*. Therefore, we usually "identify" the element $y \in H$ with the functional $\mu_y \in H^*$. We write simply $y = \mu_y$ and say that y "is" a linear functional on H, when we actually mean that y determines the functional $\mu_y(x) = \langle x, y \rangle$. The fact that y is an element of H while μ_y is a functional on H usually causes no confusion, and the meaning is clear from context. In the same way, we identify H with H^*, and write $H = H^*$. In this sense, all Hilbert spaces are self-dual; this is not true for non-Hilbert spaces. Again, there is the possible source of confusion deriving from the fact that if $\mu_y(x) = \langle x, y \rangle$ then the identification $y \mapsto \mu_y$ is anti-linear (because $\mu_{cy} = \bar{c}\mu_y$). However, this is not a problem in practice. $\diamondsuit$

Since X^* is a Banach space, we can consider its dual space.

Definition 1.32. Since X^* is a Banach space, we can consider its dual space $X^{**} = (X^*)^*$. Each element $x \in X$ determines an element $\pi(x) \in X^{**}$ by the formula $\langle x^*, \pi(x) \rangle = \langle x, x^* \rangle$ for $x^* \in X^*$. This mapping $\pi \colon X \to X^{**}$ is called the *canonical embedding* of X into X^{**}, since it identifies X with a subspace $\pi(X) \subset X^{**}$. If π is a bijection then we write $X = X^{**}$ and say that X is *reflexive.* $\diamondsuit$

Example 1.33. $L^p(E)$ and ℓ^p are reflexive if $1 < p < \infty$, but not for $p = 1$ or $p = \infty$. $\diamondsuit$

1.6. Adjoints. The duality between Banach spaces and their dual spaces allows us to define the "dual" of an operator $S\colon X \to Y$.

Definition 1.34. Let X and Y be Banach spaces, and let $S\colon X \to Y$ be a bounded linear operator. Fix $y^* \in Y^*$, and define a functional $x^*\colon X \to \mathbf{F}$ by

$$\langle x, x^* \rangle \;=\; \langle Sx, y^* \rangle, \qquad x \in X.$$

Then x^* is linear since S and y^* are linear. Further,

$$|\langle x, x^* \rangle| \;=\; |\langle Sx, y^* \rangle| \;\le\; \|Sx\|_Y \, \|y^*\|_{Y^*},$$

so

$$\|x^*\| \;=\; \sup_{\|x\|_X=1} |\langle x, x^* \rangle| \;\le\; \|y^*\|_{Y^*} \sup_{\|x\|_X=1} \|Sx\|_Y \;=\; \|y^*\|_{Y^*} \, \|S\| \;<\; \infty. \tag{1.4}$$

Hence x^* is bounded, so $x^* \in X^*$. Thus, for each $y^* \in Y^*$ we have defined a functional $x^* \in X^*$. Therefore, we can define an operator $S^*\colon Y^* \to X^*$ by setting $S^*(y^*) = x^*$. This mapping S^* is linear, and by (1.4) we have

$$\|S^*\| \;=\; \sup_{\|y^*\|_{Y^*}=1} \|S^*(y^*)\|_{X^*} \;=\; \sup_{\|y^*\|_{Y^*}=1} \|x^*\|_{X^*} \;\le\; \sup_{\|y^*\|_{Y^*}=1} \|y^*\|_{Y^*} \, \|S\| \;=\; \|S\|.$$

In fact, it is true that $\|S^*\| = \|S\|$. This operator S^* is called the *adjoint* of S.

The fundamental property of the adjoint can be restated as follows: $S^* : Y^* \to X^*$ is the unique mapping which satisfies

$$\forall\, x \in X, \quad \forall\, y^* \in Y^*, \quad \langle Sx, y^* \rangle \;=\; \langle x, S^*(y^*) \rangle. \quad \diamondsuit \tag{1.5}$$

Definition 1.35. Assume that $X = H$ and $Y = K$ are Hilbert spaces. Then $H = H^*$ and $K = K^*$. Therefore, if $S\colon H \to K$ then its adjoint S^* maps K back to H. Moreover, by (1.5), the adjoint $S^*\colon K \to H$ is the unique mapping which satisfies

$$\forall\, x \in H, \quad \forall\, y \in K, \quad \langle Sx, y \rangle \;=\; \langle x, S^*y \rangle. \quad \diamondsuit \tag{1.6}$$

We make the following further definitions specifically for operators $S\colon H \to H$ which map a Hilbert space H into itself.

Definition 1.36. Let H be a Hilbert space.

(a) $S\colon H \to H$ is *self-adjoint* if $S = S^*$. By (1.6),

$$S \text{ is self-adjoint} \qquad \Longleftrightarrow \qquad \forall\, x, y \in H, \quad \langle Sx, y \rangle \;=\; \langle x, Sy \rangle.$$

It can be shown that if S is self-adjoint, then $\langle Sx, x \rangle$ is real for every x, and

$$\|S\| \;=\; \sup_{\|x\|=1} |\langle Sx, x \rangle|.$$

(b) $S\colon H \to H$ is *positive*, denoted $S \ge 0$, if $\langle Sx, x \rangle$ is real and $\langle Sx, x \rangle \ge 0$ for every $x \in H$. It can be shown that a positive operator on a complex Hilbert space is self-adjoint.

(c) $S: H \to H$ is *positive definite*, denoted $S > 0$, if $\langle Sx, x\rangle$ is real and $\langle Sx, x\rangle > 0$ for every $x \neq 0$.

(d) If $S, T: H \to H$, then we write $S \geq T$ if $S - T \geq 0$. Similarly, $S > T$ if $S - T > 0$. ♢

As an example, consider the finite-dimensional Hilbert spaces $H = \mathbf{C}^n$ and $K = \mathbf{C}^m$. A linear operator $S: \mathbf{C}^n \to \mathbf{C}^m$ is simply an $m \times n$ matrix with complex entries, and its adjoint $S^*: \mathbf{C}^m \to \mathbf{C}^n$ is simply the $n \times m$ matrix given by the conjugate transpose of S. In this case, the matrix S^* is often called the *Hermitian* of the matrix S.

1.7. The Hahn–Banach Theorem. In this section we list several extremely useful theorems about Banach spaces. Our statements of these results are adapted from [RS80], where they are stated for the case of complex scalars, and [Roy68], where they are stated for real scalars.

The following result is fundamental.

Theorem 1.37 (Hahn–Banach). *Let X be a vector space, and let p be a real-valued function on X such that*

$$\forall\, x, y \in X, \quad \forall\, a, b \in \mathbf{C}, \quad |a| + |b| = 1 \implies p(ax + by) \leq |a|\, p(x) + |b|\, p(y).$$

Let λ be a linear functional on a subspace Y of X, and suppose that λ satisfies

$$\forall\, x \in Y, \quad |\lambda(x)| \leq p(x).$$

Then there exists a linear functional Λ on X such that

$$\forall\, x \in X, \quad |\Lambda(x)| \leq p(x) \qquad \text{and} \qquad \forall\, x \in Y, \quad \Lambda(x) = \lambda(x). \quad \diamondsuit$$

The following corollaries of the Hahn–Banach theorem are often more useful in practice than Theorem 1.37 itself. Therefore, they are often referred to individually as "the" Hahn–Banach Theorem, even though they are only consequences of Theorem 1.37.

Corollary 1.38. *Let X be a normed linear space, let Y be a subspace of X, and let $\lambda \in Y^*$. Then there exists $\Lambda \in X^*$ such that*

$$\forall\, x \in Y, \quad \langle x, \Lambda\rangle = \langle x, \lambda\rangle \qquad \text{and} \qquad \|\Lambda\|_{X^*} = \|\lambda\|_{Y^*}. \quad \diamondsuit$$

Corollary 1.39. *Let X be a normed linear space, and let $y \in X$. Then there exists $\Lambda \in X^*$ such that*

$$\langle y, \Lambda\rangle = \|\Lambda\|_{X^*}\, \|y\|_X.$$

In particular, there exists $\Lambda \in X^*$ such that

$$\|\Lambda\|_{X^*} = 1 \qquad \text{and} \qquad \langle y, \Lambda\rangle = \|y\|_X. \quad \diamondsuit$$

Corollary 1.40. *Let Z be a subspace of a normed linear space X, and let $y \in X$. Let $d = dist(y,Z) = \inf_{z \in Z} \|y-z\|_X$. Then there exists $\Lambda \in X^*$ such that:*

(a) $\|\Lambda\|_{X^*} \le 1$,

(b) $\langle y, \Lambda\rangle = d$,

(c) $\forall\, z \in Z, \quad \langle z, \Lambda\rangle = 0. \quad \diamondsuit$

We will have occasion to use the following corollary often.

Corollary 1.41. *Let X be a Banach space. Then $\{x_n\} \subset X$ is complete if and only if the only $x^* \in X^*$ satisfying $\langle x_n, x^*\rangle = 0$ for all n is $x^* = 0$.*

Proof. $\Rightarrow$. Suppose that $\{x_n\}$ is complete, i.e., $\overline{\text{span}}\{x_n\} = X$, and suppose that $x^* \in X^*$ satisfies $\langle x_n, x^*\rangle = 0$ for all n. Since x^* is linear, we therefore have $\langle x, x^*\rangle = 0$ for every $x = \sum_{n=1}^N c_n x_n \in \text{span}\{x_n\}$. However, x^* is continuous, so this implies $\langle x, x^*\rangle = 0$ for every $x \in \overline{\text{span}}\{x_n\} = X$. Hence x^* is the zero functional.

$\Leftarrow$. Suppose now that the only $x^* \in X^*$ satisfying $\langle x_n, x^*\rangle = 0$ for every n is $x^* = 0$. Define $Z = \overline{\text{span}}\{x_n\}$, and suppose that $Z \ne X$. Then we can find an element $y \in X$ such that $y \notin Z$. Since Z is a closed subset of X, we therefore have $d = \text{dist}(y,Z) > 0$. By the Hahn–Banach Theorem (Corollary 1.40), there exists a functional $\Lambda \in X^*$ satisfying $\langle y, \Lambda\rangle = d \ne 0$ and $\langle z, \Lambda\rangle = 0$ for every $z \in Z$. However, this implies that $\langle x_n, \Lambda\rangle = 0$ for every n. By hypothesis, Λ must then be the zero functional, contradicting the fact that $\langle y, \Lambda\rangle \ne 0$. Hence, we must in fact have that $Z = X$, so $\{x_n\}$ is complete in X. $\square$

If H is a Hilbert space then $H^* = H$. Therefore, Corollary 1.41 implies that a sequence $\{x_n\}$ in a Hilbert space H is complete in H if and only if the only element $y \in H$ satisfying $\langle x_n, y\rangle = 0$ for all n is $y = 0$.

Next we list several related major results.

Theorem 1.42 (Uniform Boundedness Principle). *Let X be a Banach space and let Y be a normed linear space. Let $\{T_\gamma\}_{\gamma\in\Gamma}$ be a family of bounded linear operators mapping X into Y. Then,*

$$\Bigl(\forall\, x \in X, \quad \sup_{\gamma\in\Gamma} \|T_\gamma(x)\|_Y < \infty\Bigr) \quad\implies\quad \sup_{\gamma\in\Gamma} \|T_\gamma\| < \infty. \quad \diamondsuit$$

Theorem 1.43 (Open Mapping Theorem). *Let $T\colon X \to Y$ be a bounded linear operator from a Banach space X onto another Banach space Y. Then $T(U) = \{T(x) : x \in U\}$ is an open set in Y whenever U is an open set in X.* $\diamondsuit$

Theorem 1.44 (Inverse Mapping Theorem). *A continuous bijection $T\colon X \to Y$ of one Banach space X onto another Banach space Y has a continuous inverse $T^{-1}\colon Y \to X$.* $\diamondsuit$

Definition 1.45. A *topological isomorphism* between Banach spaces X and Y is a linear bijection $S\colon X \to Y$ that is continuous. If S is a norm-preserving in addition, then we say that S is an *isometric isomorphism*. Banach spaces X and Y are *isomorphic* if there exists a topological isomorphism mapping X onto Y. $\diamondsuit$

For example, it can be shown that every finite-dimensional Banach space is topologically isomorphic to $\mathbf{C}^n$ for some n if the scalars are complex, or to $\mathbf{R}^n$ for some n if the scalars are real.

By the Inverse Mapping Theorem (Theorem 1.44), a topological isomorphism must have a continuous inverse. For simplicity, topological isomorphisms are sometimes simply called "isomorphisms" or even just "invertible mappings."

Theorem 1.46 (Closed Graph Theorem). *Let $T\colon X \to Y$ be a linear mapping of one Banach space X onto another Banach space Y. Then T is bounded if and only if*

$$\operatorname{graph}(T) \;=\; \{(x,y) \in X \times Y : y = T(x)\}$$

is a closed set in $X \times Y$. That is, T is bounded if and only if for each $\{x_n\} \subset X$ we have

$$\Big(x_n \to x \;\text{ and }\; T(x_n) \to y\Big) \qquad \Longrightarrow \qquad y = T(x). \quad \diamondsuit$$

1.8. Weak Convergence. In this section we discuss some types of "weak convergence" that we will make use of in Chapter 10.

Definition 1.47. Let X be a Banach space.

(a) A sequence $\{x_n\}$ of elements of X *converges* to $x \in X$ if Definition 1.2(a) holds, i.e., if $\lim_{n\to\infty} \|x - x_n\| = 0$. For emphasis, we sometimes refer to this type of convergence as *strong convergence* or *norm convergence*.

(b) A sequence $\{x_n\}$ of elements of X *converges weakly* to $x \in X$ if

$$\forall\, x^* \in X^*, \quad \lim_{n\to\infty} \langle x_n, x^*\rangle \;=\; \langle x, x^*\rangle.$$

In this case, we say that $x_n \to x$ *weakly*.

(c) A sequence $\{x_n^*\}$ of functionals in X^* *converges weak** to $x^* \in X^*$ if

$$\forall\, x \in X, \quad \lim_{n\to\infty} \langle x, x_n^*\rangle \;=\; \langle x, x^*\rangle.$$

In this case, we say that $x_n \to x$ *weak**, or *in the weak* topology*. $\diamondsuit$

Note that weak* convergence only applies to convergence of functionals in a dual space X^*. However, since X^* is itself a Banach space, we could consider strong or weak convergence of functionals in X^* as well as weak* convergence of these functionals. In particular, if X is reflexive then $X = X^{**}$, and therefore $x_n^* \to x^*$ weakly in X^* if and only if $x_n^* \to x^*$ weak* in X^*. For general Banach spaces, we have the following implications.

Lemma 1.48. *Let X be a Banach space.*

(a) *Strong convergence in X implies weak convergence in X.*

(b) *Weak convergence in X^* implies weak* convergence in X^*.*

Proof. (a) Suppose that x_n, $x \in X$ and that $x_n \to x$ strongly. Fix any $x^* \in X^*$. Since x^* is continuous, we have $\lim_{n\to\infty} \langle x_n, x^* \rangle = \langle x, x^* \rangle$, so $x_n \to x$ weakly by definition.

(b) Suppose that x_n^*, $x^* \in X^*$ and that $x_n^* \to x^*$ weakly. Let $x \in X$. Then $\pi(x) \in X^{**}$, where $\pi \colon X \to X^{**}$ is the canonical embedding of X into X^{**} defined in Definition 1.32. By definition of weak convergence, we have $\lim_{n\to\infty} \langle x_n^*, x^{**} \rangle = \langle x^*, x^{**} \rangle$ for every $x^{**} \in X^{**}$. Taking $x^{**} = \pi(x)$ in particular, we therefore have

$$\lim_{n\to\infty} \langle x, x_n^* \rangle \;=\; \lim_{n\to\infty} \langle x_n^*, \pi(x) \rangle \;=\; \langle x^*, \pi(x) \rangle \;=\; \langle x, x^* \rangle.$$

Thus $x_n^* \to x^*$ in the weak* topology. $\square$

It is easy to see that strongly convergent sequences are norm-bounded above. It is more difficult to prove that the same is true of weakly convergent sequences.

Lemma 1.49. *All weakly convergent sequences are norm-bounded above. That is, if $\{x_n\} \subset X$ and $x_n \to x \in X$ weakly, then* $\sup \|x_n\| < \infty$. $\diamondsuit$

Strong, weak, and weak* convergence can all be defined in terms of topologies on X or X^*. For example, the strong topology is defined by the norm $\|\cdot\|$ on X. The weak topology on X is defined by the family of seminorms $p_{x^*}(x) = |\langle x, x^* \rangle|$, with x^* ranging through X^*. The weak* topology on X^* is defined by the family of seminorms $p_x(x^*) = |\langle x, x^* \rangle|$, with x ranging through X. These are only three specific examples of topologies on a Banach space X. There are many other topologies that are useful in specific applications. Additionally, there are many other useful vector spaces which are not Banach spaces, but for which topologies can still be defined. We shall not deal with such *topological vector spaces*, but instead refer the reader to [Con85] or [Hor66] for more information.

II. CONVERGENCE OF SERIES

2. UNCONDITIONAL CONVERGENCE OF SERIES IN BANACH SPACES

Definition 2.1. Let $\{x_n\}$ be a sequence in a Banach space X.

(a) The series $\sum x_n$ is *convergent* and equals $x \in X$ if the partial sums $s_N = \sum_{n=1}^{N} x_n$ converge to x in the norm of X, i.e., if

$$\forall\, \varepsilon > 0, \quad \exists\, N_0 > 0, \quad \forall\, N \geq N_0, \quad \|x - s_N\| \;=\; \left\| x - \sum_{n=1}^{N} x_n \right\| \;<\; \varepsilon.$$

(b) The series $\sum x_n$ is *Cauchy* if the sequence $\{s_N\}$ of partial sums is a Cauchy sequence in X, i.e., if

$$\forall\, \varepsilon > 0, \quad \exists\, N_0 > 0, \quad \forall\, N > M \geq N_0, \quad \|s_N - s_M\| \;=\; \left\| \sum_{n=M+1}^{N} x_n \right\| \;<\; \varepsilon.$$

Since X is a Banach space, a series $\sum x_n$ converges if and only if it is a Cauchy series. $\diamondsuit$

Here are some additional, more restrictive, types of convergence of series.

Definition 2.2. Let $\{x_n\}$ be a sequence in a Banach space X.

(a) A series $\sum x_n$ is *unconditionally convergent* if $\sum x_{\sigma(n)}$ converges for every permutation σ of $\mathbf{N}$.

(b) A series $\sum x_n$ is *absolutely convergent* if $\sum \|x_n\| < \infty$. $\diamondsuit$

Although Definition 2.2 does not require that $\sum x_{\sigma(n)}$ must converge to the same value for every permutation σ, we show in Corollary 2.9 that this is indeed the case.

If (c_n) is a sequence of real or complex numbers, then $\sum c_n$ converges unconditionally if and only if it converges absolutely (Lemma 2.3). In a general Banach space, it is true that absolute convergence implies unconditional convergence (Lemma 2.4), but the converse is not always true (Example 2.5). In fact, it can be shown that unconditional convergence is equivalent to absolute convergence only for finite-dimensional Banach spaces.

Lemma 2.3. [Rud64, p. 68]. *Let (c_n) be a sequence of real or complex scalars. Then,*

$$\sum_n c_n \text{ converges absolutely} \quad \Longleftrightarrow \quad \sum_n c_n \text{ converges unconditionally.}$$

Proof. $\Rightarrow$. Suppose that $\sum |c_n| < \infty$, and choose any $\varepsilon > 0$. Then there exists $N_0 > 0$ such that $\left|\sum_{n=M+1}^{N} c_n\right| < \varepsilon$ whenever $N > M \geq N_0$. Let σ be any permutation of $\mathbf{N}$, and let

$$N_1 \;=\; \max\{\sigma^{-1}(1), \dots, \sigma^{-1}(N_0)\}.$$

Suppose that $N > M \geq N_1$. If $M+1 \leq n \leq N$, then $n > N_1$. Therefore $n \neq \sigma^{-1}(1), \dots, \sigma^{-1}(N_0)$, so $\sigma(n) \neq 1, \dots, N_0$. Hence $\sigma(n) > N_0$. In particular, $K = \min\{\sigma(M+1), \dots, \sigma(N)\} > N_0$ and $L = \max\{\sigma(M+1), \dots, \sigma(N)\} \geq K$, so

$$\left|\sum_{n=M+1}^{N} c_{\sigma(n)}\right| \leq \sum_{n=M+1}^{N} |c_{\sigma(n)}| \leq \sum_{n=K}^{L} |c_n| < \varepsilon.$$

Hence $\sum c_{\sigma(n)}$ is a Cauchy series of scalars, and therefore must converge.

$\Leftarrow$. Suppose first that $\sum c_n$ is a sequence of *real* scalars that does not converge absolutely. Let (p_n) be the sequence of nonnegative terms of (c_n) in order, and let (q_n) be the sequence of negative terms of (c_n) in order. If $\sum p_n$ and $\sum q_n$ both converge, then it is easy to see that $\sum |c_n|$ converges and equals $\sum p_n - \sum q_n$, which is a contradiction. Hence either $\sum p_n$ or $\sum q_n$ must diverge.

Suppose that $\sum p_n$ diverges. Since $p_n \geq 0$ for every n, there must exist an $m_1 > 0$ such that

$$p_1 + \cdots + p_{m_1} > 1.$$

Then, there must exist an $m_2 > m_1$ such that

$$p_1 + \cdots + p_{m_1} - q_1 + p_{m_1+1} + \cdots + p_{m_2} > 2.$$

Continuing in this way, we see that

$$p_1 + \cdots + p_{m_1} - q_1 + p_{m_1+1} + \cdots + p_{m_2} - q_2 + \cdots$$

is a rearrangement of $\sum c_n$ which diverges. Hence $\sum c_n$ cannot converge unconditionally. A similar proof applies if $\sum q_n$ diverges.

Thus we have shown, by a contrapositive argument, that if $\sum c_n$ is a series of real scalars that converges unconditionally, then it must converge absolutely. Suppose now that $\sum c_n$ is a series of complex scalars that converges unconditionally. We will show that the real part and the imaginary part of $\sum c_n$ must each converge unconditionally as well. Write $c_n = a_n + ib_n$, and let σ be any permutation of $\mathbf{N}$. Then $c = \sum c_{\sigma(n)}$ must converge. Write $c = a + ib$. Then $\left|a - \sum_{n=1}^{N} a_{\sigma(n)}\right| \leq \left|c - \sum_{n=1}^{N} c_{\sigma(n)}\right|$, so $a = \sum a_{\sigma(n)}$ converges. Since this is true for every permutation σ, the series $\sum a_n$ must converge unconditionally. Since this is a series of real scalars, it therefore must converge absolutely. Similarly, $\sum b_n$ must converge absolutely. Hence, $\sum |c_n| = \sum |a_n + ib_n| \leq \sum |a_n| + \sum |b_n| < \infty$, so $\sum c_n$ converges absolutely. □

Lemma 2.4. *Let $\{x_n\}$ be a sequence of elements of a Banach space X. If $\sum x_n$ converges absolutely then it converges unconditionally.*

Proof. Assume that $\sum \|x_n\| < \infty$. If $M < N$, then

$$\left\|\sum_{n=M+1}^{N} x_n\right\| \leq \sum_{n=M+1}^{N} \|x_n\|.$$

Since $\sum \|x_n\|$ is a Cauchy series of real numbers, it follows that $\sum x_n$ is a Cauchy series in X. Hence $\sum x_n$ must converge in X. Moreover, we can repeat this argument for any permutation σ of $\mathbf{N}$ since we always have $\sum \|x_{\sigma(n)}\| < \infty$ by Lemma 2.3. Therefore $\sum x_n$ is unconditionally convergent. □

Example 2.5. Let $\{e_n\}$ be an infinite orthonormal sequence in an infinite-dimensional Hilbert space H. Then by Theorem 1.19(a), $\sum c_n e_n$ converges if and only if $\sum |c_n|^2 < \infty$. However, by Lemma 2.3, this occurs if and only if $\sum |c_{\sigma(n)}|^2 < \infty$ for every permutation σ of $\mathbf{N}$. Since $\{e_{\sigma(n)}\}$ is also an orthonormal sequence, this implies that $\sum c_n e_n$ converges if and only if it converges unconditionally, and that this occurs exactly for $(c_n) \in \ell^2$.

On the other hand, since $\|e_n\| = 1$, we have that $\sum c_n e_n$ converges absolutely if and only if $\sum |c_n| < \infty$. Hence absolute convergence holds exactly for $(c_n) \in \ell^1$. Since ℓ^1 is a proper subset of ℓ^2, it follows that there are series $\sum c_n e_n$ which converge unconditionally but not absolutely. ♢

Note that in this example, we were able to exactly characterize the collection of coefficients (c_n) such that $\sum c_n e_n$ converges, because we knew that $\{e_n\}$ was an orthonormal sequence in a Hilbert space. For arbitrary sequences $\{x_n\}$ in Hilbert or Banach spaces, it is usually much more difficult to characterize explicitly those coefficients (c_n) such that $\sum c_n x_n$ converges or converges unconditionally.

Next, we define another restricted form of convergence of the series $\sum x_n$. We will see in Theorem 2.8 that this notion of convergence is equivalent to unconditional convergence.

Definition 2.6. The finite subsets of $\mathbf{N}$ form a net when ordered by inclusion. We can therefore define a convergence notion with respect to the net. If $\{\sum_{n\in F} x_n : \text{all finite } F \subset \mathbf{N}\}$ has a limit with respect to this net of finite subsets of $\mathbf{N}$, then we denote the limit by $\lim_F \sum_{n\in F} x_n$. To be precise, $x = \lim_F \sum_{n\in F} x_n$ exists if and only if

$$\forall\, \varepsilon > 0, \quad \exists \text{ finite } F_0 \subset \mathbf{N}, \quad \forall \text{ finite } F \supset F_0, \quad \Bigl\| x - \sum_{n\in F} x_n \Bigr\| < \varepsilon. \quad \diamondsuit$$

Proposition 2.7. *If $x = \lim_F \sum_{n\in F} x_n$ exists, then $\sum x_n$ is convergent and $x = \sum x_n$.*

Proof. Suppose $x = \lim_F \sum_{n\in F} x_n$ exists, and choose $\varepsilon > 0$. Then there is a finite set $F_0 \subset \mathbf{N}$ such that

$$\forall \text{ finite } F \supset F_0, \quad \Bigl\| x - \sum_{n\in F} x_n \Bigr\| \;<\; \varepsilon.$$

Let $N_0 = \max(F_0)$. Then, if $N > N_0$ then $F_0 \subset \{1,\dots,N\}$. Hence $\|x - \sum_{n=1}^{N} x_n\| < \varepsilon$, so $x = \sum x_n$. □

We collect in the following result several equivalent definitions of unconditional convergence, including the fact that $\lim_F \sum_{n\in F} x_n$ exists if and only if $\sum x_n$ converges unconditionally.

Theorem 2.8. *Given a sequence $\{x_n\}$ in a Banach space X, the following statements are equivalent.*

(a) *$\sum x_n$ converges unconditionally.*

(b) *$\lim_F \sum_{n\in F} x_n$ exists.*

(c) $\forall\, \varepsilon > 0, \quad \exists\, N > 0, \quad \forall$ *finite* $F \subset \mathbf{N}, \quad \min(F) > N \implies \|\sum_{n\in F} x_n\| < \varepsilon.$

(d) $\sum x_{n_j}$ *converges for every increasing sequence* $0 < n_1 < n_2 < \cdots$.

(e) $\sum \varepsilon_n x_n$ *converges for every choice of signs* $\varepsilon_n = \pm 1$.

(f) $\sum \lambda_n x_n$ *converges for every bounded sequence of scalars* (λ_n).

(g) $\sum |\langle x_n, x^* \rangle|$ *converges uniformly with respect to the unit ball* $\{x^* \in X^* : \|x^*\| \le 1\}$ *in* X^*. *That is,*

$$\lim_{N\to\infty} \sup \left\{ \sum_{n=N}^{\infty} |\langle x_n, x^* \rangle| \,:\, x^* \in X^*,\, \|x^*\| \le 1 \right\} \;=\; 0.$$

Proof. (a) $\Rightarrow$ (b). Suppose $x = \sum x_n$ converges unconditionally but that $\lim_F \sum_{n\in F} x_n$ does not exist. Then there is an $\varepsilon > 0$ such that

$$\forall \text{ finite } F_0, \quad \exists \text{ finite } F \supset F_0 \text{ such that } \left\| x - \sum_{n\in F} x_n \right\| \;\ge\; \varepsilon. \tag{2.1}$$

Since $\sum x_n$ converges, there is an integer $M_1 > 0$ such that

$$\forall\, N \ge M_1, \quad \left\| x - \sum_{1}^{N} x_n \right\| \;<\; \frac{\varepsilon}{2}.$$

Define $F_1 = \{1, \dots, M_1\}$. Then, by (2.1), there is a $G_1 \supset F_1$ such that $\|x - \sum_{n\in G_1} x_n\| \ge \varepsilon$. Let M_2 be the largest integer in G_1 and let $F_2 = \{1, \dots, M_2\}$. Continuing in this way we obtain a sequence of finite sets $F_1 \subset G_1 \subset F_2 \subset G_2 \subset \cdots$ such that

$$\left\| x - \sum_{n\in F_N} x_n \right\| \;<\; \frac{\varepsilon}{2} \qquad \text{and} \qquad \left\| x - \sum_{n\in G_N} x_n \right\| \;\ge\; \varepsilon.$$

Hence

$$\begin{aligned} \left\| \sum_{n\in G_N \setminus F_N} x_n \right\| &= \left\| \sum_{n\in G_N} x_n \;-\; \sum_{n\in F_N} x_n \right\| \\ &\ge \left\| x - \sum_{n\in G_N} x_n \right\| \;-\; \left\| x - \sum_{n\in F_N} x_n \right\| \;\ge\; \varepsilon \;-\; \frac{\varepsilon}{2} \;=\; \frac{\varepsilon}{2}. \end{aligned}$$

Therefore, we must have $F_N \ne G_N$, so $|F_N| < |G_N|$. Let σ be any permutation of $\mathbf{N}$ obtained by enumerating in turn the elements of F_1, then $G_1 \setminus F_1$, then $F_2 \setminus G_1$, then $G_2 \setminus F_2$, etc. Then for each N we have

$$\left\| \sum_{n=|F_N|+1}^{|G_N|} x_{\sigma(n)} \right\| \;=\; \left\| \sum_{n\in G_N \setminus F_N} x_n \right\| \;\ge\; \frac{\varepsilon}{2}.$$

Since $|F_N|, |G_N| \to \infty$ as N increases, we see that $\sum x_{\sigma(n)}$ is not Cauchy, and hence not convergent, a contradiction.

(b) ⇒ (c). Suppose $x = \lim_F \sum_{n\in F} x_n$ exists, and choose $\varepsilon > 0$. By definition, there must be a finite set $F_0 \subset \mathbf{N}$ such that

$$\forall \text{ finite } F \supset F_0, \qquad \Bigl\| x - \sum_{n\in F} x_n \Bigr\| \; < \; \frac{\varepsilon}{2}.$$

Let $N = \max(F_0)$, and suppose we have a finite $G \subset \mathbf{N}$ with $\min(G) > N$. Then since $F_0 \cap G = \emptyset$,

$$\begin{aligned}
\Bigl\| \sum_{n\in G} x_n \Bigr\| \; &= \; \Bigl\| \Bigl(x - \sum_{n\in F_0} x_n \Bigr) \; - \; \Bigl(x - \sum_{n\in F_0\cup G} x_n \Bigr) \Bigr\| \\
&\le \; \Bigl\| x - \sum_{n\in F_0} x_n \Bigr\| \; + \; \Bigl\| x - \sum_{n\in F_0\cup G} x_n \Bigr\| \\
&< \; \frac{\varepsilon}{2} \; + \; \frac{\varepsilon}{2} \; = \; \varepsilon.
\end{aligned}$$

Therefore statement (c) holds.

(c) ⇒ (a). Assume that statement (c) holds, and let σ be any permutation of $\mathbf{N}$. We need only show that $\sum x_{\sigma(n)}$ is Cauchy. So, choose $\varepsilon > 0$, and let N be the number whose existence is implied by statement (c). Define

$$N_0 \; = \; \max\{\sigma^{-1}(1), \dots, \sigma^{-1}(N)\}.$$

Assume that $L > K \ge N_0$, and set $F = \{\sigma(K+1), \dots, \sigma(L)\}$. Then

$$\min(F) \; = \; \min\{\sigma(K+1), \dots, \sigma(L)\} \; > \; N,$$

since if $k \ge K+1$ then $k > N_0$, so $k \ne \sigma^{-1}(1), \dots, \sigma^{-1}(N)$ and therefore $\sigma(k) \ne 1, \dots, N$. Hypothesis (c) therefore implies that

$$\Bigl\| \sum_{n=K+1}^{L} x_{\sigma(n)} \Bigr\| \; = \; \Bigl\| \sum_{n\in F} x_n \Bigr\| \; < \; \varepsilon,$$

so $\sum x_{\sigma(n)}$ is Cauchy and therefore must converge.

(c) ⇒ (d). Assume that statement (c) holds, and let $0 < n_1 < n_2 < \cdots$ be any increasing set of integers. We will show that $\sum x_{n_i}$ is Cauchy, hence convergent. Given $\varepsilon > 0$ let N be the number whose existence is implied by statement (c). Let j be such that $n_j > N$. If $\ell > k \ge j$ then

$$\min\{n_{k+1}, \dots, n_\ell\} \; \ge \; n_j \; > \; N,$$

so statement (c) implies $\bigl\| \sum_{i=k+1}^{\ell} x_{n_i} \bigr\| < \varepsilon$, as desired.

(c) ⇒ (g). Assume that statement (c) holds, and choose $\varepsilon > 0$. Let N be the integer whose existence is guaranteed by statement (c). Given $L \ge K > N$ and any $x^* \in X^*$ with $\|x^*\| \le 1$, define

$$\begin{aligned}
F^+ \; &= \; \{n \in \mathbf{N} : K \le n \le L \text{ and } \operatorname{Re}(\langle x_n, x^* \rangle) \ge 0\}, \\
F^- \; &= \; \{n \in \mathbf{N} : K \le n \le L \text{ and } \operatorname{Re}(\langle x_n, x^* \rangle) < 0\}.
\end{aligned}$$

Note that $\min(F^+) \geq K > N$, so

$$\begin{aligned}\sum_{n\in F^+} |\mathrm{Re}(\langle x_n, x^*\rangle)| \;&=\; \mathrm{Re}\biggl(\sum_{n\in F^+} \langle x_n, x^*\rangle\biggr) \\ &=\; \mathrm{Re}\biggl(\biggl\langle \sum_{n\in F^+} x_n,\; x^*\biggr\rangle\biggr) \\ &\leq\; \biggl|\biggl\langle \sum_{n\in F^+} x_n,\; x^*\biggr\rangle\biggr| \;\leq\; \|x^*\| \; \biggl\|\sum_{n\in F^+} x_n\biggr\| \;<\; \varepsilon.\end{aligned}$$

A similar inequality holds for F^-, so $\sum_{n=K}^{L} |\mathrm{Re}(\langle x_n, x^*\rangle)| < 2\varepsilon$. Working then with the imaginary parts, we obtain $\sum_{n=K}^{L} |\langle x_n, x^*\rangle| < 4\varepsilon$. Letting $L \to \infty$, we conclude that

$$K \;>\; N \quad\Longrightarrow\quad \sup\biggl\{\sum_{n=K}^{\infty} |\langle x_n, x^*\rangle| : x^* \in X^*,\, \|x^*\| \leq 1\biggr\} \;\leq\; 4\varepsilon,$$

from which statement (g) follows.

(d) $\Rightarrow$ (c) *and* (a) $\Rightarrow$ (c). Assume that statement (c) does not hold. Then there exists an $\varepsilon > 0$ such that for each $N \in \mathbf{N}$ there exists a finite set of integers F_N such that $\min(F_N) > N$ yet $\bigl\|\sum_{n\in F_N} x_n\bigr\| \geq \varepsilon$.

Let $G_1 = F_1$ and $N_1 = \max(G_1)$. Then let $G_2 = F_{N_1}$ and $N_2 = \max(G_2)$. Continuing in this way, we obtain a sequence of finite sets G_K such that for each K,

$$\max(G_K) \;<\; \min(G_{K+1}) \qquad \text{and} \qquad \biggl\|\sum_{n\in G_K} x_n\biggr\| \;\geq\; \varepsilon. \tag{2.2}$$

Now let $0 < n_1 < n_2 < \cdots$ be the complete listing of $\bigcup G_K$. It is clear then from (2.2) that $\sum x_{n_j}$ is not Cauchy, hence not convergent, so statement (d) does not hold.

Finally, let σ be any permutation of $\mathbf{N}$ obtained by enumerating in turn the elements of

$$G_1, \quad \{1,\dots,\max(G_1)\}\backslash G_1, \quad G_2, \quad \{\max(G_1)+1,\dots,\max(G_2)\}\backslash G_2, \quad G_3, \quad \dots$$

As this is a complete listing of $\mathbf{N}$, it follows from (2.2) that $\sum x_{\sigma(n)}$ is not Cauchy, so statement (a) does not hold either.

(d) $\Rightarrow$ (e). Assume that statement (d) holds and let (ε_n) be any sequence of signs $\varepsilon_n = \pm 1$. Define

$$F^+ \;=\; \{n : \varepsilon_n = 1\} \qquad \text{and} \qquad F^- \;=\; \{n : \varepsilon_n = -1\}.$$

Let $F^+ = \{n_j^+\}$ and $F^- = \{n_j^-\}$ be the complete listing of elements of F^+ and F^- in increasing order, respectively. By hypothesis, both $\sum x_{n_j^+}$ and $\sum x_{n_j^-}$ converge, whence $\sum \varepsilon_n x_n = \sum x_{n_j^+} - \sum x_{n_j^-}$ converges as well. Thus statement (e) holds.

(e) $\Rightarrow$ (d). Suppose that statement (e) holds, and that we are given an increasing sequence of integers $0 < n_1 < n_2 < \cdots$. Define $\varepsilon_n = 1$ for all n, and set

$$\eta_n = \begin{cases} 1, & \text{if } n = n_j \text{ for some } j, \\ -1, & \text{if } n \neq n_j \text{ for any } j. \end{cases}$$

By hypothesis, both $\sum \varepsilon_n x_n$ and $\sum \eta_n x_n$ converge, whence

$$\sum_j x_{n_j} \;=\; \frac{1}{2}\Bigl(\sum_n \varepsilon_n x_n \;+\; \sum_n \eta_n x_n\Bigr)$$

converges as well. Thus statement (d) holds.

(f) $\Rightarrow$ (e). Every sequence of signs (ε_n) is a bounded sequence of scalars.

(g) $\Rightarrow$ (f). Suppose that statement (g) holds, and let (λ_n) be any sequence of scalars with each $|\lambda_n| \le 1$. Given $\varepsilon > 0$, there exists by hypothesis a number N_0 such that

$$\forall\, N \ge N_0, \quad \sup\biggl\{\sum_{n=N}^{\infty} |\langle x_n, x^*\rangle| : x^* \in X^*,\ \|x^*\| \le 1\biggr\} \;<\; \varepsilon.$$

Suppose that $N > M \ge N_0$. By the Hahn–Banach theorem (Corollary 1.39), we can find a functional $x^* \in X^*$ such that $\|x^*\| = 1$ and

$$\biggl\langle \sum_{n=M+1}^{N} \lambda_n x_n,\ x^* \biggr\rangle \;=\; \biggl\|\sum_{n=M+1}^{N} \lambda_n x_n\biggr\|.$$

Therefore,

$$\biggl\|\sum_{n=M+1}^{N} \lambda_n x_n\biggr\| \;=\; \sum_{n=M+1}^{N} \lambda_n \langle x_n, x^*\rangle \;\le\; \sum_{n=M+1}^{N} |\lambda_n|\, |\langle x_n, x^*\rangle| \;\le\; \sum_{n=M+1}^{N} |\langle x_n, x^*\rangle| \;<\; \varepsilon.$$

Hence $\sum \lambda_n x_n$ is Cauchy, and therefore must converge. Thus statement (f) holds. □

Corollary 2.9. *If the series $\sum x_n$ is unconditionally convergent, then $\sum x_{\sigma(n)} = \sum x_n$ for every permutation σ of* $\mathbf{N}$.

Proof. Suppose that $\sum x_n$ is unconditionally convergent. Then $x = \lim_F \sum_{n\in F} x_n$ exists by Theorem 2.8. Let σ be any permutation of $\mathbf{N}$, and choose $\varepsilon > 0$. Then, by Definition 2.6, there is a finite set $F_0 \subset N$ such that

$$\forall \text{ finite } F \supset F_0, \quad \biggl\|x - \sum_{n\in F} x_n\biggr\| \;<\; \varepsilon. \tag{2.3}$$

Let N_0 be large enough that $F_0 \subset \{\sigma(1), \dots, \sigma(N_0)\}$. Choose any $N \ge N_0$, and define $F = \{\sigma(1), \dots, \sigma(N)\}$. Then $F \supset F_0$, so by (2.3),

$$\biggl\|x - \sum_{n=1}^{N} x_{\sigma(n)}\biggr\| \;=\; \biggl\|x - \sum_{n\in F} x_n\biggr\| \;<\; \varepsilon.$$

Hence $x = \sum x_{\sigma(n)}$, with x independent of σ. □

Notation 2.10. Given a sequence $\{x_n\}$ in a Banach space X, we will let R, $R_\mathcal{E}$, and R_Λ denote the following numbers (defined in the extended real sense):

$$R = \sup\left\{\left\|\sum_{n\in F} x_n\right\| : \text{all finite } F\subset N\right\},$$

$$R_\mathcal{E} = \sup\left\{\left\|\sum_{n\in F} \varepsilon_n x_n\right\| : \text{all finite } F\subset N \text{ and all } \mathcal{E}=(\varepsilon_n) \text{ with every } \varepsilon_n=\pm 1\right\},$$

$$R_\Lambda = \sup\left\{\left\|\sum_{n\in F} \lambda_n x_n\right\| : \text{all finite } F\subset N \text{ and all } \Lambda=(\lambda_n) \text{ with every } |\lambda_n|\le 1\right\}.$$

Note that we always have $0\le R\le R_\mathcal{E}\le R_\Lambda\le +\infty$. $\diamondsuit$

We will show in Theorem 2.13 that each of R, $R_\mathcal{E}$, and R_Λ are finite if $\sum x_n$ converges unconditionally. However, Example 2.14 shows that the finiteness of any or all of R, $R_\mathcal{E}$, or R_Λ does not imply that $\sum x_n$ converges unconditionally.

The following standard result is due to Caratheodory.

Theorem 2.11. *Given real numbers $\lambda_1,\dots,\lambda_N$, each with $|\lambda_n|\le 1$, there exist real numbers $c_k\ge 0$ and signs $\varepsilon_k^n=\pm1$, where $k=1,\dots,N+1$ and $n=1,\dots,N$, such that*

(a) $\displaystyle\sum_{k=1}^{N+1} c_k = 1$, *and*

(b) $\displaystyle\sum_{k=1}^{N+1} \varepsilon_k^n\, c_k = \lambda_n$ *for* $n=1,\dots,N$. $\diamondsuit$

Proposition 2.12. *Given a sequence $\{x_n\}$ in a Banach space X, the following relations hold in the extended real sense:*

(a) $R\le R_\mathcal{E}\le 2R$,

(b) $R_\mathcal{E}=R_\Lambda$ *if the scalar field is* $\mathbf{R}$,

(c) $R_\mathcal{E}\le R_\Lambda\le 2R_\mathcal{E}$ *if the scalar field is* $\mathbf{C}$.

As a consequence, any one of R, $R_\mathcal{E}$, R_Λ is finite if and only if the other two are.

Proof. Recall that we always have the inequalities $0\le R\le R_\mathcal{E}\le R_\Lambda\le+\infty$.

(a) Given any finite set $F\subset\mathbf{N}$ and any sequence of signs $\varepsilon_n=\pm1$, define

$$F^+ = \{n:\varepsilon_n=1\} \qquad\text{and}\qquad F^- = \{n:\varepsilon_n=-1\}.$$

Then

$$\left\|\sum_{n\in F}\varepsilon_n x_n\right\| = \left\|\sum_{n\in F^+} x_n - \sum_{n\in F^-} x_n\right\| \le \left\|\sum_{n\in F^+} x_n\right\| + \left\|\sum_{n\in F^-} x_n\right\| \le 2R.$$

Taking suprema, we obtain $R_\mathcal{E}\le 2R$.

(b) Choose any finite $F \subset \mathbf{N}$ and any sequence $\Lambda = (\lambda_n)$ of real scalars such that $|\lambda_n| \le 1$ for every n. Let N be the cardinality of F. Since the λ_n are real, it follows from Caratheodory's Theorem (Theorem 2.11) that there exist real numbers $c_k \ge 0$ and signs $\varepsilon_k^n = \pm 1$, where the indices range over $k = 1, \dots, N+1$ and $n \in F$, such that

$$\sum_{k=1}^{N+1} c_k \;=\; 1 \qquad \text{and} \qquad \sum_{k=1}^{N+1} \varepsilon_k^n c_k \;=\; \lambda_n \quad \text{for } n \in F.$$

Therefore,

$$\Bigl\|\sum_{n\in F} \lambda_n x_n\Bigr\| \;=\; \Bigl\|\sum_{n\in F}\sum_{k=1}^{N+1} \varepsilon_k^n c_k x_n\Bigr\| \;\le\; \sum_{k=1}^{N+1} c_k \,\Bigl\|\sum_{n\in F} \varepsilon_k^n x_n\Bigr\| \;\le\; \sum_{k=1}^{N+1} c_k R_{\mathcal{E}} \;=\; R_{\mathcal{E}}.$$

Taking suprema, we obtain $R_\Lambda \le R_{\mathcal{E}}$.

(c) Choose any finite $F \subset \mathbf{N}$ and any sequence $\Lambda = (\lambda_n)$ of complex scalars such that $|\lambda_n| \le 1$ for every n. Write $\lambda_n = \alpha_n + i\beta_n$ with α_n, β_n real. Then, as in the proof for part (b), we obtain

$$\Bigl\|\sum_{n\in F} \alpha_n x_n\Bigr\| \;\le\; R_{\mathcal{E}} \qquad \text{and} \qquad \Bigl\|\sum_{n\in F} \beta_n x_n\Bigr\| \;\le\; R_{\mathcal{E}}.$$

Therefore $\|\sum_{n\in F} \lambda_n x_n\| \le 2R_{\mathcal{E}}$, from which it follows that $R_\Lambda \le 2R_{\mathcal{E}}$.

Alternative proof of (b) *and* (c). We will give another proof of statements (b) and (c) which uses the Hahn–Banach Theorem instead of Caratheodory's Theorem. Assume first that the scalar field is real. Let $F \subset \mathbf{N}$ be finite, and let $\Lambda = (\lambda_n)$ be any sequence of real scalars such that $|\lambda_n| \le 1$ for each n. By the Hahn–Banach theorem (Corollary 1.39), there exists an $x^* \in X^*$ such that

$$\|x^*\| \;=\; 1 \qquad \text{and} \qquad \Bigl\langle \sum_{n\in F} \lambda_n x_n,\; x^* \Bigr\rangle \;=\; \Bigl\|\sum_{n\in F} \lambda_n x_n\Bigr\|.$$

Since x^* is a real-valued functional, we have that $\langle x_n, x^* \rangle$ is real for every n. Define

$$\varepsilon_n \;=\; \begin{cases} 1, & \text{if } \langle x_n, x^* \rangle \ge 0, \\ -1, & \text{if } \langle x_n, x^* \rangle < 0. \end{cases}$$

Then

$$\begin{aligned} \Bigl\|\sum_{n\in F} \lambda_n x_n\Bigr\| \;&=\; \sum_{n\in F} \lambda_n \langle x_n, x^* \rangle \\ &\le\; \sum_{n\in F} |\lambda_n \langle x_n, x^* \rangle| \\ &\le\; \sum_{n\in F} |\langle x_n, x^* \rangle| \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n\in F} \varepsilon_n \langle x_n, x^* \rangle \\
&= \Big\langle \sum_{n\in F} \varepsilon_n x_n,\; x^* \Big\rangle \\
&\le \|x^*\| \, \Big\| \sum_{n\in F} \varepsilon_n x_n \Big\| \;=\; \Big\| \sum_{n\in F} \varepsilon_n x_n \Big\|.
\end{aligned}
$$

Taking suprema, we obtain $R_\Lambda \le R_{\mathcal{E}}$, as desired.

The complex case is now proved as before by splitting into real and imaginary parts. The only trouble is finding a *real-valued* functional x^* with the desired properties. This is accomplished by considering X as a Banach space over the real field instead of the complex field. □

Theorem 2.13. *If $\sum x_n$ converges unconditionally then R, $R_{\mathcal{E}}$, and R_Λ are all finite.*

Proof. By Proposition 2.12, we need only show that any one of R, $R_{\mathcal{E}}$, or R_Λ is finite. However, we choose to give separate proofs of the finiteness of R and R_Λ.

Proof that $R < \infty$. Assume that $\sum x_n$ converges unconditionally. Then, by Theorem 2.8(c), we can find an $N > 0$ such that

$$\forall \text{ finite } G \subset \mathbf{N}, \quad \min(G) \;>\; N \;\implies\; \Big\| \sum_{n\in G} x_n \Big\| \;<\; 1.$$

Define $F_0 = \{1, \dots, N\}$ and set

$$M \;=\; \max_{F\subset F_0} \Big\| \sum_{n\in F} x_n \Big\|.$$

Note that $M < \infty$ since F_0 contains only finitely many subsets.

Now choose any finite $F \subset \mathbf{N}$, and write $F = (F \cap F_0) \cup (F \backslash F_0)$. Then

$$\Big\| \sum_{n\in F} x_n \Big\| \;\le\; \Big\| \sum_{n\in F\cap F_0} x_n \Big\| \;+\; \Big\| \sum_{n\in F\backslash F_0} x_n \Big\| \;\le\; M + 1.$$

Hence $R \le M + 1 < \infty$, as desired.

Proof that $R_\Lambda < \infty$. Assume that $\sum x_n$ converges unconditionally. For each finite $F \subset \mathbf{N}$ and each sequence $\Lambda = (\lambda_n)$ satisfying $|\lambda_n| \le 1$ for all n, define a functional $T_{F,\Lambda} \colon X^* \to \mathbf{F}$ by

$$T_{F,\Lambda}(x^*) \;=\; \Big\langle \sum_{n\in F} \lambda_n x_n,\; x^* \Big\rangle.$$

Then, by definition of the operator norm and by Theorem 1.27, we have

$$\|T_{F,\Lambda}\| \;=\; \sup_{\|x^*\|=1} |T_{F,\Lambda}(x^*)| \;=\; \sup_{\|x^*\|=1} \Big| \Big\langle \sum_{n\in F} \lambda_n x_n,\; x^* \Big\rangle \Big| \;=\; \Big\| \sum_{n\in F} \lambda_n x_n \Big\|.$$

Therefore, R_Λ is realized by the formula

$$R_\Lambda \;=\; \sup_{F,\Lambda} \|T_{F,\Lambda}\|.$$

Now let $x^* \in X^*$ be fixed. Then, by the continuity of x^* and the unconditional convergence of $\sum x_n$, we have that $\sum \langle x_{\sigma(n)}, x^*\rangle = \bigl\langle \sum x_{\sigma(n)},\; x^*\bigr\rangle$ converges for every permutation σ of $\mathbf{N}$. Therefore, the series $\sum \langle x_n, x^*\rangle$ converges unconditionally. However, the terms $\langle x_n, x^*\rangle$ in this series are scalars. By Lemma 2.3, unconditional convergence of a series of scalars is equivalent to absolute convergence of the series. Therefore,

$$|T_{F,\Lambda}(x^*)| \;=\; \biggl|\biggl\langle \sum_{n\in F} \lambda_n x_n,\; x^* \biggr\rangle\biggr| \;\le\; \sum_{n\in F} |\lambda_n|\,|\langle x_n, x^*\rangle| \;\le\; \sum_{n\in F} |\langle x_n, x^*\rangle| \;<\; \infty.$$

Hence

$$\sup_{F,\Lambda} |T_{F,\Lambda}(x^*)| \;\le\; \sum_{n=1}^{\infty} |\langle x_n, x^*\rangle| \;<\; \infty.$$

The Uniform Boundedness Principle (Theorem 1.42) therefore implies that $R_\Lambda = \sup_{F,\Lambda} \|T_{F,\Lambda}\| < \infty$. □

The following example shows that the converse of Theorem 2.13 is false in general, i.e., finiteness of R, $R_{\mathcal{E}}$, or R_Λ need not imply that the series $\sum x_n$ converges unconditionally, or even that the series converges at all.

Example 2.14. Let X be the Banach space ℓ^∞, and let $e_n = (\delta_{mn})_{m\in\mathbf{N}}$ be the sequence in ℓ^∞ consisting of all zeros except for a single 1 at position n. Then for every finite set $F \subset \mathbf{N}$ we have $\|\sum_{n\in F} e_n\|_{\ell^\infty} = 1$. Thus $R = 1$ (and similarly $R_{\mathcal{E}} = R_\Lambda = 1$). However, by the same reasoning, $\sum e_n$ is not a Cauchy series, hence does not converge in ℓ^∞. ◊

3. UNCONDITIONAL CONVERGENCE OF SERIES IN HILBERT SPACES

The following result provides a necessary condition for the unconditional convergence of a series in a Hilbert space [Orl33].

Theorem 3.1 (Orlicz's Theorem). *If $\{x_n\}$ is a sequence in a Hilbert space H, then*

$$\sum_{n=1}^{\infty} x_n \text{ converges unconditionally} \quad \Longrightarrow \quad \sum_{n=1}^{\infty} \|x_n\|^2 < \infty. \quad \diamondsuit$$

The analogue of Orlicz's Theorem for Banach spaces is false in general. Further, the following example shows that, even in a Hilbert space, the converse of Orlicz's Theorem is false in general.

Example 3.2. Let H be a Hilbert space, and fix any $x \in H$ with $\|x\| = 1$. Then

$$\Bigl\| \sum_{n=M+1}^{N} c_n x \Bigr\| \;=\; \Bigl| \sum_{n=M+1}^{N} c_n \Bigr| \, \|x\| \;=\; \Bigl| \sum_{n=M+1}^{N} c_n \Bigr|,$$

so $\sum c_n x$ converges in H if and only if $\sum c_n$ converges as a series of scalars. Likewise, $\sum c_n x$ converges unconditionally in H if and only if $\sum c_n$ converges unconditionally. Therefore, if $(c_n) \in \ell^2$ is such that $\sum c_n$ converges conditionally, then $\sum c_n x$ converges conditionally even though $\sum \|c_n x\|^2 = \sum |c_n|^2 < \infty$. For example, this is the case if $c_n = (-1)^n/n$. $\square$

We will give three proofs of Orlicz's Theorem. The first is simpler, but the second and third give improved bounds on the value of $\sum \|x_n\|^2$. We will use the numbers R, $R_{\mathcal{E}}$, and R_Λ defined in Notation 2.10. By Theorem 2.13, if $\sum x_n$ converges unconditionally, then R, $R_{\mathcal{E}}$, and R_Λ are all finite.

The first proof requires the following simple lemma.

Lemma 3.3. [GK69, p. 315]. *Let H be a Hilbert space, and suppose $x_1, \dots, x_N \in H$. Then there exist scalars $\lambda_1, \dots, \lambda_N$, each with $|\lambda_n| \le 1$, such that*

$$\sum_{n=1}^{N} \|x_n\|^2 \;\le\; \Bigl\| \sum_{n=1}^{N} \lambda_n x_n \Bigr\|^2.$$

Proof. This is clear for $N = 1$. For $N = 2$, define $\lambda_1 = 1$ and $\lambda_2 = e^{i \arg(\langle x_1, x_2 \rangle)}$. Then

$$\begin{aligned} \|\lambda_1 x_1 + \lambda_2 x_2\|^2 \;&=\; \|x_1\|^2 \;+\; 2\,\mathrm{Re}\bigl(\lambda_1 \bar{\lambda}_2 \, \langle x_1, x_2 \rangle\bigr) \;+\; \|x_2\|^2 \\ &=\; \|x_1\|^2 \;+\; 2\,|\langle x_1, x_2 \rangle| \;+\; \|x_2\|^2 \\ &\ge\; \|x_1\|^2 \;+\; \|x_2\|^2. \end{aligned}$$

An easy induction establishes the full result. $\square$

We can now give our first proof of Orlicz's Theorem.

Theorem 3.4. [GK69, p. 315]. *If $\{x_n\}$ is a sequence in a Hilbert space H then*

$$\sum_{n=1}^{\infty} \|x_n\|^2 \ \le \ {R_\Lambda}^2.$$

In particular, if $\sum x_n$ converges unconditionally then both of these quantities are finite.

Proof. Fix any $N > 0$. Then by Lemma 3.3, we can find scalars λ_n with $|\lambda_n| \le 1$ such that

$$\sum_{n=1}^{N} \|x_n\|^2 \ \le \ \left\|\sum_{n=1}^{N} \lambda_n x_n\right\|^2 \ \le \ {R_\Lambda}^2.$$

Letting $N \to \infty$ therefore gives the result. □

The second proof uses the following lemma.

Lemma 3.5. [LT77, p. 18]. *If $x_1, \dots, x_N$ are elements of a Hilbert space H, then*

$$\text{Average}\left\{\left\|\sum_{n=1}^{N} \varepsilon_n x_n\right\|^2 : \textit{all } \varepsilon_n = \pm 1\right\} \ = \ \sum_{n=1}^{N} \|x_n\|^2. \tag{3.1}$$

Proof. For each N, define $\mathcal{S}_N = \{(\varepsilon_1, \dots, \varepsilon_N) : \text{all } \varepsilon_n = \pm 1\}$. Note that $|\mathcal{S}_N| = 2^N$.

We will proceed by induction on N. For $N = 1$ we have

$$\text{Average}\left\{\left\|\sum_{n=1}^{1} \varepsilon_n x_n\right\|^2 : (\varepsilon_n) \in \mathcal{S}_1\right\} \ = \ \frac{1}{2}\left(\|x_1\|^2 \ + \ \|-x_1\|^2\right) \ = \ \|x_1\|^2.$$

Therefore (3.1) holds when $N = 1$.

Suppose now that (3.1) holds for some $N \ge 1$. Recall the Parallelogram law in Hilbert spaces (Theorem 1.16):

$$\forall\, x, y \in H, \quad \|x + y\|^2 \ + \ \|x - y\|^2 \ = \ 2\left(\|x\|^2 \ + \ \|y\|^2\right).$$

Therefore,

$$\begin{aligned}
&\text{Average}\left\{\left\|\sum_{n=1}^{N+1} \varepsilon_n x_n\right\|^2 : (\varepsilon_n) \in \mathcal{S}_{N+1}\right\} \\
&\quad = \ \frac{1}{2^{N+1}} \sum_{(\varepsilon_n) \in \mathcal{S}_{N+1}} \left\|\sum_{n=1}^{N+1} \varepsilon_n x_n\right\|^2 \\
&\quad = \ \frac{1}{2^{N+1}} \sum_{(\varepsilon_n) \in \mathcal{S}_{N}} \ \sum_{\varepsilon_{N+1} = \pm 1} \left\|\sum_{n=1}^{N+1} \varepsilon_n x_n\right\|^2
\end{aligned}$$

$$= \frac{1}{2^{N+1}} \sum_{(\varepsilon_n)\in\mathcal{S}_N} \left(\left\| \sum_{n=1}^{N} \varepsilon_n x_n + x_{N+1} \right\|^2 + \left\| \sum_{n=1}^{N} \varepsilon_n x_n - x_{N+1} \right\|^2 \right)$$

$$= \frac{1}{2^{N+1}} \sum_{(\varepsilon_n)\in\mathcal{S}_N} 2 \left(\left\| \sum_{n=1}^{N} \varepsilon_n x_n \right\|^2 + \|x_{N+1}\|^2 \right)$$

$$= \frac{1}{2^{N}} \sum_{(\varepsilon_n)\in\mathcal{S}_N} \left\| \sum_{n=1}^{N} \varepsilon_n x_n \right\|^2 + \frac{1}{2^{N}} \sum_{(\varepsilon_n)\in\mathcal{S}_N} \|x_{N+1}\|^2$$

$$= \left(\sum_{n=1}^{N} \|x_n\|^2 \right) + \|x_{N+1}\|^2,$$

the last equality following from the induction hypothesis. Thus (3.1) holds for $N+1$ as well. □

We can now give a second proof of Orlicz's Theorem. Since $R_{\mathcal{E}} \le R_{\Lambda}$, the bound on the value of $\sum \|x_n\|^2$ in the following is sharper in general than the corresponding bound in Theorem 3.4.

Theorem 3.6. [LT77, p. 18]. *If $\{x_n\}$ is a sequence in a Hilbert space H then*

$$\sum \|x_n\|^2 \le R_{\mathcal{E}}{}^2.$$

In particular, if $\sum x_n$ converges unconditionally then both of these quantities are finite.

Proof. Fix any $N > 0$. Then by Lemma 3.5,

$$\sum_{n=1}^{N} \|x_n\|^2 = \text{Average}\left\{ \left\| \sum_{n=1}^{N} \varepsilon_n x_n \right\|^2 : \text{all } \varepsilon_n = \pm 1 \right\} \le \text{Average}\{R_{\mathcal{E}}{}^2 : \text{all } \varepsilon_n = \pm 1\} = R_{\mathcal{E}}{}^2.$$

Letting $N \to \infty$ therefore gives the result. □

Our final proof uses the Rademacher system (a sequence of orthonormal functions in $L^2[0,1]$) to derive Orlicz's Theorem in the special case $H = L^2(E)$. However, since all separable Hilbert spaces are isometrically isomorphic, this proves Orlicz's Theorem for all separable Hilbert spaces. The first four Rademacher functions are pictured in Figure 3.1.

Definition 3.7. The *Rademacher system* is the sequence of functions $\{R_n\}_{n=0}^{\infty}$, each with domain $[0,1]$, defined by

$$R_n(t) = \operatorname{sign}(\sin 2^n \pi t) = \begin{cases} 1, & t \in \bigcup_{k=0}^{2^{n-1}-1} \left(\frac{2k}{2^n}, \frac{2k+1}{2^n} \right), \\ 0, & t = \frac{k}{2^n},\ k = 0, \dots, 2^n, \\ -1, & t \in \bigcup_{k=0}^{2^{n-1}-1} \left(\frac{2k+1}{2^n}, \frac{2k+2}{2^n} \right). \end{cases} \quad \diamondsuit$$

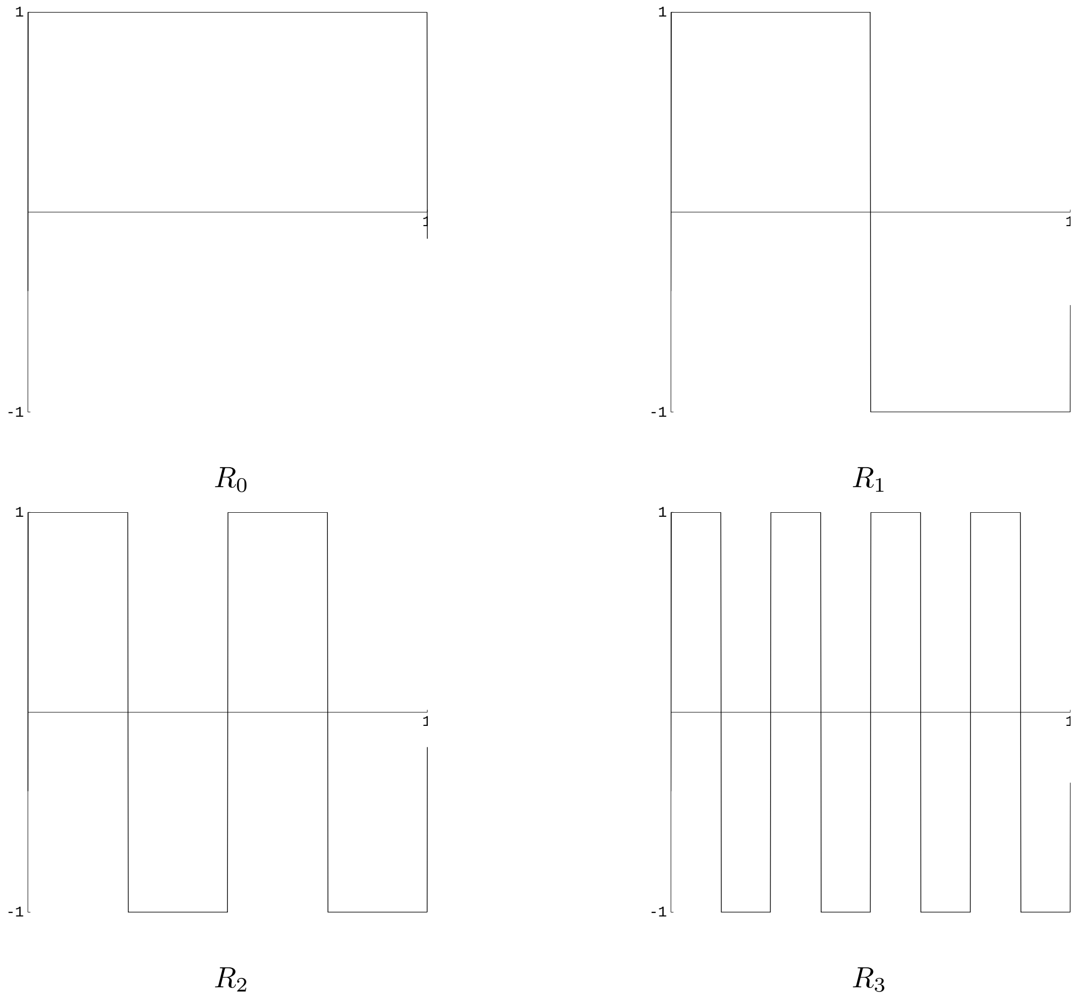

FIGURE 3.1. The Rademacher functions R_0, R_1, R_2, and R_3.

Proposition 3.8. *The Rademacher system is a orthonormal sequence in* $L^2[0,1]$*, but it is not complete in* $L^2[0,1]$*.*

Proof. Since $|R_n(t)| = 1$ almost everywhere on $[0,1]$ we have $\|R_n\|_2 = 1$. Thus, Rademacher functions are normalized. To show the orthogonality, define

$$S_n^+ = \{t \in [0,1] : R_n(t) > 0\} \qquad \text{and} \qquad S_n^- = \{t \in [0,1] : R_n(t) < 0\}.$$

If $m \neq n$ then we have

$$\langle R_m, R_n \rangle = |S_m^+ \cap S_n^+| - |S_m^+ \cap S_n^-| - |S_m^- \cap S_n^+| + |S_m^- \cap S_n^-| = \frac{1}{4} - \frac{1}{4} - \frac{1}{4} + \frac{1}{4} = 0.$$

Thus $\{R_n\}_{n=0}^\infty$ is an orthonormal sequence in $L^2[0,1]$.

Finally, consider the function $w(t) = R_1(t)\, R_2(t)$, pictured in Figure 3.2. Reasoning similar to the above shows that $\langle w, R_n \rangle = 0$ for every $n \geq 0$. Hence $\{R_n\}_{n=0}^\infty$ is incomplete in $L^2[0,1]$. $\square$

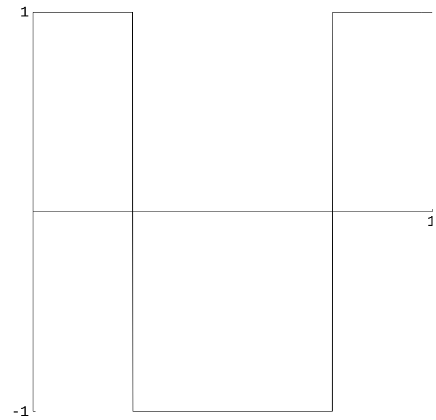

FIGURE 3.2. The function $w(t) = R_1(t)\, R_2(t)$.

Although the Rademacher system is not complete, it is the starting point for the construction of the *Walsh system*, which is a complete orthonormal basis for $L^2[0,1]$. Elements of the Walsh system are formed by taking finite products of Rademacher functions. The Rademacher and Walsh systems are closely related to the *Haar system*, which is the simplest wavelet orthonormal basis for $L^2(\mathbf{R})$ [Dau92].

We can now give our final proof of Orlicz's Theorem.

Theorem 3.9. [Mar69, p. 83]. *Let $E \subset \mathbf{R}$. If $\{f_n\}$ is a sequence of functions in $L^2(E)$ then*

$$\sum \|f_n\|^2_{L^2(E)} \;\le\; R_{\mathcal{E}}{}^2.$$

In particular, if $\sum f_n$ converges unconditionally then both of these quantities are finite.

Proof. Let $\{R_n\}_{n=0}^{\infty}$ be the Rademacher system (Definition 3.7). Let $Z = \bigcup_{n=1}^{\infty} \{x \in E : |f_n(x)| = +\infty\}$. Then Z has Lebesgue measure zero, i.e., $|Z| = 0$, since each f_n is square-integrable. Since $\{R_n\}$ is an orthonormal system, we have by the Plancherel formula (Theorem 1.20) that

$$\forall\, x \notin Z, \qquad \left\|\sum_{n=1}^{N} f_n(x)\, R_n\right\|^2_{L^2[0,1]} \;=\; \sum_{n=1}^{N} |f_n(x)|^2.$$

Moreover, since $R_n(t) = \pm 1$ for a.e. t, we have

$$\left\|\sum_{n=1}^{N} R_n(t)\, f_n\right\|_{L^2(E)} \;\le\; R_{\mathcal{E}} \quad \text{for a.e. } t. \tag{3.2}$$

Therefore,

$$\begin{aligned}
\sum_{n=1}^{N} \|f_n\|^2_{L^2(E)} &= \int_X \sum_{n=1}^{N} |f_n(x)|^2 \, dx \\
&= \int_X \left\|\sum_{n=1}^{N} f_n(x)\, R_n\right\|^2_{L^2[0,1]} dx
\end{aligned}$$

$$= \int_X \int_0^1 \left| \sum_{n=1}^N f_n(x)\, R_n(t) \right|^2 dt\, dx$$

$$= \int_0^1 \int_X \left| \sum_{n=1}^N f_n(x)\, R_n(t) \right|^2 dx\, dt \qquad \text{(by Tonelli's theorem)}$$

$$= \int_0^1 \left\| \sum_{n=1}^N R_n(t)\, f_n \right\|_{L^2(E)}^2 dt$$

$$\leq \int_0^1 {R_{\mathcal{E}}}^2\, dt$$

$$= {R_{\mathcal{E}}}^2,$$

where Tonelli's Theorem [WZ77, p. 92] allows us to interchange the order of integration at the point indicated because the integrands are nonnegative. Letting $N \to \infty$ therefore gives the result. □

Suppose that in the proof of Theorem 3.9, we substitute for the Rademacher system any orthonormal basis $\{e_n\}$ for $L^2[0,1]$ whose elements are uniformly bounded, say $\|e_n\|_{L^\infty} \leq M$ for all n. Then in place of (3.2), we would have $\left\|\sum_{n=1}^N e_n(t)\, f_n\right\|_{L^2(E)} \leq MR_\Lambda$ for almost every t. The remainder of the proof would then remain valid if R_n is changed to e_n, except that the final conclusion would be that $\sum_{n=1}^N \|f_n\|_{L^2(E)}^2 \leq (MR_\Lambda)^2$. For example, if we took $e_n(t) = e^{2\pi i n t}$, then we would have $M = 1$.

III. BASES IN BANACH SPACES

4. BASES IN BANACH SPACES

Since a Banach space X is a vector space, it must possess a *Hamel*, or *vector space*, *basis*, i.e., a subset $\{x_\gamma\}_{\gamma\in\Gamma}$ whose finite linear span is all of X and which has the property that every finite subcollection is linearly independent. Any element $x \in X$ can therefore be written as some *finite* linear combination of x_γ. However, even a separable infinite-dimensional Banach space would require an *uncountable* Hamel basis. Moreover, the proof of the existence of Hamel bases for arbitrary infinite-dimensional spaces requires the Axiom of Choice (in fact, in can be shown that the statement "Every vector space has a Hamel basis" is equivalent to the Axiom of Choice). Hence for most Banach spaces there is no constructive method of producing a Hamel basis.

Example 4.1. [Gol66, p. 101]. We will use the existence of Hamel bases to show that if X is an infinite-dimensional Banach space, then there exist linear functionals on X which are not continuous. Let $\{x_\gamma\}_{\gamma\in\Gamma}$ be a Hamel basis for an infinite-dimensional Banach space X, normalized so that $\|x_\gamma\| = 1$ for every $\gamma \in \Gamma$. Let $\Gamma_0 = \{\gamma_1, \gamma_2, \dots\}$ be any countable subsequence of Γ. Define $\mu\colon X \to \mathbf{C}$ by setting $\mu(\gamma_n) = n$ for $n \in \mathbf{N}$ and $\mu(\gamma) = 0$ for $\gamma \in \Gamma\backslash\Gamma_0$, and then extending μ linearly to X. Then this μ is a linear functional on X, but it is not bounded. $\diamondsuit$

More useful than a Hamel basis is a *countable* sequence $\{x_n\}$ such that every element $x \in X$ can be written as some unique *infinite* linear combination $x = \sum c_n x_n$. This leads to the following definition.

Definition 4.2.

(a) A sequence $\{x_n\}$ in a Banach space X is a *basis* for X if

$$\forall\, x \in X, \quad \exists \text{ unique scalars } a_n(x) \text{ such that } x \;=\; \sum_n a_n(x)\, x_n. \tag{4.1}$$

(b) A basis $\{x_n\}$ is an *unconditional basis* if the series in (4.1) converges unconditionally for each $x \in X$.

(c) A basis $\{x_n\}$ is an *absolutely convergent basis* if the series in (4.1) converges absolutely for each $x \in X$.

(d) A basis $\{x_n\}$ is a *bounded basis* if $\{x_n\}$ is norm-bounded both above and below, i.e., if $0 < \inf \|x_n\| \le \sup \|x_n\| < \infty$.

(e) A basis $\{x_n\}$ is a *normalized basis* if $\{x_n\}$ is normalized, i.e, if $\|x_n\| = 1$ for every n. $\diamondsuit$

Absolutely convergent bases are studied in detail in Chapter 5. Unconditional bases are studied in detail in Chapter 9.

Note that if $\{x_n\}$ is a basis, then the fact that each $x \in X$ can be written uniquely as $x = \sum a_n(x)\, x_n$ implies that $x_n \ne 0$ for every n. As a consequence, $\{x_n/\|x_n\|\}$ is a normalized basis for X.

If X possesses a basis $\{x_n\}$ then X must be separable, since the set of all finite linear combinations $\sum_{n=1}^{N} c_n x_n$ with rational c_n (or rational real and imaginary parts if the c_n are complex) forms a countable, dense subset of X. The question of whether every separable Banach space possesses a basis was a longstanding problem known as the *Basis Problem*. It was shown by Enflo [Enf73] that there do exist separable, reflexive Banach spaces which do not possess any bases.

Notation 4.3. Note that the coefficients $a_n(x)$ defined in (4.1) are linear functions of x. Moreover, they are uniquely determined by the basis, i.e., the basis $\{x_n\}$ determines a unique collection of linear functionals $a_n \colon X \to \mathbf{F}$. We therefore call $\{a_n\}$ the *associated sequence of coefficient functionals*. Since these functionals are uniquely determined, we often do not declare them explicitly. When we do need to refer explicitly to both the basis and the associated coefficient functionals, we will write "$(\{x_n\}, \{a_n\})$ is a basis" to mean that $\{x_n\}$ is a basis with associated coefficient functionals $\{a_n\}$. We show in Theorem 4.11 that the coefficient functionals for any basis must be continuous, i.e., $\{a_n\} \subset X^*$.

Further, note that since $x_m = \sum a_n(x)\, x_n$ and $x_m = \sum \delta_{mn}\, x_n$ are two expansions of x_m, we must have $a_m(x_n) = \delta_{mn}$ for every m and n. We therefore say that the sequences $\{x_n\} \subset X$ and $\{a_n\} \subset X^*$ are *biorthogonal*, and we often say that $\{a_n\}$ is the *biorthogonal system* associated with $\{x_n\}$. General biorthogonal systems are considered in more detail in Chapter 7. In particular, we show there that the fact that $\{x_n\}$ is a basis implies that $\{a_n\}$ is the unique sequence in X^* that is biorthogonal to $\{x_n\}$. $\diamondsuit$

Example 4.4. Fix $1 \le p < \infty$, and consider the space $X = \ell^p$ defined in Example 1.6. Define sequences $e_n = (\delta_{mn})_{m=1}^{\infty} = (0, \dots, 0, 1, 0, \dots)$, where the 1 is in the nth position. Then $\{e_n\}$ is a basis for ℓ^p, often called the *standard basis* for ℓ^p. Note that $\{e_n\}$ is its own sequence of coefficient functionals.

On the other hand, $\{e_n\}$ is not a basis for ℓ^∞, and indeed ℓ^∞ has no bases whatsoever since it is not separable. Using the ℓ^∞ norm, the sequence $\{e_n\}$ is a basis for the space c_0 defined in Example 1.6(c). $\diamondsuit$

We are primarily interested in bases for which the coefficient functionals $\{a_n\}$ are *continuous*. We therefore give such bases a special name.

Definition 4.5. A basis $(\{x_n\}, \{a_n\})$ is a *Schauder basis* if each coefficient functional a_n is continuous. In this case, each a_n is an element of the dual space, i.e., $a_n \in X^*$ for every n. $\diamondsuit$

We shall see in Theorem 4.11 that *every* basis is a Schauder basis, i.e., the coefficient functionals a_n are *always* continuous. First, however, we require some definitions and miscellaneous facts. In particular, the following operators play a key role in analyzing bases.

Notation 4.6. The *partial sum operators*, or the *natural projections*, associated with the basis $(\{x_n\}, \{a_n\})$ are the mappings $S_N \colon X \to X$ defined by

$$S_N x \;=\; \sum_{n=1}^{N} a_n(x)\, x_n. \quad \diamondsuit$$

The partial sum operators are clearly linear. We will show in Corollary 4.8 that if $\{x_n\}$ is a basis then each partial sum operator S_N is a bounded mapping of X into itself. Then the fact that all bases are Schauder bases will follow from the continuity of the partial sum operators (Theorem 4.11). The next proposition will be a key tool in this analysis. It states that if $\{x_n\}$ is a basis, then it is possible to endow the space Y of all sequences (c_n) such that $\sum c_n x_n$ converges with a norm so that it becomes a Banach space isomorphic to X. In general, however, it is difficult or impossible to explicitly describe the space Y. One exception was discussed in Example 2.5: if $\{e_n\}$ is an orthonormal basis for a Hilbert space H, then $\sum c_n e_n$ converges if and only if $(c_n) \in \ell^2$.

Recall that a *topological isomorphism* between Banach spaces X and Y is a linear bijection $S: X \to Y$ that is continuous. By the Inverse Mapping Theorem (Theorem 1.44), every topological isomorphism has a continuous inverse $S^{-1}: Y \to X$.

Proposition 4.7. [Sin70, p. 18]. *Let $\{x_n\}$ be a sequence in a Banach space X, and assume that $x_n \neq 0$ for every n. Define $Y = \{(c_n) : \sum c_n x_n \text{ converges in } X\}$, and set*

$$\|(c_n)\|_Y \;=\; \sup_N \biggl\|\sum_{n=1}^{N} c_n x_n\biggr\|.$$

Then the following statements hold.

(a) *Y is a Banach space.*

(b) *If $\{x_n\}$ is a basis for X then Y is topologically isomorphic to X via the mapping $(c_n) \mapsto \sum c_n x_n$.*

Proof. (a) It is clear that Y is a linear space. If $(c_n) \in Y$ then $\sum c_n x_n = \lim_{N\to\infty} \sum_{n=1}^{N} c_n x_n$ converges. Since convergent sequences are bounded, we therefore have $\|(c_n)\|_Y < \infty$ for each $(c_n) \in Y$. Thus $\|\cdot\|_Y$ is well-defined. It is easy to see that $\|(c_n) + (d_n)\|_Y \le \|(c_n)\|_Y + \|(d_n)\|_Y$ and $\|a\,(c_n)\|_Y = |a|\,\|(c_n)\|_Y$ for every scalar a, so $\|\cdot\|_Y$ is at least a seminorm on Y. Suppose that $\|(c_n)\|_Y = 0$. Then $\|\sum_{n=1}^{N} c_n x_n\| = 0$ for every N. In particular, $\|c_1 x_1\| = 0$, so we must have $c_1 = 0$ since we have assumed $x_1 \neq 0$, But then $\|c_2 x_2\| = \|\sum_{n=1}^{2} c_n x_n\| = 0$, so $c_2 = 0$, etc. Hence $\|\cdot\|_Y$ is a norm on Y.

It remains only to show that Y is complete in this norm. Let $A_N = (c_n^N)$ be any collection of sequences from Y which form a Cauchy sequence with respect to the norm $\|\cdot\|_Y$. Then for n fixed, we have

$$|c_n^M - c_n^N|\,\|x_n\| \;=\; \|(c_n^M - c_n^N)\,x_n\| \;\le\; \biggl\|\sum_{k=1}^{n} (c_k^M - c_k^N)\,x_k\biggr\| + \biggl\|\sum_{k=1}^{n-1} (c_k^M - c_k^N)\,x_k\biggr\| \;\le\; 2\,\|A_M - A_N\|_Y.$$

Since $\{A_N\}$ is Cauchy and $x_n \neq 0$, we conclude that $(c_n^N)_{N=1}^{\infty}$ is a Cauchy sequence of scalars, so must converge to some scalar c_n as $N \to \infty$.

Choose now any $\varepsilon > 0$. Then since $\{A_N\}$ is Cauchy in Y, there exists an integer $N_0 > 0$ such that

$$\forall\; M, N \ge N_0, \quad \|A_M - A_N\|_Y \;=\; \sup_L \biggl\|\sum_{n=1}^{L} (c_n^M - c_n^N)\,x_n\biggr\| \;<\; \varepsilon. \tag{4.2}$$

Fix $N \ge N_0$ and any $L > 0$, and set $y_M = \sum_{n=1}^{L} (c_n^M - c_n^N)\, x_n$. Then $\|y_M\| < \varepsilon$ for each $M \ge N_0$ by (4.2). However, $y_M \to y = \sum_{n=1}^{L} (c_n - c_n^N)\, x_n$, so we must have $\|y\| \le \varepsilon$. Thus, we have shown that

$$\forall\ N \ge N_0, \quad \sup_L \biggl\| \sum_{n=1}^{L} (c_n - c_n^N)\, x_n \biggr\| \ \le\ \varepsilon. \tag{4.3}$$

Further, $(c_n^{N_0})_{n=1}^{\infty} \in Y$, so $\sum_n c_n^{N_0}\, x_n$ converges by definition. Hence, there is an $M_0 > 0$ such that

$$\forall\ N > M \ge M_0, \quad \biggl\| \sum_{n=M+1}^{N} c_n^{N_0}\, x_n \biggr\| \ <\ \varepsilon.$$

Therefore, if $N > M \ge M_0,\ N_0$ then

$$\begin{aligned} \biggl\| \sum_{n=M+1}^{N} c_n x_n \biggr\| \ &=\ \biggl\| \sum_{n=1}^{N} (c_n - c_n^{N_0})\, x_n \ -\ \sum_{n=1}^{M} (c_n - c_n^{N_0})\, x_n \ +\ \sum_{n=M+1}^{N} c_n^{N_0}\, x_n \biggr\| \\ &\le\ \biggl\| \sum_{n=1}^{N} (c_n - c_n^{N_0})\, x_n \biggr\| \ +\ \biggl\| \sum_{n=1}^{M} (c_n - c_n^{N_0})\, x_n \biggr\| \ +\ \biggl\| \sum_{n=M+1}^{N} c_n^{N_0}\, x_n \biggr\| \\ &\le\ \varepsilon \ +\ \varepsilon \ +\ \varepsilon \ =\ 3\varepsilon. \end{aligned}$$

Therefore $\sum c_n x_n$ converges in X, so $A = (c_n) \in Y$. Finally, by (4.3), we know that $A_N \to A$ in the norm of Y, so Y is complete.

(b) Suppose now that $\{x_n\}$ is a basis for X. Define the map $T\colon Y \to X$ by $T(c_n) = \sum c_n x_n$. This mapping is well-defined by the definition of Y. It is clearly linear, and it is bijective because $\{x_n\}$ is a basis. Finally, if $(c_n) \in Y$ then

$$\|T(c_n)\| \ =\ \biggl\| \sum_{n=1}^{\infty} c_n x_n \biggr\| \ =\ \lim_{N\to\infty} \biggl\| \sum_{n=1}^{N} c_n x_n \biggr\| \ \le\ \sup_N \biggl\| \sum_{n=1}^{N} c_n x_n \biggr\| \ =\ \|(c_n)\|_Y.$$

Therefore T is bounded, hence is a topological isomorphism of Y onto X. □

An immediate consequence of Proposition 4.7 is that the partial sum operators S_N are bounded.

Corollary 4.8. *Let $(\{x_n\}, \{a_n\})$ be a basis for a Banach space X. Then:*

(a) $\sup \|S_N x\| < \infty$ *for each* $x \in X$,

(b) $C = \sup \|S_N\| < \infty$, *and*

(c) $|\!|\!| x |\!|\!| = \sup \|S_N x\|$ *forms a norm on X equivalent to the initial norm $\|\cdot\|$ for X, and satisfies* $\|\cdot\| \le |\!|\!|\cdot|\!|\!| \le C\,\|\cdot\|$.

Proof. (a) Let Y be as in Proposition 4.7. Then $T\colon X \to Y$ defined by $T(c_n) = \sum c_n x_n$ is a topological isomorphism of X onto Y. Suppose that $x \in X$. Then we have by definition that $x = \sum a_n(x)\, x_n$ and that the scalars $a_n(x)$ are unique, so we must have $T^{-1}x = (a_n(x))$. Hence

$$\sup_N \|S_N x\| \ =\ \sup_N \biggl\| \sum_{n=1}^{N} a_n(x)\, x_n \biggr\| \ =\ \bigl\|(a_n(x))\bigr\|_Y \ =\ \|T^{-1}x\|_Y \ \le\ \|T^{-1}\|\, \|x\| \ <\ \infty. \tag{4.4}$$

(b) From (4.4), we see that $\sup \|S_N\| \le \|T^{-1}\| < \infty$.

(c) It is easy to see that $|||\cdot|||$ has the properties of at least a seminorm. Now, given $x \in X$ we have

$$|||x||| \;=\; \sup_N \|S_N x\| \;\le\; \sup_N \|S_N\|\,\|x\| \;=\; C\,\|x\|$$

and

$$\|x\| \;=\; \lim_{N\to\infty} \|S_N x\| \;\le\; \sup_N \|S_N x\| \;=\; |||x|||.$$

It follows from these two statements that $|||\cdot|||$ is in fact a norm, and is equivalent to $\|\cdot\|$. □

The number C appearing in Corollary 4.8 is important enough to be dignified with a name of its own.

Definition 4.9. If $(\{x_n\},\{a_n\})$ is a basis for a Banach space X, then its *basis constant* is the finite number $C = \sup \|S_N\|$. The basis constant satisfies $C \ge 1$. If the basis constant is $C = 1$, then the basis is said to be *monotone*. ◇

The basis constant does depend on the norm. Unless otherwise specified, the basis constant is always taken with respect to the original norm on X. Changing to an equivalent norm for X will not change the fact that $\{x_n\}$ is a basis, but it can change the basis constant for $\{x_n\}$. For example, we show now that the basis constant in the norm $|||\cdot|||$ is always 1.

Proposition 4.10. *Every basis is monotone with respect to the equivalent norm* $|||\cdot|||$ *defined in* Corollary 4.8(c).

Proof. Note first that the composition of the partial sum operators S_M and S_N satisfies the rule

$$S_M\,S_N \;=\; \begin{cases} S_M, & \text{if } M \le N,\\ S_N, & \text{if } M \ge N.\end{cases}$$

Therefore,

$$|||S_N x||| \;=\; \sup_M \|S_M S_N x\| \;=\; \sup\,\{\|S_1 x\|, \dots, \|S_N x\|\}.$$

Hence,

$$\sup_N |||S_N x||| \;=\; \sup_N \|S_N x\| \;=\; |||x|||.$$

It follows from this that $\sup |||S_N||| = 1$. □

Now we can prove our main result: the coefficient functionals for *every* basis are continuous!

Theorem 4.11. [Sin70, p. 20]. *Every basis* $(\{x_n\},\{a_n\})$ *for a Banach space* X *is a Schauder basis for* X*. In fact, the coefficients functionals* a_n *are continuous linear functionals on* X *which satisfy*

$$1 \;\le\; \|a_n\|\,\|x_n\| \;\le\; 2C, \tag{4.5}$$

where C *is the basis constant for* $(\{x_n\},\{a_n\})$.

Proof. Since each a_n is a linear functional on X, we need only show that each a_n is bounded and that (4.5) is satisfied. Given $x \in X$, we compute

$$\begin{aligned} |a_n(x)| \, \|x_n\| = \|a_n(x)\, x_n\| &= \left\| \sum_{n=1}^{n} a_k(x)\, x_k - \sum_{n=1}^{n-1} a_k(x)\, x_k \right\| \\ &\le \left\| \sum_{n=1}^{n} a_k(x)\, x_k \right\| + \left\| \sum_{n=1}^{n-1} a_k(x)\, x_k \right\| \\ &= \|S_n x\| + \|S_{n-1} x\| \\ &\le 2C \, \|x\|. \end{aligned}$$

Since each x_n is nonzero, we conclude that $\|a_n\| \le 2C/\|x_n\| < \infty$. The final inequality follows from computing $1 = a_n(x_n) \le \|a_n\| \, \|x_n\|$. ☐

Since the coefficient functionals a_n are therefore elements of X^*, we use the notations $a_n(x) = \langle x, a_n \rangle$ interchangeably. In fact, from this point onward our preferred notation is $\langle x, a_n \rangle$.

We end this chapter with several useful results concerning the invariance of bases under topological isomorphisms.

Lemma 4.12. [You80, p. 30]. *Bases are preserved by topological isomorphisms. That is, if $\{x_n\}$ is a basis for a Banach space X and $S\colon X \to Y$ is a topological isomorphism, then $\{Sx_n\}$ is a basis for Y.*

Proof. If y is any element of Y then $S^{-1}y \in X$, so there are unique scalars (c_n) such that $S^{-1}y = \sum c_n x_n$. Since S is continuous, this implies $y = S(S^{-1}y) = \sum c_n S x_n$. Suppose that $y = \sum b_n S x_n$ was another representation of y. Then the fact that S^{-1} is also continuous implies that $S^{-1}y = \sum b_n x_n$, and hence that $b_n = c_n$ for each n. Thus $\{Sx_n\}$ is a basis for Y. ☐

This lemma motivates the following definition.

Definition 4.13. Let X and Y be Banach spaces. A basis $\{x_n\}$ for X is *equivalent* to a basis $\{y_n\}$ for Y if there exists a topological isomorphism $S\colon X \to Y$ such that $Sx_n = y_n$ for all n. If $X = Y$ then we write $\{x_n\} \sim \{y_n\}$ to mean that $\{x_n\}$ and $\{y_n\}$ are equivalent bases for X. ◊

It is clear that $\sim$ is an equivalence relation on the set of all bases of a Banach space X.

Note that we could define, more generally, that a basis $\{x_n\}$ for X is equivalent to a *sequence* $\{y_n\}$ in Y if there exists a topological isomorphism $S\colon X \to Y$ such that $Sx_n = y_n$. However, by Lemma 4.12, it follows immediately that such a sequence must be a basis for Y.

Pelczynski and Singer showed in 1964 that there exist uncountably many nonequivalent normalized conditional bases in every infinite dimensional Banach space which has a basis.

We show below in Corollary 4.15 that all orthonormal bases in a Hilbert space are equivalent. More generally, we show in Chapter 11 that all bounded unconditional bases in a Hilbert space are equivalent (and hence must be equivalent to orthonormal bases). Lindenstrauss and Pelczynski showed in 1968 that a non-Hilbert space H in which all bounded unconditional bases are equivalent must be isomorphic either to the sequence space c_0 or to the sequence space ℓ^1.

We can now give a characterization of equivalent bases.

Theorem 4.14. [You80, p. 30]. *Let X and Y be Banach spaces. Let $\{x_n\}$ be a basis for X and let $\{y_n\}$ be a basis for Y. Then the following two statements are equivalent.*

(a) *$\{x_n\}$ is equivalent to $\{y_n\}$.*

(b) *$\sum c_n x_n$ converges in X if and only if $\sum c_n y_n$ converges in Y.*

Proof. (a) $\Rightarrow$ (b). Suppose that $\{x_n\}$ is equivalent to $\{y_n\}$. Then there is a topological isomorphism $S\colon X \to Y$ such that $Sx_n = y_n$ for every n. Since S is continuous, the convergence of $\sum c_n x_n$ in X therefore implies the convergence of $\sum c_n Sx_n$ in Y. Similarly, S^{-1} is continuous, so the convergence of $\sum c_n y_n$ in Y implies the convergence of $\sum c_n S^{-1} y_n$ in X. Therefore (b) holds.

(b) $\Rightarrow$ (a). Suppose that (b) holds. Let $\{a_n\} \subset X^*$ be the coefficient functionals for the basis $\{x_n\}$, and let $\{b_n\} \subset Y^*$ be the coefficient functionals for the basis $\{y_n\}$. Suppose that $x \in X$ is given. Then $x = \sum \langle x, a_n\rangle\, x_n$ converges in X, so $Sx = \sum \langle x, a_n\rangle\, y_n$ converges in Y. Clearly S defined in this way is linear. The fact that the expansion $x = \sum \langle x, a_n\rangle\, x_n$ is unique ensures that S is well-defined. Further, if $Sx = 0$ then $\sum 0\, y_n = 0 = Sx = \sum \langle x, a_n\rangle\, y_n$, and therefore $\langle x, a_n\rangle = 0$ for every n since $\{y_n\}$ is a basis. This implies $x = \sum \langle x, a_n\rangle\, x_n = 0$, so we conclude that S is injective. Next, if y is any element of y, then $y = \sum \langle y, b_n\rangle\, y_n$ converges in Y, so $x = \sum \langle y, b_n\rangle\, x_n$ converges in X. Since $x = \sum \langle x, a_n\rangle\, x_n$ and $\{x_n\}$ is a basis, this forces $\langle y, b_n\rangle = \langle x, a_n\rangle$ for every n. Hence $Sx = y$ and therefore S is surjective. Thus S is a bijection of X onto Y.

It remains only to show that S is continuous. For each N, define $T_N\colon X \to Y$ by $T_N x = \sum_{n=1}^N \langle x, a_n\rangle\, y_n$. Since each functional a_n is continuous, we conclude that each T_N is continuous. In fact,

$$\|T_N x\| \;=\; \left\|\sum_{n=1}^N \langle x, a_n\rangle\, y_n\right\| \;\le\; \sum_{n=1}^N |\langle x, a_n\rangle|\, \|y_n\| \;\le\; \|x\| \sum_{n=1}^N \|a_n\|\, \|y_n\|.$$

Since $T_N x \to Sx$, we conclude that $\|Sx\| \le \sup \|T_N x\| < \infty$ for each individual $x \in X$. By the Uniform Boundedness Principle (Theorem 1.42), it follows that $\sup \|T_N\| < \infty$. However, $\|S\| \le \sup \|T_N\|$, so S is a bounded mapping. □

Corollary 4.15. *All orthonormal bases in a Hilbert space are equivalent.*

Proof. Suppose that $\{e_n\}$ and $\{f_n\}$ are both orthonormal bases for a Hilbert space H. Then, by Theorem 1.19(a),

$$\sum_n c_n e_n \text{ converges} \quad\Longleftrightarrow\quad \sum_n |c_n|^2 < \infty \quad\Longleftrightarrow\quad \sum_n c_n f_n \text{ converges.}$$

Hence $\{e_n\} \sim \{f_n\}$ by Theorem 4.14. □

5. ABSOLUTELY CONVERGENT BASES IN BANACH SPACES

It is often desirable to have a basis $\{x_n\}$ such that the series $x = \sum \langle x, a_n \rangle x_n$ has some special convergence properties. In this section we study those bases which have the property that this series is always absolutely convergent. We will see that this is a highly restrictive condition, which implies that X is isomorphic to ℓ^1. In Chapter 9 we will study those bases for which the series $x = \sum \langle x, a_n \rangle x_n$ is always unconditionally convergent.

Definition 5.1. A basis $(\{x_n\}, \{a_n\})$ for a Banach space X is *absolutely convergent* if the series $x = \sum \langle x, a_n \rangle x_n$ converges absolutely in X for each $x \in X$. That is, we require that

$$\forall\, x \in X, \quad \sum_n |\langle x, a_n \rangle| \, \|x_n\| < \infty. \quad \diamondsuit$$

Theorem 5.2. [Mar69, p. 42]. *If a Banach space X possesses an absolutely convergent basis then X is topologically isomorphic to ℓ^1.*

Proof. Suppose that $(\{x_n\}, \{a_n\})$ is an absolutely convergent basis for X. Define the mapping $T\colon X \to \ell^1$ by $Tx = (\langle x, a_n \rangle \|x_n\|)$. Certainly T is a well-defined, injective, and linear map. Suppose that $y_N \in X$, that $y_N \to y \in X$, and that $Ty_N \to (c_n) \in \ell^1$. Then

$$\lim_{N\to\infty} \sum_n \Bigl| \langle y_N, a_n \rangle \|x_n\| \; - \; c_n \Bigr| \;=\; \lim_{N\to\infty} \|Ty_N - (c_n)\|_{\ell^1} \;=\; 0. \tag{5.1}$$

Since the coefficient functionals a_n are continuous, we have by (5.1) that

$$\langle y, a_n \rangle \|x_n\| \;=\; \lim_{N\to\infty} \langle y_N, a_n \rangle \|x_n\| \;=\; c_n.$$

Therefore $Ty = (c_n)$, so T is a closed mapping. We conclude from the Closed Graph Theorem (Theorem 1.46) that T is continuous.

Now choose any $(c_n) \in \ell^1$. Then $(\|c_n x_n\|/\|x_n\|) \in \ell^1$, so $x = \sum \frac{c_n}{\|x_n\|} x_n \in X$. However, $Tx = (c_n)$, so T is surjective. Therefore T is a topological isomorphism of X onto ℓ^1. In particular, it follows from the Inverse Mapping Theorem (Theorem 1.44) that T^{-1} is continuous. Alternatively, we can see this directly from the calculation

$$\|x\| \;=\; \Bigl\| \sum_n \langle x, a_n \rangle x_n \Bigr\| \;\le\; \sum_n |\langle x, a_n \rangle| \, \|x_n\| \;=\; \|Tx\|_{\ell^1}. \quad \square$$

Example 5.3. Let H be a separable, infinite-dimensional Hilbert space, and let $\{e_n\}$ be any orthonormal basis for H. We saw in Example 2.5 that $\sum c_n e_n$ converges if and only if $(c_n) \in \ell^2$, and that in this case the convergence is unconditional. On the other hand, since $\|e_n\| = 1$, we see that $\sum c_n e_n$ converges absolutely if and only if $(c_n) \in \ell^1$. Since ℓ^1 is a proper subset of ℓ^2, this implies that $\{e_n\}$ is not an absolutely convergent basis for H. Moreover, since H is topologically isomorphic to ℓ^2, and since ℓ^2 is not topologically isomorphic to ℓ^1, it follows from Theorem 5.2 that H does not possess any absolutely convergent bases. $\diamondsuit$

6. SOME TYPES OF LINEAR INDEPENDENCE OF SEQUENCES

In an infinite-dimensional Banach space, there are several possible types of linear independence of sequences. We list three of these in the following definition. We will consider minimal sequences in particular in more detail in Chapter 7.

Definition 6.1. A sequence $\{x_n\}$ in a Banach space X is:

(a) *finitely independent* if $\sum_{n=1}^{N} c_n x_n = 0$ implies $c_1 = \cdots = c_N = 0$,

(b) *ω-independent* if $\sum_{n=1}^{\infty} c_n x_n$ converges and equals 0 only when $c_n = 0$ for every n,

(c) *minimal* if $x_m \notin \overline{\text{span}}\{x_n\}_{n \neq m}$ for every m. ◇

Theorem 6.2. *Let $\{x_n\}$ be a sequence in a Banach space X. Then:*

(a) *$\{x_n\}$ is a basis $\implies$ $\{x_n\}$ is minimal and complete.*

(b) *$\{x_n\}$ is minimal $\implies$ $\{x_n\}$ is ω-independent.*

(c) *$\{x_n\}$ is ω-independent $\implies$ $\{x_n\}$ is finitely independent.*

Proof. (a) Assume that $(\{x_n\}, \{a_n\})$ is a basis for a Banach space X. Then $\{x_n\}$ is certainly complete, so we need only show that it is minimal. Fix m, and define $E = \text{span}\{x_n\}_{n \neq m}$. Then, since $\{x_n\}$ and $\{a_n\}$ are biorthogonal, we have $\langle x, a_m \rangle = 0$ for every $x \in E$. Since a_m is continuous, this implies $\langle x, a_m \rangle = 0$ for every $x \in \bar{E} = \overline{\text{span}}\{x_n\}_{n \neq m}$. However, we know that $\langle x_m, a_m \rangle = 1$, so we conclude that $x_m \notin \bar{E}$. Hence $\{x_n\}$ is minimal.

(b) Suppose that $\{x_n\}$ is minimal and that $\sum c_n x_n$ converges and equals 0. Let m be such that $c_m \neq 0$. Then $x_m = -\frac{1}{c_m} \sum_{m \neq n} c_n x_n \in \overline{\text{span}}\{x_n\}_{n \neq m}$, a contradiction.

(c) Clear. □

None of the implications in Theorem 6.2 are reversible, as the following examples show.

Example 6.3. [Sin70, p. 24]. *Minimal and complete $\not\Longrightarrow$ basis.*

Define $C(\mathbf{T}) = \{f \in C(\mathbf{R}) : f(t+1) = f(t)\}$, the space of all continuous, 1-periodic functions. Then $C(\mathbf{T})$ is a Banach space under the uniform norm $\|\cdot\|_{L^\infty}$. Consider the functions $e_n(t) = e^{2\pi int}$ for $n \in \mathbf{Z}$. Not only are these functions elements of $C(\mathbf{T})$, but they define continuous linear functionals on $C(\mathbf{T})$ via the inner product $\langle f, e_n \rangle = \int_0^1 f(t)\, e^{-2\pi int}\, dt$. Further, $\{e_n\}_{n \in \mathbf{Z}}$ is its own biorthogonal system since $\langle e_m, e_n \rangle = \delta_{mn}$. Lemma 7.2 below therefore implies that $\{e_n\}_{n \in \mathbf{Z}}$ is minimal in $C(\mathbf{T})$. The Weierstrass Approximation Theorem [Kat68, p. 15] states that if $f \in C(\mathbf{T})$ then $\left\|f - \sum_{n=-N}^{N} c_n e_n\right\|_{L^\infty} < \varepsilon$ for some scalars c_n. Hence $\text{span}\{e_n\}_{n \in \mathbf{Z}}$ is dense in $C(\mathbf{T})$, and therefore $\{e_n\}_{n \in \mathbf{Z}}$ is complete in $C(\mathbf{T})$. Alternatively, we can demonstrate the completeness as follows. Suppose that $f \in C(\mathbf{T})$ satisfies $\langle f, e_n \rangle = 0$ for every n. Since $C(\mathbf{T}) \subset L^2(\mathbf{T})$ and since $\{e_n\}_{n \in \mathbf{Z}}$ is an orthonormal basis for $L^2(\mathbf{T})$, this implies that f is the zero function in the space

$L^2(\mathbf{T})$, hence is zero almost everywhere. Since f is continuous, it follows that $f(t) = 0$ for all t. Hence $\{e_n\}_{n\in\mathbf{Z}}$ is complete both $C(\mathbf{T})$ and $L^2(\mathbf{T})$ by Corollary 1.41.

Thus, $\{e_n\}_{n\in\mathbf{Z}}$ is both minimal and complete in $C(\mathbf{T})$. Further, if $f = \sum c_n e_n$ converges in $C(\mathbf{T})$, then it is easy to see from the orthonormality of the e_n that $c_n = \langle f, e_n\rangle$. However, it is known that there exist continuous functions $f \in C(\mathbf{T})$ whose Fourier series $f = \sum \langle f, e_n\rangle\, e_n$ do not converge uniformly [Kat68, p. 51]. Therefore, $\{e_n\}_{n\in\mathbf{Z}}$ cannot be a basis for $C(\mathbf{T})$. $\diamondsuit$

Example 6.4. [Sin70, p. 24]. *ω-independent $\not\Rightarrow$ minimal.*

Let X be a Banach space such that there exists a sequence $\{x_n\}$ that is both minimal and complete in X but is not a basis for X (for example, we could use $X = C(\mathbf{T})$ and $x_n(t) = e_n(t) = e^{2\pi int}$ as in Example 6.3). Since $\{x_n\}$ is minimal, it follows from Lemma 7.2 that there exists a sequence $\{a_n\} \subset X^*$ that is biorthogonal to $\{x_n\}$. Since $\{x_n\}$ is not a basis, there must exist some $y \in X$ such that the series $\sum \langle y, a_n\rangle\, x_n$ does not converge in X. Consider the sequence $\{y\} \cup \{x_n\}$. This new sequence is certainly complete, and since $y \in \overline{\text{span}}\{x_n\}$, it cannot be minimal. However, we will show that $\{y\} \cup \{x_n\}$ is ω-independent. Assume that $cy + \sum c_n x_n = 0$, i.e., the summation converges and equals zero. If $c \neq 0$ then we would have $y = -\frac{1}{c}\sum c_n x_n$. The biorthogonality of $\{x_n\}$ and $\{a_n\}$ then implies that $\langle y, a_n\rangle = -c_n/c$. But then $\sum \langle y, a_n\rangle\, x_n$ converges, which is a contradiction. Therefore, we must have $c = 0$, and therefore $\sum c_n x_n = 0$. However, $\{x_n\}$ is minimal, and therefore is ω-independent, so this implies that every c_n is zero. Thus $\{y\} \cup \{x_n\}$ is ω-independent and complete, but not minimal.

Alternatively, we can give a Hilbert space example of a complete ω-independent sequence that is not minimal [VD97]. Let $\{e_n\}$ be any orthonormal basis for any separable Hilbert space H, and define $f_1 = e_1$ and $f_n = e_1 + e_n/n$ for $n \geq 2$. Then $\{f_n\}$ is certainly complete since $\text{span}\{f_n\} = \text{span}\{e_n\}$. However, $\|f_1 - f_n\| = \|e_n/n\| = 1/n \to 0$. Therefore $f_1 \in \overline{\text{span}}\{f_n\}_{n\geq 2}$, so $\{f_n\}$ is not minimal. To see that $\{f_n\}$ is ω-independent, suppose that $\sum c_n f_n$ converges and equals zero. Then

$$\sum_{n=1}^{N} c_n f_n \;=\; \Bigl(\sum_{n=1}^{N} c_n\Bigr) e_1 \;+\; \sum_{n=2}^{N} c_n e_n \;\to\; 0.$$

Therefore,

$$\biggl\|\Bigl(\sum_{n=1}^{N} c_n\Bigr) e_1 \;+\; \sum_{n=2}^{N} c_n e_n\biggr\|^2 \;=\; \biggl|\sum_{n=1}^{N} c_n\biggr|^2 \;+\; \sum_{n=2}^{N} |c_n|^2 \;\to\; 0.$$

This implies immediately that $c_n = 0$ for each $n \geq 2$, and therefore $c_1 = 0$ as well. $\diamondsuit$

Example 6.5. [Sin70, p. 25]. *Finitely independent $\not\Rightarrow$ ω-independent.*

Let $(\{x_n\}, \{a_n\})$ be a basis for a Banach space X, and let $x \in X$ be any element such that $\langle x, a_n\rangle \neq 0$ for every n. For example, we could take $x = \sum \frac{x_n}{2^n\,\|x_n\|}$. Note that x cannot equal any x_n because $\langle x_n, a_m\rangle = 0$ when $m \neq n$. Consider then the new sequence $\{x\} \cup \{x_n\}$. This is certainly complete, and $-x + \sum \langle x, a_n\rangle\, x_n = 0$, so it is not ω-independent. However, we will show that it is finitely independent. Suppose that $cx + \sum_{n=1}^{N} c_n x_n = 0$. Substituting the fact that

$x = \sum \langle x, a_n \rangle\, x_n$, it follows that

$$\sum_{n=1}^{N} \bigl(c\langle x, a_n\rangle + c_n\bigr)\, x_n \;+\; \sum_{n=N+1}^{\infty} c\langle x, a_n\rangle\, x_n \;=\; 0.$$

However, $\{x_n\}$ is a basis, so this is only possible if $c\langle x, a_n\rangle + c_n = 0$ for $n = 1, \dots, N$ and $c\langle x, a_n\rangle = 0$ for $n > N$. Since no $\langle x, a_n\rangle$ is zero we therefore must have $c = 0$. But then $c_1 = \cdots = c_N = 0$, so $\{x\} \cup \{x_n\}$ is finitely independent. $\diamondsuit$

7. BIORTHOGONAL SYSTEMS IN BANACH SPACES

A basis $\{x_n\}$ and its associated coefficient functionals $\{a_n\}$ are an example of biorthogonal sequences. We study the properties of general biorthogonal systems in this chapter.

Definition 7.1. Given a Banach space X and given sequences $\{x_n\} \subset X$ and $\{a_n\} \subset X^*$, we say that $\{a_n\}$ is *biorthogonal* to $\{x_n\}$, or that $(\{x_n\}, \{a_n\})$ is a *biorthogonal system*, if $\langle x_m, a_n \rangle = \delta_{mn}$ for every m, n. We associate with each biorthogonal system $(\{x_n\}, \{a_n\})$ the *partial sum operators* $S_N \colon X \to X$ defined by

$$S_N x \;=\; \sum_{n=1}^{N} \langle x, a_n \rangle \, x_n. \quad \diamondsuit$$

We show now that the existence of sequence biorthogonal to $\{x_n\}$ is equivalent to the statement that $\{x_n\}$ is minimal.

Lemma 7.2. [You80, p. 28], [Sin70, p. 53]. *Let X be a Banach space, and let $\{x_n\} \subset X$. Then:*

(a) $\exists\ \{a_n\} \subset X^*$ *biorthogonal to* $\{x_n\} \iff \{x_n\}$ *is minimal.*

(b) $\exists$ *unique* $\{a_n\} \subset X^*$ *biorthogonal to* $\{x_n\} \iff \{x_n\}$ *is minimal and complete.*

Proof. (a) $\Rightarrow$. Suppose that $\{a_n\} \subset X^*$ is biorthogonal to $\{x_n\}$. Fix any m, and choose $z \in \operatorname{span}\{x_n\}_{n \neq m}$, say $z = \sum_{j=1}^{N} c_{n_j} x_{n_j}$. Then $\langle z, a_m \rangle = \sum_{j=1}^{N} c_{n_j} \langle x_{n_j}, a_m \rangle = 0$ since $x_{n_j} \neq x_m$ for all j. Since a_m is continuous, we then have $\langle z, a_m \rangle = 0$ for all $z \in \overline{\operatorname{span}}\{x_n\}_{n \neq m}$. However $\langle x_m, a_m \rangle = 1$, so we must have $x_m \notin \overline{\operatorname{span}}\{x_n\}_{n \neq m}$. Therefore $\{x_n\}$ is minimal.

$\Leftarrow$. Suppose that $\{x_n\}$ is minimal. Fix m, and define $E = \overline{\operatorname{span}}\{x_n\}_{n \neq m}$. This is a closed subspace of X which does not contain x_m. Therefore, by the Hahn–Banach Theorem (Corollary 1.40) there is a functional $a_m \in X^*$ such that

$$\langle x_m, a_m \rangle \;=\; 1 \qquad \text{and} \qquad \langle x, a_m \rangle \;=\; 0 \text{ for } x \in E.$$

Repeating this for all m we obtain a sequence $\{a_n\}$ that is biorthogonal to $\{x_n\}$.

(b) $\Rightarrow$. Suppose there is a unique sequence $\{a_n\} \subset X^*$ that is biorthogonal to $\{x_n\}$. We know that $\{x_n\}$ is minimal by part (a), so it remains only to show that $\{x_n\}$ is complete. Suppose that $x^* \in X^*$ is a continuous linear functional such that $\langle x_n, x^* \rangle = 0$ for every n. Then

$$\langle x_m, x^* + a_n \rangle \;=\; \langle x_m, x^* \rangle + \langle x_m, a_n \rangle \;=\; 0 + \delta_{mn} = \delta_{mn}.$$

Thus $\{x^* + a_n\}$ is also biorthogonal to $\{x_n\}$. By our uniqueness assumption, we must have $x^* = 0$. The Hahn–Banach Theorem (Corollary 1.41) therefore implies that $\overline{\operatorname{span}}\{x_n\} = X$, so $\{x_n\}$ is complete.

$\Leftarrow$. Suppose that $\{x_n\}$ is both minimal and complete. By part (a) we know that there exists at least one sequence $\{a_n\} \subset X^*$ that is biorthogonal to $\{x_n\}$, so we need only show that this sequence is unique. Suppose that $\{b_n\} \subset X^*$ is also biorthogonal to $\{x_n\}$. Then $\langle x_n, a_m - b_m\rangle = \delta_{mn} - \delta_{mn} = 0$ for every m and n. However, $\{x_n\}$ is complete, so the Hahn–Banach Theorem (Corollary 1.41) implies that $a_m - b_m = 0$ for every m. Thus $\{a_n\}$ is unique. □

Next, we characterize the additional properties that a minimal sequence must possess in order to be a basis.

Theorem 7.3. [Sin70, p. 25]. *Let $\{x_n\}$ be a sequence in a Banach space X. Then the following statements are equivalent.*

(a) $\{x_n\}$ *is a basis for* X.

(b) *There exists a biorthogonal sequence* $\{a_n\} \subset X^*$ *such that*

$$\forall\, x \in X, \quad x = \sum_n \langle x, a_n\rangle\, x_n.$$

(c) $\{x_n\}$ *is complete and there exists a biorthogonal sequence* $\{a_n\} \subset X^*$ *such that*

$$\forall\, x \in X, \quad \sup_N \|S_N x\| < \infty.$$

(d) $\{x_n\}$ *is complete and there exists a biorthogonal sequence* $\{a_n\} \subset X^*$ *such that*

$$\sup_N \|S_N\| < \infty.$$

Proof. (a) $\Rightarrow$ (b). Follows immediately from the definition of basis and the fact that every basis is a Schauder basis (Theorem 4.11).

(b) $\Rightarrow$ (a). Assume that statement (b) holds. We need only show that the representation $x = \sum \langle x, a_n\rangle\, x_n$ is unique. However, each a_m is continuous, so if $x = \sum c_n x_n$, then $\langle x, a_m\rangle = \sum c_n \langle x_n, a_m\rangle = \sum c_n\, \delta_{mn} = c_m$.

(b) $\Rightarrow$ (c). Assume that statement (b) holds. Then the fact that every x can be written $x = \sum \langle x, a_n\rangle\, x_n$ implies that $\mathrm{span}\{x_n\}$ is dense is X, hence that $\{x_n\}$ is complete. Further, it implies that $x = \lim_{N\to\infty} S_N x$, i.e., that the sequence $\{S_N x\}$ is convergent. Therefore statement (c) holds since all convergent sequences are bounded.

(c) $\Rightarrow$ (d). Each S_N is a bounded linear operator mapping X into itself. Therefore, this implication follows immediately from the Uniform Boundedness Principle (Theorem 1.42).

(d) $\Rightarrow$ (b). Assume that statement (d) holds, and choose any $x \in \mathrm{span}\{x_n\}$, say $x = \sum_{n=1}^M c_n x_n$. Then, since S_N is linear and $\{x_n\}$ and $\{a_n\}$ are biorthogonal, we have for each $N \geq M$ that

$$S_N x = S_N\Biggl(\sum_{m=1}^M c_m x_m\Biggr) = \sum_{m=1}^M c_m\, S_N x_m = \sum_{m=1}^M c_m \sum_{n=1}^N \langle x_m, a_n\rangle\, x_n = \sum_{m=1}^M c_m x_m = x.$$

Therefore, we trivially have $x = \lim_{N\to\infty} S_N x = \sum \langle x, a_n\rangle\, x_n$ when $x \in \mathrm{span}\{x_n\}$.

Now we will show that $x = \lim_{N\to\infty} S_N x$ for arbitrary $x \in X$. Let $C = \sup \|S_N\|$, and let x be an arbitrary element of X. Since $\{x_n\}$ is complete, $\mathrm{span}\{x_n\}$ is dense in X. Therefore, given $\varepsilon > 0$ we can find an element $y \in \mathrm{span}\{x_n\}$ with $\|x - y\| < \varepsilon/(1+C)$, say $y = \sum_{m=1}^{M} c_m x_m$. Then for $N \ge M$ we have

$$\begin{aligned}\|x - S_N x\| &\le \|x-y\| \;+\; \|y - S_N y\| \;+\; \|S_N y - S_N x\| \\ &\le \|x-y\| \;+\; 0 \;+\; \|S_N\|\,\|x-y\| \\ &\le (1+C)\,\|x-y\| \\ &< \varepsilon.\end{aligned}$$

Thus $x = \lim_{N\to\infty} S_N x = \sum \langle x, a_n\rangle\, x_n$ for arbitrary $x \in X$, as desired. □

The next two theorems give a characterization of minimal sequences and bases in terms of the size of *finite* linear combinations of the sequence elements.

Theorem 7.4. [Sin70, p. 54]. *Given a sequence $\{x_n\}$ in a Banach space X with all $x_n \neq 0$, the following two statements are equivalent.*

(a) $\{x_n\}$ *is minimal.*

(b) $\forall\, M,\ \exists\, C_M \ge 1$ *such that*

$$\forall\, N \ge M, \quad \forall\, c_0, \dots, c_N, \quad \Bigl\|\sum_{n=1}^{M} c_n x_n\Bigr\| \;\le\; C_M \Bigl\|\sum_{n=1}^{N} c_n x_n\Bigr\|.$$

Proof. (a) ⇒ (b). Assume that $\{x_n\}$ is minimal. Then there exists a sequence $\{a_n\} \subset X^*$ that is biorthogonal to $\{x_n\}$. Let $\{S_N\}$ be the partial sum operators associated with $(\{x_n\}, \{a_n\})$. Suppose that $N \ge M$, and that $c_0, \dots, c_N$ are any scalars. Then

$$\Bigl\|\sum_{n=1}^{M} c_n x_n\Bigr\| \;=\; \Bigl\|S_M\Bigl(\sum_{n=1}^{N} c_n x_n\Bigr)\Bigr\| \;\le\; \|S_M\|\,\Bigl\|\sum_{n=1}^{N} c_n x_n\Bigr\|.$$

Therefore statement (b) follows with $C_M = \|S_M\|$.

(b) ⇒ (a). Assume that statement (b) holds, and let $E = \mathrm{span}\{x_n\}$. Given $x = \sum_{n=1}^{N} c_n x_n \in E$ and $M \le N$, we have

$$\begin{aligned}|c_M|\,\|x_M\| \;=\; \|c_M x_M\| &\le \Bigl\|\sum_{n=1}^{M} c_n x_n\Bigr\| \;+\; \Bigl\|\sum_{n=1}^{M-1} c_n x_n\Bigr\| \\ &\le C_M \Bigl\|\sum_{n=1}^{N} c_n x_n\Bigr\| \;+\; C_{M-1} \Bigl\|\sum_{n=1}^{N} c_n x_n\Bigr\| \;=\; (C_M + C_{M-1})\,\|x\|.\end{aligned}$$

As $x_M \neq 0$, we therefore have

$$|c_M| \leq \frac{(C_M + C_{M-1})\,\|x\|}{\|x_M\|}. \tag{7.1}$$

In particular, $x = 0$ implies $c_1 = \cdots = c_N = 0$. Thus $\{x_n\}$ is finitely linearly independent. Since E is the finite linear span of $\{x_n\}$, this implies that every element of E has a *unique* representation of the form $x = \sum_{n=1}^{N} c_n x_n$. As a consequence, we can define a scalar-valued mapping a_m on the set E by $a_m\bigl(\sum_{n=1}^{N} c_n x_n\bigr) = c_m$ (where we set $c_m = 0$ if $m > N$). By (7.1), we have $|a_m(x)| \leq (C_m + C_{m-1})\,\|x\|/\|x_m\|$ for every $x \in E$, so a_m is continuous on E. Since E is dense in X, the Hahn–Banach Theorem (Corollary 1.38) implies that there is a continuous extension of a_m to all of X. This extended a_m is therefore a continuous linear functional on X which is biorthogonal to $\{x_n\}$. Lemma 7.2 therefore implies that $\{x_n\}$ is minimal. □

For an arbitrary minimal sequence, the constants C_M is Theorem 7.4 need not be uniformly bounded. Compare this to the situation for bases given in the following result.

Theorem 7.5. [LT77, p. 2]. *Let $\{x_n\}$ be a sequence in a Banach space X. Then the following statements are equivalent.*

(a) *$\{x_n\}$ is a basis for X.*

(b) *$\{x_n\}$ is complete, $x_n \neq 0$ for all n, and there exists $C \geq 1$ such that*

$$\forall\, N \geq M, \quad \forall\, c_1, \dots, c_N, \quad \Bigl\|\sum_{n=1}^{M} c_n x_n\Bigr\| \;\leq\; C \Bigl\|\sum_{n=1}^{N} c_n x_n\Bigr\|. \tag{7.2}$$

In this case, the best constant C in (7.2) is the basis constant $C = \sup \|S_N\|$.

Proof. (a) ⇒ (b). Suppose that $\{x_n\}$ is a basis for X, and let $C = \sup \|S_N\|$ be the basis constant. Then $\{x_n\}$ is complete and $x_n \neq 0$ for every n. Fix $N \geq M$, and choose any $c_1, \dots, c_N$. Then

$$\Bigl\|\sum_{n=1}^{M} c_n x_n\Bigr\| \;=\; \Bigl\|S_M\Bigl(\sum_{n=1}^{N} c_n x_n\Bigr)\Bigr\| \;\leq\; \|S_M\|\,\Bigl\|\sum_{n=1}^{N} c_n x_n\Bigr\| \;\leq\; C\,\Bigl\|\sum_{n=1}^{N} c_n x_n\Bigr\|.$$

(b) ⇒ (a). Suppose that statement (b) holds. It then follows from Theorem 7.4 that $\{x_n\}$ is minimal, so by Lemma 7.2 there exists a biorthogonal system $\{a_n\} \subset X^*$. Let S_N denote the partial sum operators associated with $(\{x_n\}, \{a_n\})$. Since $\{x_n\}$ is complete, it suffices by Theorem 7.3 to show that $\sup \|S_N\| < \infty$.

So, suppose that $x = \sum_{n=1}^{M} c_n x_n \in \operatorname{span}\{x_n\}$. Then:

$$N \leq M \implies \|S_N x\| \;=\; \Bigl\|\sum_{n=1}^{N} c_n x_n\Bigr\| \;\leq\; C\,\Bigl\|\sum_{n=1}^{M} c_n x_n\Bigr\| \;=\; C\,\|x\|,$$

$$N > M \implies \|S_N x\| \;=\; \Bigl\|\sum_{n=1}^{M} c_n x_n\Bigr\| \;=\; \|x\|.$$

As $C \geq 1$ we therefore have $\|S_N x\| \leq C\,\|x\|$ for all N whenever $x \in \operatorname{span}\{x_n\}$. However, each S_N is continuous and $\operatorname{span}\{x_n\}$ is dense in X, so this inequality must therefore hold for all $x \in X$. Thus $\sup \|S_N\| \leq C < \infty$, as desired. This inequality also shows that the smallest possible value for C in (7.2) is $C = \sup \|S_N\|$. □

The following result is an application of Theorem 7.5. Given a basis $\{x_n\}$ for a Banach space X, it is often useful to have some bound on how much the elements x_n can be perturbed so that the resulting sequence remains a basis for X, or at least a basis for its closed linear span. The following result is classical, and is a typical example of perturbation theorems that apply to general bases. For specific types of bases in specific Banach spaces, it is often possible to derive sharper results. For a survey of results on basis perturbations, we refer to [RH71].

Theorem 7.6. *Let $(\{x_n\}, \{a_n\})$ be a basis for a Banach space X, with basis constant C. If $\{y_n\} \subset X$ is such that*

$$R \;=\; \sum_n \|a_n\|\,\|x_n - y_n\| \;<\; 1,$$

then $\{y_n\}$ is a basis for $\overline{\operatorname{span}}\{y_n\}$, and has basis constant $C' \leq \frac{1+R}{1-R}\,C$. Moreover, in this case, the basis $\{x_n\}$ for X and the basis $\{y_n\}$ for $Y = \overline{\operatorname{span}}\{y_n\}$ are equivalent in the sense of Definition 4.13.

Proof. Note that, by definition, $\{y_n\}$ is complete in $Y = \overline{\operatorname{span}}\{y_n\}$. Further, if some $y_n = 0$ then we would have $R \geq \|a_n\|\,\|x_n\| \geq 1$ by (4.5), which contradicts the fact that $R < 1$. Therefore, each y_n must be nonzero. By Theorem 7.5, it therefore suffices to show that there exists a constant B such that

$$\forall\, N \geq M, \quad \forall\, c_1, \dots, c_N, \qquad \Bigl\|\sum_{n=1}^{M} c_n y_n\Bigr\| \;\leq\; B\,\Bigl\|\sum_{n=1}^{N} c_n y_n\Bigr\|. \tag{7.3}$$

Further, if (7.3) holds, then Theorem 7.5 also implies that the basis constant C' for $\{y_n\}$ satisfies $C' \leq B$.

So, assume that $N \geq M$ and that $c_1, \dots, c_N$ are given. Before showing the existence of the constant B, we will establish several useful inequalities. First, since $\{x_n\}$ and $\{a_n\}$ are biorthogonal, we have that

$$\forall\, K \geq m, \quad |c_m| \;=\; \Bigl|\Bigl\langle \sum_{n=1}^{K} c_n x_n,\; a_m \Bigr\rangle\Bigr| \;\leq\; \|a_m\|\,\Bigl\|\sum_{n=1}^{K} c_n x_n\Bigr\|.$$

Therefore, for each $K > 0$ we have

$$\begin{aligned}
\Bigl\|\sum_{m=1}^{K} c_m\,(x_m - y_m)\Bigr\| \;&\leq\; \sum_{m=1}^{K} |c_m|\,\|x_m - y_m\| \\
&\leq\; \sum_{m=1}^{K} \Bigl(\|a_m\|\,\Bigl\|\sum_{n=1}^{K} c_n x_n\Bigr\|\Bigr)\,\|x_m - y_m\|
\end{aligned}$$

$$= \left\| \sum_{n=1}^{K} c_n x_n \right\| \sum_{m=1}^{K} \|a_m\| \, \|x_m - y_m\|$$

$$\leq R \left\| \sum_{n=1}^{K} c_n x_n \right\|. \tag{7.4}$$

As a consequence,

$$\left\| \sum_{n=1}^{M} c_n y_n \right\| \leq \left\| \sum_{n=1}^{M} c_n x_n \right\| + \left\| \sum_{n=1}^{M} c_n (y_n - x_n) \right\| \leq (1+R) \left\| \sum_{n=1}^{M} c_n x_n \right\|. \tag{7.5}$$

Further, (7.4) implies that

$$\left\| \sum_{n=1}^{N} c_n x_n \right\| \leq \left\| \sum_{n=1}^{N} c_n y_n \right\| + \left\| \sum_{n=1}^{N} c_n (x_n - y_n) \right\| \leq \left\| \sum_{n=1}^{N} c_n y_n \right\| + R \left\| \sum_{n=1}^{N} c_n x_n \right\|.$$

Therefore,

$$\left\| \sum_{n=1}^{N} c_n y_n \right\| \geq (1-R) \left\| \sum_{n=1}^{N} c_n x_n \right\|. \tag{7.6}$$

Finally, since $\{x_n\}$ is a basis with basis constant C, Theorem 7.5 implies that

$$\left\| \sum_{n=1}^{M} c_n x_n \right\| \leq C \left\| \sum_{n=1}^{N} c_n x_n \right\|. \tag{7.7}$$

Combining (7.5), (7.6), and (7.7), we obtain

$$\left\| \sum_{n=1}^{M} c_n y_n \right\| \leq (1+R) \left\| \sum_{n=1}^{M} c_n x_n \right\| \leq (1+R)C \left\| \sum_{n=1}^{N} c_n x_n \right\| \leq \frac{1+R}{1-R} C \left\| \sum_{n=1}^{N} c_n y_n \right\|.$$

Hence (7.3) holds with $B = \frac{1+R}{1-R} C$, and therefore $\{y_n\}$ is a basis for $\overline{\text{span}}\{y_n\}$ with basis constant $C' \leq B$.

Finally, calculations similar to (7.5) and (7.6) imply that

$$(1-R) \left\| \sum_{n=M+1}^{N} c_n x_n \right\| \leq \left\| \sum_{n=M+1}^{N} c_n y_n \right\| \leq (1+R) \left\| \sum_{n=M+1}^{N} c_n x_n \right\|.$$

Hence $\sum c_n x_n$ converges if and only if $\sum c_n y_n$ converges. It therefore follows from Theorem 4.14 that $\{x_n\}$ is equivalent to $\{y_n\}$. □

8. DUALITY FOR BASES IN BANACH SPACES

Let π denote the canonical embedding of X into X^{**} described in Definition 1.32. That is, if $x \in X$ then $\pi(x) \in X^{**}$ is the continuous linear functional on X^* defined by $\langle x^*, \pi(x)\rangle = \langle x, x^*\rangle$ for $x^* \in X^*$.

Suppose that $(\{x_n\}, \{a_n\})$ is a biorthogonal system in a Banach space X. Then $(\{a_n\}, \{\pi(x_n)\})$ is a biorthogonal system in X^*. Suppose in addition that $(\{x_n\}, \{a_n\})$ is a basis for X. Then it is natural to ask whether $(\{a_n\}, \{\pi(x_n)\})$ is a basis for X^*. In general the answer must be no, since X^* may be nonseparable even though X is separable (e.g., $X = \ell^1$, $X^* = \ell^\infty$), and therefore $\{a_n\}$ could not be complete in X^* in this case. However, $\{a_n\}$ is always complete in its closed linear span $\overline{\operatorname{span}}\{a_n\}$ in X^*, and the next theorem shows that $(\{a_n\}, \{\pi(x_n)\})$ is always a basis for the subspace $\overline{\operatorname{span}}\{a_n\}$ in X^*.

Theorem 8.1. *Let X be a Banach space.*

(a) *If $(\{x_n\}, \{a_n\})$ is a basis for X, then $(\{a_n\}, \{\pi(x_n)\})$ is a basis for $\overline{\operatorname{span}}\{a_n\}$ in X^*.*

(b) *If $(\{x_n\}, \{a_n\})$ is an unconditional basis for X, then $(\{a_n\}, \{\pi(x_n)\})$ is an unconditional basis for $\overline{\operatorname{span}}\{a_n\}$ in X^*.*

(c) *If $(\{x_n\}, \{a_n\})$ is a bounded basis for X, then $(\{a_n\}, \{\pi(x_n)\})$ is a bounded basis for $\overline{\operatorname{span}}\{a_n\}$ in X^*.*

Proof. (a) Suppose that $(\{x_n\}, \{a_n\})$ is a basis for X. By definition, $\{a_n\}$ is complete in the closed subspace $\overline{\operatorname{span}}\{a_n\} \subset X^*$. Further, $(\{a_n\}, \{\pi(x_n)\})$ is a biorthogonal system in X^* since $\langle a_m, \pi(x_n)\rangle = \langle x_n, a_m\rangle = \delta_{mn}$. Therefore, by Theorem 7.3, we need only show that $\sup \|T_N\| < \infty$, where the T_N are the partial sum operators associated with $(\{a_n\}, \{\pi(x_n)\})$, i.e.,

$$T_N(x^*) \;=\; \sum_{n=1}^{N} \langle x^*, \pi(x_n)\rangle\, a_n \;=\; \sum_{n=1}^{N} \langle x_n, x^*\rangle\, a_n, \qquad \text{for } x^* \in \overline{\operatorname{span}}\{a_n\}.$$

As usual, let S_N denote the partial sum operators associated with the basis $(\{x_n\}, \{a_n\})$ for X. Since S_N is a continuous linear mapping of X into itself, it has an adjoint mapping $S_N^*\colon X^* \to X^*$. Since the norm of an adjoint equals the norm of the original operator, we have $\|S_N^*\| = \|S_N\|$ (see Definition 1.34). Now, if $x \in X$ and $x^* \in X^*$ then we have by (1.5) that

$$\langle x, S_N^*(x^*)\rangle \;=\; \langle S_N x, x^*\rangle \;=\; \left\langle \sum_{n=1}^{N} \langle x, a_n\rangle\, x_n,\; x^* \right\rangle \;=\; \left\langle x,\; \sum_{n=1}^{N} \langle x_n, x^*\rangle\, a_n \right\rangle \;=\; \langle x, T_N(x^*)\rangle.$$

Therefore $T_N = S_N^*$, so $\sup \|T_N\| = \sup \|S_N^*\| = \sup \|S_N\| < \infty$.

(b) Suppose that $(\{x_n\}, \{a_n\})$ is an unconditional basis for X. Then by part (a), we know that $(\{a_n\}, \{\pi(x_n)\})$ is a basis for $\overline{\operatorname{span}}\{a_n\}$. So, we need only show that this basis is unconditional. Therefore, fix any $x^* \in \overline{\operatorname{span}}\{a_n\}$. Then $x^* = \sum \langle x^*, \pi(x_n)\rangle\, a_n$ is the unique representation of x^*

in the basis $(\{a_n\}, \{\pi(x_n)\})$. We must show that this series converges unconditionally. Let σ be any permutation of $\mathbf{N}$. Then for any $x \in X$,

$$\begin{aligned}
\langle x, x^* \rangle &= \Big\langle x,\ \sum_n \langle x^*, \pi(x_n)\rangle\, a_n \Big\rangle && \text{since } x^* = \textstyle\sum \langle x^*, \pi(x_n)\rangle\, a_n \\
&= \sum_n \langle x^*, \pi(x_n)\rangle\, \langle x, a_n\rangle \\
&= \sum_n \langle x_n, x^*\rangle\, \langle x, a_n\rangle && \text{by definition of } \pi \\
&= \Big\langle \sum_n \langle x, a_n\rangle\, x_n,\ x^* \Big\rangle \\
&= \Big\langle \sum_n \langle x, a_{\sigma(n)}\rangle\, x_{\sigma(n)},\ x^* \Big\rangle && \text{since } x = \textstyle\sum \langle x, a_n\rangle\, x_n \text{ converges unconditionally} \\
&= \sum_n \langle x, a_{\sigma(n)}\rangle\, \langle x_{\sigma(n)}, x^*\rangle \\
&= \Big\langle x,\ \sum_n \langle x^*, \pi(x_{\sigma(n)})\rangle\, a_{\sigma(n)} \Big\rangle.
\end{aligned}$$

Therefore $x^* = \sum \langle x^*, \pi(x_{\sigma(n)})\rangle\, a_{\sigma(n)}$, so the series $x^* = \sum \langle x^*, \pi(x_n)\rangle\, a_n$ converges unconditionally.

(c) Assume that $(\{x_n\}, \{a_n\})$ is a bounded basis for X. Then, by definition, $0 < \inf \|x_n\| \le \sup \|x_n\| < \infty$. Further, by (4.5) we have $1 \le \|a_n\|\, \|x_n\| \le 2C$, where C is the basis constant for $(\{x_n\}, \{a_n\})$. Therefore $0 < \inf \|a_n\| \le \sup \|a_n\| < \infty$. Combined with part (a), this implies that $(\{a_n\}, \{\pi(x_n)\})$ is a bounded basis. □

Corollary 8.2. *If $(\{x_n\}, \{a_n\})$ is a basis, unconditional basis, or bounded basis for a reflexive Banach space X, then $(\{a_n\}, \{\pi(x_n)\})$ is a basis, unconditional basis, or bounded basis for X^*.*

Proof. Assume $(\{x_n\}, \{a_n\})$ is a basis for X. Then Theorem 8.1 implies that $(\{a_n\}, \{\pi(x_n)\})$ is a basis for for $\overline{\text{span}}\{a_n\}$ in X^*, so we need only show that $\{a_n\}$ is complete in X^*. Suppose that $x^{**} \in X^{**}$ satisfied $\langle a_n, x^{**}\rangle = 0$ for every n. Since X is reflexive, $X^{**} = \pi(X)$. Therefore $x^{**} = \pi(x)$ for some $x \in X$. But then $\langle x, a_n\rangle = \langle a_n, \pi(x)\rangle = \langle a_n, x^{**}\rangle = 0$ for every n. Therefore $x = \sum \langle x, a_n\rangle\, x_n = 0$, so $x^{**} = \pi(x) = 0$. The Hahn–Banach Theorem (Corollary 1.41) therefore implies that $\{a_n\}$ is complete in X^*. The statements for an unconditional or bounded basis then follow as an immediate consequence. □

Corollary 8.3. *Let H be a Hilbert space. Then $(\{x_n\}, \{y_n\})$ is a basis, unconditional basis, or bounded basis for H if and only if the same is true of $(\{y_n\}, \{x_n\})$.*

Proof. The result follows from Corollary 8.2 and the fact that Hilbert spaces are self-dual, i.e., $H^* = H$. □

9. UNCONDITIONAL BASES IN BANACH SPACES

Recall from Definition 4.2 that a basis $(\{x_n\}, \{a_n\})$ is *unconditional* if the series $x = \sum \langle x, a_n \rangle\, x_n$ converges unconditionally for every $x \in X$. Additionally, a basis is *bounded* if $0 < \inf \|x_n\| \le \sup \|x_n\| < \infty$.

Lemma 9.1. [Sin70, p. 461]. *Given a sequence $\{x_n\}$ in a Banach space X, the following two statements are equivalent.*

(a) *$\{x_n\}$ is an unconditional basis for X.*

(b) *$\{x_{\sigma(n)}\}$ is a basis for X for every permutation σ of $\mathbf{N}$.*

Proof. (a) $\Rightarrow$ (b). Assume that $(\{x_n\}, \{a_n\})$ is an unconditional basis for X, and let σ be any permutation of $\mathbf{N}$. Choose any $x \in X$. Then the series $x = \sum \langle x, a_n \rangle\, x_n$ converges unconditionally, so $x = \sum \langle x, a_{\sigma(n)} \rangle\, x_{\sigma(n)}$ converges by Corollary 2.9. We must show that this is the unique representation of x in terms of the $x_{\sigma(n)}$. Suppose that we also had $x = \sum c_n x_{\sigma(n)}$ for some scalars (c_n). Then, since $\{x_n\}$ and $\{a_n\}$ are biorthogonal, we have

$$\langle x, a_{\sigma(m)} \rangle \;=\; \sum_n c_n \, \langle x_{\sigma(n)}, a_{\sigma(m)} \rangle \;=\; \sum_n c_n \, \delta_{\sigma(n),\sigma(m)} \;=\; \sum_n c_n \, \delta_{nm} \;=\; c_m.$$

which shows that the representation is unique.

(b) $\Rightarrow$ (a). Assume that $\{x_{\sigma(n)}\}$ is a basis for every permutation σ of $\mathbf{N}$. Let $\{a_n\}$ be the sequence of coefficient functionals associated with the basis $\{x_n\}$. We must show that for each $x \in X$ the representation $x = \sum \langle x, a_n \rangle\, x_n$ converges unconditionally. Fix any permutation σ of $\mathbf{N}$. Since $\{x_{\sigma(n)}\}$ is a basis, there exist unique scalars c_n such that $x = \sum c_n \, x_{\sigma(n)}$. However, each a_m is continuous and $\{a_n\}$ is biorthogonal to $\{x_n\}$, so

$$\langle x, a_{\sigma(m)} \rangle \;=\; \sum_n c_n \, \langle x_{\sigma(n)}, a_{\sigma(m)} \rangle \;=\; \sum_n c_n \, \delta_{\sigma(n),\sigma(m)} \;=\; \sum_n c_n \, \delta_{nm} \;=\; c_m.$$

Therefore $x = \sum c_n x_n = \sum \langle x, a_{\sigma(n)} \rangle\, x_{\sigma(n)}$ converges for every permutation σ, so $x = \sum \langle x, a_n \rangle\, x_n$ converges unconditionally. □

Example 9.2. [Mar69, p. 83]. Let H be a Hilbert space. We will show that every orthonormal basis $\{e_n\}$ for H is a bounded unconditional basis for H.

Let σ be any permutation of $\mathbf{N}$. Then $\{e_{\sigma(n)}\}$ is still an orthonormal sequence in H. Choose any $x \in H$. Then $\sum |\langle x, e_n \rangle|^2 = \|x\|^2 < \infty$ by Theorem 1.20. The series of real numbers $\sum |\langle x, x_n \rangle|^2$ therefore converges absolutely, and hence converges unconditionally by Lemma 2.3. Therefore

$$\sum_n |\langle x, e_{\sigma(n)} \rangle|^2 \;=\; \sum_n |\langle x, e_n \rangle|^2 \;=\; \|x\|^2.$$

Theorem 1.20 therefore implies that $\{e_{\sigma(n)}\}$ is an orthonormal basis for H. As this is true for every σ, we have by Lemma 9.1 that $\{e_n\}$ is an unconditional basis for H. Finally, this basis is bounded since $\|e_n\| = 1$ for every n. $\diamondsuit$

We will study bounded unconditional bases in Hilbert spaces in detail in Chapter 11.

Recall from Lemma 4.12 that bases are preserved by topological isomorphisms. We next show that the same is true of unconditional bases.

Lemma 9.3.

(a) *Unconditional bases are preserved by topological isomorphisms. That is, if $\{x_n\}$ is an unconditional basis for a Banach space X and $S: X \to Y$ is a topological isomorphism, then $\{Sx_n\}$ is an unconditional basis for Y.*

(b) *Bounded unconditional bases are likewise preserved by topological isomorphisms.*

Proof. (a) If σ is any permutation of $\mathbf{N}$ then we know that $\{x_{\sigma(n)}\}$ is a basis for X. However, bases are preserved by topological isomorphisms (Lemma 4.12), so $\{Sx_{\sigma(n)}\}$ is a basis for Y. As this is true for every σ, the basis $\{Sx_{\sigma(n)}\}$ is unconditional.

(b) In light of part (a), we need only show that $\{Sx_n\}$ is bounded if $\{x_n\}$ is bounded. This follows from the facts $\|Sx_n\| \le \|S\| \, \|x_n\|$ and $\|x_n\| = \|S^{-1}Sx_n\| \le \|S^{-1}\| \, \|Sx_n\|$. $\square$

Recall from Definition 4.13 that two bases are *equivalent* if there exists a topological isomorphism S such that $Sx_n = y_n$ for every n. We will see in Chapter 11 that all bounded unconditional bases in a Hilbert space are equivalent, and are equivalent to orthonormal bases. Up to isomorphisms, the only other Banach spaces in which all bounded unconditional bases are equivalent are the sequence spaces c_0 and ℓ^1.

Notation 9.4. We associate three types of partial sum operators with each unconditional basis $(\{x_n\}, \{a_n\})$. First, with each finite set $F \subset \mathbf{N}$ we associate the partial sum operator $S_F: X \to X$ defined by

$$S_F(x) \;=\; \sum_{n \in F} \langle x, a_n \rangle \, x_n, \qquad x \in X.$$

Second, with each finite set $F \subset \mathbf{N}$ and each set $\mathcal{E} = \{\varepsilon_n\}_{n \in F}$ satisfying $\varepsilon_n = \pm 1$ for each n, we associate the operator $S_{F,\mathcal{E}}: X \to X$ defined by

$$S_{F,\mathcal{E}}(x) \;=\; \sum_{n \in F} \varepsilon_n \, \langle x, a_n \rangle \, x_n, \qquad x \in X.$$

Finally, with each finite set $F \subset \mathbf{N}$ and each collection of bounded scalars $\Lambda = \{\lambda_n\}_{n \in F}$ satisfying $|\lambda| \le 1$ for each n, we associate the operator $S_{F,\Lambda}: X \to X$ defined by

$$S_{F,\Lambda}(x) \;=\; \sum_{n \in F} \lambda_n \, \langle x, a_n \rangle \, x_n, \qquad x \in X. \quad \diamondsuit$$

Although the operators S_F are projections, the operators $S_{F,\mathcal{E}}$ and $S_{F,\Lambda}$ are not projections since they are not idempotent, i.e., $(S_{F,\mathcal{E}})^2$ need not equal $S_{F,\mathcal{E}}$.

The following result is the analogue of Corollary 4.8 for the case of unconditional bases.

Theorem 9.5. *Let $(\{x_n\},\{a_n\})$ be an unconditional basis for a Banach space X. Then the following statements hold.*

(a) *The following three quantities are finite for each $x \in X$:*

$$|||x||| \;=\; \sup_F \|S_F(x)\|, \qquad |||x|||_{\mathcal{E}} \;=\; \sup_{F,\mathcal{E}} \|S_{F,\mathcal{E}}(x)\|, \qquad |||x|||_{\Lambda} \;=\; \sup_{F,\Lambda} \|S_{F,\Lambda}(x)\|.$$

(b) *The following three numbers are finite:*

$$K \;=\; \sup_F \|S_F\|, \qquad K_{\mathcal{E}} \;=\; \sup_{F,\mathcal{E}} \|S_{F,\mathcal{E}}\|, \qquad K_{\Lambda} \;=\; \sup_{F,\Lambda} \|S_{F,\Lambda}\|.$$

(c) $|||\cdot||| \le |||\cdot|||_{\mathcal{E}} \le 2\,|||\cdot|||$ *and* $K \le K_{\mathcal{E}} \le 2K$.

(d) *If* $\mathbf{F} = \mathbf{R}$ *then* $|||\cdot|||_{\mathcal{E}} = |||\cdot|||_{\Lambda}$ *and* $K_{\mathcal{E}} = K_{\Lambda}$.

(e) *If* $\mathbf{F} = \mathbf{C}$ *then* $|||\cdot|||_{\mathcal{E}} \le |||\cdot|||_{\Lambda} \le 2\,|||\cdot|||_{\mathcal{E}}$ *and* $K_{\mathcal{E}} \le K_{\Lambda} \le 2K_{\mathcal{E}}$.

(f) $|||\cdot|||$, $|||\cdot|||_{\mathcal{E}}$, *and* $|||\cdot|||_{\Lambda}$ *form norms on X equivalent to the initial norm $\|\cdot\|$. In fact,*

$$\|\cdot\| \;\le\; |||\cdot||| \;\le\; K\,\|\cdot\|, \qquad \|\cdot\| \;\le\; |||\cdot|||_{\mathcal{E}} \;\le\; K_{\mathcal{E}}\,\|\cdot\|, \qquad \|\cdot\| \;\le\; |||\cdot|||_{\Lambda} \;\le\; K_{\Lambda}\,\|\cdot\|.$$

Proof. (a), (c), (d), (e). These follow from the fact that $x = \sum \langle x, a_n\rangle\, x_n$ and that this series converges unconditionally.

(b) Follows from (a) by the Uniform Boundedness Principle (Theorem 1.42).

(f) Follows from (a) and (b). □

Notation 9.6. If $(\{x_n\},\{a_n\})$ is an unconditional basis for a Banach space X, then we let $|||\cdot|||$, $|||\cdot|||_{\mathcal{E}}$, and $|||\cdot|||_{\Lambda}$ denote the equivalent norms for X defined in Theorem 9.5(a), and we let K, $K_{\mathcal{E}}$, and K_{Λ} be the numbers defined in Theorem 9.5(b). In particular, $K_{\mathcal{E}}$ is the *unconditional basis constant* for $(\{x_n\},\{a_n\})$. ◇

Comparing the number K and the unconditional basis constant $K_{\mathcal{E}}$ to the basis constant C from Definition 4.9, we see that $C \le K \le K_{\mathcal{E}}$. In fact, if we let C_σ be the basis constant for the permuted basis $\{x_{\sigma(n)}\}$, then $K = \sup C_\sigma$.

Just as for the basis constant C, the unconditional basis constant $K_{\mathcal{E}}$ does depend implicitly on the norm for X, and changing the norm to some other equivalent norm may change the value of the basis constant. For example, the unconditional basis constant for $(\{x_n\},\{a_n\})$ in the equivalent norm $|||\cdot|||_{\mathcal{E}}$ is 1 (compare Proposition 4.10 for the analogous statement for the basis constant).

Next we give several equivalent definitions of unconditional bases.

Theorem 9.7. [Sin70, p. 461]. *Let $\{x_n\}$ be a complete sequence in a Banach space X such that $x_n \ne 0$ for every n. Then the following statements are equivalent.*

(a) *$\{x_n\}$ is an unconditional basis for X.*

(b) $\exists\, C_1 \geq 1, \quad \forall\, c_1, \dots, c_N, \quad \forall\, \varepsilon_1, \dots, \varepsilon_N = \pm 1,$

$$\biggl\|\sum_{n=1}^N \varepsilon_n c_n x_n\biggr\| \;\leq\; C_1 \,\biggl\|\sum_{n=1}^N c_n x_n\biggr\|.$$

(c) $\exists\, C_2 \geq 1, \quad \forall\, b_1, \dots, b_N, \quad \forall\, c_1, \dots, c_N,$

$$|b_1| \leq |c_1|, \dots, |b_N| \leq |c_N| \implies \biggl\|\sum_{n=1}^N b_n x_n\biggr\| \;\leq\; C_2 \,\biggl\|\sum_{n=1}^N c_n x_n\biggr\|.$$

(d) $\exists\, 0 < C_3 \leq 1 \leq C_4 < \infty, \quad \forall\, c_1, \dots, c_N,$

$$C_3 \,\biggl\|\sum_{n=1}^N |c_n|\, x_n\biggr\| \;\leq\; \biggl\|\sum_{n=1}^N c_n x_n\biggr\| \;\leq\; C_4 \,\biggl\|\sum_{n=1}^N |c_n|\, x_n\biggr\|.$$

(e) $\{x_n\}$ *is a basis, and for each bounded sequence of scalars* $\Lambda = (\lambda_n)$ *there exists a continuous linear operator* $T_\Lambda \colon X \to X$ *such that* $T_\Lambda(x_n) = \lambda_n x_n$ *for all* n.

Proof. (a) ⇒ (b). Suppose that $(\{x_n\}, \{a_n\})$ is an unconditional basis for X. Choose any scalars $c_1, \dots, c_N$ and any signs $\varepsilon_1, \dots, \varepsilon_N = \pm 1$, and set $x = \sum_{n=1}^N c_n x_n$. Then $\langle x, a_n\rangle = c_n$ if $n \leq N$ while $\langle x, a_n\rangle = 0$ if $n > N$. Therefore

$$\sum_{n=1}^N \varepsilon_n c_n x_n \;=\; \sum_{n \in F} \varepsilon_n \langle x, a_n\rangle\, x_n \;=\; S_{F,\mathcal{E}}(x),$$

where $F = \{1, \dots, N\}$ and $\mathcal{E} = \{\varepsilon_1, \dots, \varepsilon_N\}$. By definition of $|\!|\!| \cdot |\!|\!|_{\mathcal{E}}$ and by Theorem 9.5(f), we therefore have

$$\biggl\|\sum_{n=1}^N \varepsilon_n c_n x_n\biggr\| \;=\; \|S_{F,\mathcal{E}}(x)\| \;=\; |\!|\!| x |\!|\!|_{\mathcal{E}} \;\leq\; K_{\mathcal{E}}\, \|x\| \;=\; K_{\mathcal{E}} \,\biggl\|\sum_{n=1}^N c_n x_n\biggr\|.$$

Thus statement (b) holds with $C_1 = K_{\mathcal{E}}$.

(b) ⇒ (a). Suppose that statement (b) holds, and let σ be any permutation of $\mathbf{N}$. We must show that $\{x_{\sigma(n)}\}$ is a basis for X. By hypothesis, $\{x_{\sigma(n)}\}$ is complete with every element nonzero. Therefore, by Theorem 7.5 it suffices to show that there is a constant C_σ such that

$$\forall\, N \geq M, \quad \forall\, c_{\sigma(1)}, \dots, c_{\sigma(N)}, \quad \biggl\|\sum_{n=1}^M c_{\sigma(n)} x_{\sigma(n)}\biggr\| \;\leq\; C_\sigma \,\biggl\|\sum_{n=1}^N c_{\sigma(n)} x_{\sigma(n)}\biggr\|.$$

To this end, fix any $N \geq M$ and choose any scalars $c_{\sigma(1)}, \dots, c_{\sigma(N)}$. Define $c_n = 0$ for $n \notin \{\sigma(1), \dots, \sigma(N)\}$. Let $L = \max\{\sigma(1), \dots, \sigma(N)\}$, and define

$$\varepsilon_n = 1 \qquad \text{and} \qquad \eta_n \;=\; \begin{cases} 1, & \text{if } n \in \{\sigma(1), \dots, \sigma(M)\}, \\ 0, & \text{otherwise.} \end{cases}$$

Then,

$$\begin{aligned}\left\|\sum_{n=1}^{M} c_{\sigma(n)}x_{\sigma(n)}\right\| &= \left\|\sum_{n=1}^{L}\left(\frac{\varepsilon_n+\eta_n}{2}\right)c_nx_n\right\| \\ &\le \frac{1}{2}\left\|\sum_{n=1}^{L}\varepsilon_nc_nx_n\right\| + \frac{1}{2}\left\|\sum_{n=1}^{L}\eta_nc_nx_n\right\| \\ &\le \frac{C_1}{2}\left\|\sum_{n=1}^{L}c_nx_n\right\| + \frac{C_1}{2}\left\|\sum_{n=1}^{L}c_nx_n\right\| \\ &= C_1\left\|\sum_{n=1}^{N}c_{\sigma(n)}x_{\sigma(n)}\right\|.\end{aligned}$$

This is the desired result, with $C_\sigma = C_1$.

(a) $\Rightarrow$ (c). Suppose that $(\{x_n\},\{a_n\})$ is an unconditional basis for X. Choose any scalars $c_1,\dots,c_N$ and $b_1,\dots,b_N$ such that $|b_n| \le |c_n|$ for every n. Define $x = \sum_{n=1}^{N} c_nx_n$, and note that $c_n = \langle x, a_n\rangle$. Let λ_n be such that $b_n = \lambda_nc_n$. Since $|b_n| \le |c_n|$ we have $|\lambda_n| \le 1$ for every n. Therefore, if we define $F = \{1,\dots,N\}$ and $\Lambda = \{\lambda_1,\dots,\lambda_N\}$, then

$$\sum_{n=1}^{N} b_nx_n = \sum_{n\in F}\lambda_nc_nx_n = \sum_{n\in F}\lambda_n\langle x,a_n\rangle x_n = S_{F,\Lambda}(x).$$

Hence

$$\left\|\sum_{n=1}^{N} b_nx_n\right\| = \|S_{F,\Lambda}(x)\| = |||x|||_\Lambda \le K_\Lambda\|x\| = K_\Lambda\left\|\sum_{n=1}^{N}c_nx_n\right\|.$$

Thus statement (c) holds with $C_2 = K_\Lambda$.

(c) $\Rightarrow$ (a). Suppose that statement (c) holds, and let σ be any permutation of $\mathbf{N}$. We must show that $\{x_{\sigma(n)}\}$ is a basis for X. By hypothesis, $\{x_{\sigma(n)}\}$ is complete in X and every element $x_{\sigma(n)}$ is nonzero. Therefore, by Theorem 7.5 it suffices to show that there is a constant C_σ such that

$$\forall\, N \ge M, \quad \forall\, c_{\sigma(1)},\dots,c_{\sigma(N)}, \quad \left\|\sum_{n=1}^{M}c_{\sigma(n)}x_{\sigma(n)}\right\| \le C\left\|\sum_{n=1}^{N}c_{\sigma(n)}x_{\sigma(n)}\right\|.$$

To this end, fix any $N \ge M$ and choose any scalars $c_{\sigma(1)},\dots,c_{\sigma(N)}$. Define $c_n = 0$ for $n \notin \{\sigma(1),\dots,\sigma(N)\}$. Let $L = \max\{\sigma(1),\dots,\sigma(N)\}$ and define

$$\lambda_n = \begin{cases} 1, & \text{if } n \in \{\sigma(1),\dots,\sigma(M)\},\\ 0, & \text{otherwise.}\end{cases}$$

Then,

$$\left\|\sum_{n=1}^{M}c_{\sigma(n)}x_{\sigma(n)}\right\| = \left\|\sum_{n=1}^{L}\lambda_nc_nx_n\right\| \le C_2\left\|\sum_{n=1}^{L}c_nx_n\right\| = C_2\left\|\sum_{n=1}^{N}c_{\sigma(n)}x_{\sigma(n)}\right\|.$$

This is the desired result, with $C_\sigma = C_2$.

(c) $\Rightarrow$ (b). Clear.

(b) $\Rightarrow$ (c). Suppose that statement (b) holds. Choose any $N > 0$, and any scalars b_n, c_n such that $|b_n| \le |c_n|$ for each $n = 1, \dots, N$. Let λ_n be such that $b_n = \lambda_n c_n$. Then we certainly have $|\lambda_n| \le 1$ for each n. Let $\alpha_n = \mathrm{Re}(\lambda_n)$ and $\beta_n = \mathrm{Im}(\lambda_n)$. Since the α_n are real and satisfy $|\alpha_n| \le 1$, Caratheodory's Theorem (Theorem 2.11) implies that we can find scalars $t_n \ge 0$ and signs $\varepsilon_m^n = \pm 1$, for $m = 1, \dots, N+1$ and $n = 1, \dots, N$, such that

$$\sum_{m=1}^{N+1} t_m = 1 \qquad \text{and} \qquad \sum_{m=1}^{N+1} \varepsilon_m^n \, t_m = \alpha_n \quad \text{for } n = 1, \dots, N.$$

Hence,

$$\begin{aligned}
\left\|\sum_{n=1}^{N} \alpha_n c_n x_n\right\| &= \left\|\sum_{n=1}^{N} \sum_{m=1}^{N+1} \varepsilon_m^n t_m c_n x_n\right\| \\
&= \left\|\sum_{m=1}^{N+1} t_m \sum_{n=1}^{N} \varepsilon_m^n c_n x_n\right\| \\
&\le \sum_{m=1}^{N+1} t_m \left\|\sum_{n=1}^{N} \varepsilon_m^n c_n x_n\right\| \\
&\le \sum_{m=1}^{N+1} t_m \, C_1 \left\|\sum_{n=1}^{N} c_n x_n\right\| = C_1 \left\|\sum_{n=1}^{N} c_n x_n\right\|.
\end{aligned}$$

A similar formula holds for the imaginary parts β_n, so

$$\left\|\sum_{n=1}^{N} b_n x_n\right\| = \left\|\sum_{n=1}^{N} \lambda_n c_n x_n\right\| \le \left\|\sum_{n=1}^{N} \alpha_n c_n x_n\right\| + \left\|\sum_{n=1}^{N} \beta_n c_n x_n\right\| \le 2C_1 \left\|\sum_{n=1}^{N} c_n x_n\right\|.$$

Therefore statement (c) holds with $C_2 = 2C_1$.

(c) $\Rightarrow$ (d). Assume that statement (c) holds, and choose any scalars $c_1, \dots, c_N$. Let $b_n = |c_n|$. Then we have both $|b_n| \le |c_n|$ and $|c_n| \le |b_n|$, so statement (c) implies

$$\left\|\sum_{n=1}^{N} b_n x_n\right\| \le C_2 \left\|\sum_{n=1}^{N} c_n x_n\right\| \qquad \text{and} \qquad \left\|\sum_{n=1}^{N} c_n x_n\right\| \le C_2 \left\|\sum_{n=1}^{N} b_n x_n\right\|.$$

Therefore (d) holds with $C_3 = 1/C_2$ and $C_4 = C_2$.

(d) $\Rightarrow$ (c). Assume that statement (d) holds. Choose any scalars $c_1, \dots, c_N$ and any signs $\varepsilon_1, \dots, \varepsilon_N = \pm 1$. Then, by statement (d),

$$\left\|\sum_{n=1}^{N} \varepsilon_n c_n x_n\right\| \le C_4 \left\|\sum_{n=1}^{N} |\varepsilon_n c_n| \, x_n\right\| = C_4 \left\|\sum_{n=1}^{N} |c_n| \, x_n\right\| \le \frac{C_4}{C_3} \left\|\sum_{n=1}^{N} c_n x_n\right\|.$$

Hence statement (c) holds with $C_1 = C_4/C_3$.

(a) $\Rightarrow$ (e). Let $(\{x_n\},\{a_n\})$ be an unconditional basis for X. Let (λ_n) be any bounded sequence of scalars, and let $M = \sup|\lambda_n|$. Fix any $x \in X$. Then the series $x = \sum \langle x, a_n\rangle\, x_n$ converges unconditionally. Hence, by Theorem 2.8(f), the series $T_\Lambda(x) = \sum \lambda_n \langle x, a_n\rangle\, x_n$ converges. Clearly $T_\Lambda \colon X \to X$ defined in this way is linear, and we have

$$\|T_\Lambda(x)\| \;=\; M \left\| \sum_n \frac{\lambda_n}{M} \langle x, a_n\rangle\, x_n \right\| \;\le\; M K_\Lambda \left\| \sum_n \langle x, a_n\rangle\, x_n \right\| \;=\; M K_\Lambda\, \|x\|.$$

Therefore T_Λ is continuous. Finally, the biorthogonality of $\{x_n\}$ and $\{a_n\}$ ensures that $T_\Lambda(x_n) = \lambda_n x_n$ for every n.

(e) $\Rightarrow$ (a). Suppose that statement (e) holds. Since $\{x_n\}$ is a basis, there exists a biorthogonal sequence $\{a_n\} \subset X^*$ such that the series $x = \sum \langle x, a_n\rangle\, x_n$ converges and is the unique expansion of x in terms of the x_n. We must show that this series converges unconditionally. Therefore, let $\Lambda = (\lambda_n)$ be any sequence of scalars such that $|\lambda_n| \le 1$ for every n. Then, by hypothesis, there exists a continuous $T_\Lambda \colon X \to X$ such that $T_\Lambda(x_n) = \lambda_n x_n$ for every n. Since T_Λ is continuous, we therefore have that

$$T_\Lambda(x) \;=\; T_\Lambda\!\left(\sum_n \langle x, a_n\rangle\, x_n\right) \;=\; \sum_n \langle x, a_n\rangle\, T_\Lambda(x_n) \;=\; \sum_n \lambda_n \langle x, a_n\rangle\, x_n$$

converges. Hence, by Theorem 2.8(f), the series $x = \sum \langle x, a_n\rangle\, x_n$ converges unconditionally. $\square$

10. WEAK AND WEAK* BASES IN BANACH SPACES

To this point we have considered sequences that are bases with respect to the strong, or norm, topology of a Banach space X. In this chapter we will briefly survey the natural generalization of bases to the case of the weak or weak* topologies. We will content ourselves with considering these topologies only, although it is clearly possible to generalize the notion of basis further to the setting of abstract topological vector spaces. We refer to [Mar69] and related sources for such generalizations,

Definition 10.1. Let X be a Banach space.

(a) A sequence $\{x_n\}$ of elements of X is a *basis* for X if Definition 4.2 holds. That is, for each $x \in X$ there must exist unique scalars $a_n(x)$ such that $x = \sum a_n(x)\, x_n$, with convergence of this series in the strong topology, i.e.,

$$\lim_{N\to\infty} \left\| x - \sum_{n=1}^{N} a_n(x)\, x_n \right\| \;=\; 0.$$

In this chapter, in order to emphasize the type of convergence required, we will often refer to a basis as a *strong basis* or a *norm basis.* By Theorem 4.11, each coefficient functional a_m associated with a strong basis is strongly continuous, i.e., if $y_n \to y$ strongly in X, then $a_m(y_n) \to a_m(y)$. Hence every strong basis is a *strong Schauder basis.*

(b) A sequence $\{x_n\}$ of elements of X is a *weak basis* for X if for each $x \in X$ there exist unique scalars $a_n(x)$ such that $x = \sum a_n(x)\, x_n$, with convergence of this series in the weak topology, i.e.,

$$\forall\, x^* \in X^*, \quad \lim_{N\to\infty} \left\langle \sum_{n=1}^{N} a_n(x)\, x_n,\; x^* \right\rangle \;=\; \langle x, x^*\rangle. \tag{10.1}$$

A weak basis is a *weak Schauder basis* if each coefficient functional a_m is weakly continuous on X, i.e., if $y_n \to y$ weakly in X implies $a_m(y_n) \to a_m(y)$.

(c) A sequence $\{x_n^*\}$ of functionals in X^* is a *weak* basis* for X^* if for each $x^* \in X^*$ there exist unique scalars $a_n^*(x^*)$ such that $x^* = \sum a_n^*(x^*)\, x_n^*$, with convergence of this series in the weak* topology, i.e.,

$$\forall\, x \in X, \quad \lim_{N\to\infty} \left\langle x,\; \sum_{n=1}^{N} a_n^*(x^*)\, x_n^* \right\rangle \;=\; \langle x, x^*\rangle.$$

A weak* basis is a *weak* Schauder basis* if each coefficient functional a_m^* is weak* continuous on X^*, i.e., if $y_n^* \to y^*$ weak* in X^* implies $a_m^*(y_n^*) \to a_m^*(y^*)$. ♢

As noted above, if $(\{x_n\}, \{a_n\})$ is a strong basis then each coefficient functional is an element of X^*. Therefore we usually write $\langle x, a_n\rangle = a_n(x)$ when dealing with the coefficient functionals associated with a strong basis. However, when dealing with functionals which are not known to be strongly continuous, we write only $a_n(x)$.

Not surprisingly, we can show that all strong bases are weak bases.

Theorem 10.2. *Let X be a Banach space. If $\{x_n\}$ is a strong basis for X, then $\{x_n\}$ is a weak basis for X. Further, in this case $\{x_n\}$ is a weak Schauder basis for X with coefficient functionals that are strongly continuous on X.*

Proof. Assume that $\{x_n\}$ is a strong basis for X. Then $\{x_n\}$ is a strong Schauder basis by Theorem 4.11, so the associated coefficient functionals $\{a_n\}$ are all strongly continuous linear functionals on X. We will show that $\{x_n\}$ is a weak basis and that $\{a_n\}$ is the sequence of coefficient functionals associated with this weak basis. Since we already know that these functionals are strongly continuous, they are necessarily weakly continuous, and hence it will follow automatically from this that $\{x_n\}$ is a weak Schauder basis.

Therefore, fix any $x \in X$. Then $x = \sum \langle x, a_n\rangle\, x_n$ converges strongly. Since strong convergence implies weak convergence, this series must also converge weakly to x. Or, to see this explicitly, simply note that if x^* is an arbitrary element of X^* then, by the continuity of x^*,

$$\lim_{N\to\infty} \Big\langle \sum_{n=1}^{N} \langle x, a_n\rangle\, x_n,\ x^*\Big\rangle \;=\; \Big\langle \lim_{N\to\infty} \sum_{n=1}^{N} \langle x, a_n\rangle\, x_n,\ x^*\Big\rangle \;=\; \Big\langle \sum_{n=1}^{\infty} \langle x, a_n\rangle\, x_n,\ x^*\Big\rangle \;=\; \langle x, x^*\rangle.$$

It therefore remains only to show that the representation $x = \sum \langle x, a_n\rangle\, x_n$ is unique. Suppose that we also had $x = \sum c_n x_n$, with weak convergence of this series. Fix any particular m. Then since $a_m \in X^*$, we have by the weak convergence of the series $x = \sum c_n x_n$ that

$$\langle x, a_m\rangle \;=\; \lim_{N\to\infty} \Big\langle \sum_{n=1}^{N} c_n x_n,\ a_m\Big\rangle \;=\; \lim_{N\to\infty} \sum_{n=1}^{N} c_n\, \langle x_n, a_m\rangle \;=\; \lim_{N\to\infty} \sum_{n=1}^{N} c_n \delta_{nm} \;=\; c_m.$$

Hence the representation is unique, and therefore $\{x_n\}$ is a weak basis for X. □

Surprisingly, the converse of this result is also true: every weak basis for a Banach space X is a strong basis for X. We prove this in Theorem 10.6 below, after establishing some basic properties of weak bases.

We let the partial sum operators for a weak basis $\{x_n\}$ be defined in the usual way, i.e., $S_N x = \sum_{n=1}^{N} a_n(x)\, x_n$ (compare Notation 4.6). The following result is the analogue of Proposition 4.7 for weak bases instead of strong bases.

Proposition 10.3. *Let $\{x_n\}$ be a sequence in a Banach space X, and assume that $x_n \neq 0$ for every n. Define $Y = \big\{(c_n) : \sum c_n x_n$ converges weakly in $X\big\}$, and set*

$$\|(c_n)\|_Y \;=\; \sup_N \Big\|\sum_{n=1}^{N} c_n x_n\Big\|.$$

Then the following statements hold.

(a) *Y is a Banach space.*

(b) *If* $\{x_n\}$ *is a weak basis for* X *then* Y *is topologically isomorphic to* X *via the mapping* $(c_n) \mapsto \sum c_n x_n$.

Proof. (a) Recall that weakly convergent sequences are bounded (Lemma 1.49). Therefore, if $(c_n) \in Y$ then $\|(c_n)\|_Y < \infty$ since $\sum c_n x_n = \lim_{N\to\infty} \sum_{n=1}^N c_n x_n$ converges weakly. The remainder of the proof is now identical to the proof of Proposition 4.7(a).

(b) Suppose that $\{x_n\}$ is a weak basis for X. Define the map $T\colon Y \to X$ by $T(c_n) = \sum c_n x_n$, where this series converges weakly. This mapping is well-defined by the definition of Y. It is clearly linear, and it is bijective because $\{x_n\}$ is a weak basis. Finally, if $(c_n) \in Y$ then

$$\|T(c_n)\| \;=\; \left\|\sum_{n=1}^{\infty} c_n x_n\right\| \;\le\; \sup_N \left\|\sum_{n=1}^{N} c_n x_n\right\| \;=\; \|(c_n)\|_Y,$$

again since weakly convergent series are bounded. Therefore T is bounded, hence is a topological isomorphism of Y onto X. □

An immediate consequence is that the partial sum operators for a weak basis are *strongly* continuous (compare Corollary 4.8).

Corollary 10.4. *Let* $(\{x_n\}, \{a_n\})$ *be a weak basis for a Banach space* X*. Then:*

(a) $\sup \|S_N x\| < \infty$ *for each* $x \in X$,

(b) *Each* S_N *is strongly continuous and* $C = \sup \|S_N\| < \infty$, *and*

(c) $|||x||| = \sup \|S_N x\|$ *forms a norm on* X *equivalent to the initial norm* $\|\cdot\|$ *for* X, *and satisfies* $\|\cdot\| \le |||\cdot||| \le C\,\|\cdot\|$.

Proof. (a) Let Y be as in Proposition 10.3. Then $T\colon X \to Y$ defined by $T(c_n) = \sum c_n x_n$ (converging weakly) is a topological isomorphism of X onto Y. Suppose that $x \in X$. Then we have by definition that $x = \sum a_n(x)\, x_n$ converges weakly and that the scalars $a_n(x)$ are unique, so we must have $T^{-1}x = (a_n(x))$. Hence

$$\sup_N \|S_N x\| \;=\; \sup_N \left\|\sum_{n=1}^{N} a_n(x)\, x_n\right\| \;=\; \big\|(a_n(x))\big\|_Y \;=\; \|T^{-1}x\|_Y \;\le\; \|T^{-1}\|\,\|x\| \;<\; \infty. \tag{10.2}$$

(b) From (10.2), we see that $\sup \|S_N\| \le \|T^{-1}\| < \infty$.

(c) It is easy to see that $|||\cdot|||$ has the properties of at least a seminorm. Now, given $x \in X$ we have

$$|||x||| \;=\; \sup_N \|S_N x\| \;\le\; \sup_N \|S_N\|\,\|x\| \;=\; C\,\|x\|$$

and

$$\|x\| \;=\; \lim_{N\to\infty} \|S_N x\| \;\le\; \sup_N \|S_N x\| \;=\; |||x|||.$$

It follows from these two statements that $|||\cdot|||$ is in fact a norm, and is equivalent to $\|\cdot\|$. □

The finite number $C = \sup \|S_N\|$ is the *weak basis constant.*

Next, we show now that all weak bases are weak Schauder bases, just as all strong bases are strong Schauder bases (Theorem 4.11). In Theorem 10.6 we will further improve this, showing that all weak bases are strong bases.

Theorem 10.5. *Every weak basis for a Banach space X is a weak Schauder basis for X. In fact, the coefficients functionals a_n are* strongly *continuous linear functionals on X which satisfy*

$$1 \;\le\; \|a_n\|\,\|x_n\| \;\le\; 2C,$$

where C is the weak basis constant.

Proof. By Corollary 10.4, we know that $C < \infty$, even though $(\{x_n\}, \{a_n\})$ is only assumed to be a weak basis for X. Since each a_n is linear, to show that a_n is strongly continuous we need only show that it is bounded. Given $x \in X$, we compute

$$\begin{aligned}
|a_n(x)|\,\|x_n\| \;=\; \|a_n(x)\,x_n\| \;&=\; \left\|\sum_{n=1}^{n} a_k(x)\,x_k \;-\; \sum_{n=1}^{n-1} a_k(x)\,x_k\right\| \\
&\le\; \left\|\sum_{n=1}^{n} a_k(x)\,x_k\right\| \;+\; \left\|\sum_{n=1}^{n-1} a_k(x)\,x_k\right\| \\
&=\; \|S_n x\| \;+\; \|S_{n-1}x\| \\
&\le\; 2C\,\|x\|.
\end{aligned}$$

Since each x_n is nonzero, we conclude that $\|a_n\| \le 2C/\|x_n\| < \infty$. The final inequality follows from computing $1 = a_n(x_n) \le \|a_n\|\,\|x_n\|$. $\square$

Now we can prove that all weak bases are strong bases.

Theorem 10.6 (Weak Basis Theorem). *Every weak basis for a Banach space X is a strong basis for X, and conversely.*

Proof. We showed in Theorem 10.2 that all strong bases are weak bases. For the converse, assume that $(\{x_n\}, \{a_n\})$ is a weak basis for X. By Theorem 10.5, each a_m is strongly continuous, and therefore $a_m \in X^*$. Moreover, by the uniqueness of the representations in (10.1), we must have $\langle x_n, a_m\rangle = \delta_{mn}$ for every m and n. Hence $(\{x_n\}, \{a_n\})$ is a biorthogonal system in the sense of Definition 7.1. Further, it follows from Corollary 10.4 that $\sup \|S_N\| < \infty$. Therefore, by Theorem 7.3, it suffices to show that $\{x_n\}$ is complete in X. Assume therefore that $x^* \in X^*$ satisfies $\langle x_n, x^*\rangle = 0$ for every n. Then for each $x \in X$, we have by (10.1) that

$$\langle x, x^*\rangle \;=\; \lim_{N\to\infty} \left\langle \sum_{n=1}^{N} \langle x, a_n\rangle\, x_n,\; x^* \right\rangle \;=\; \lim_{N\to\infty} \sum_{n=1}^{N} \langle x, a_n\rangle\,\langle x_n, x^*\rangle \;=\; 0.$$

Hence $x^* = 0$, so $\{x_n\}$ is complete. $\square$

We turn our attention to weak* bases for the remainder of this chapter. First, we give an example showing that a Banach space can possess a weak* basis even though it is not separable. By contrast, a nonseparable space cannot possess any strong bases (and therefore by Theorem 10.6 cannot possess any weak bases either).

In the following examples, we will use the sequence spaces c_0, ℓ^1, and ℓ^∞ defined in Example 1.6. Recall from Example 1.28 that $(\ell^1)^* = \ell^\infty$, in the sense that every element $y = (y_n) \in \ell^\infty$ determines a continuous linear functional on ℓ^1 by the formula

$$\langle x, y \rangle \;=\; \sum_n x_n y_n, \qquad x = (x_n) \in \ell^1, \tag{10.3}$$

and that all continuous linear functionals on ℓ^1 are obtained in this way. Additionally, $(c_0)^* = \ell^1$, with the duality defined analogously to (10.3), i.e., $y = (y_n) \in \ell^1$ acts on $x = (x_n) \in c_0$ by $\langle x, y \rangle = \sum x_n y_n$.

The sequences $e_n = (\delta_{mn})_{m=1}^\infty = (0, \dots, 0, 1, 0, \dots)$ will be useful. By Example 4.4, $\{e_n\}$ is a strong basis for ℓ^p for each $1 \le p < \infty$. This basis is called the *standard basis* for ℓ^p. Since ℓ^∞ is not separable, it does not possess any strong bases.

Example 10.7. Let $X = \ell^1$, so that $X^* = \ell^\infty$ is not separable. We will show that $\{e_n\}$ is a weak* basis for ℓ^∞, even though it cannot be a strong or weak basis for ℓ^∞. To see this, let $y = (y_n)$ be any element of ℓ^∞. Then for any $x = (x_n) \in \ell^1$, we have

$$\lim_{N\to\infty} \Big\langle x, \sum_{n=1}^N y_n e_n \Big\rangle \;=\; \lim_{N\to\infty} \sum_{n=1}^N \langle x, e_n \rangle\, y_n \;=\; \lim_{N\to\infty} \sum_{n=1}^N x_n y_n \;=\; \langle x, y \rangle.$$

Hence $y = \sum y_n e_n$ in the weak* topology (even though this series need not converge strongly), and it easy to see that this representation is unique. Therefore $\{e_n\}$ is a weak* basis for ℓ^∞. ◇

Although any strong or weak basis is a strong Schauder bases, the following example shows that a weak* basis need not be a weak* Schauder basis.

Lemma 10.8. [Sin70, p. 153]. *Let $X = c_0$, so that $X^* = \ell^1$. Let $\{e_n\}$ be the standard basis for ℓ^1, and define*

$$\begin{aligned} x_1 \;&=\; e_1, \\ x_n \;&=\; (-1)^{n+1} e_1 + e_n \;=\; ((-1)^{n+1}, 0, \dots, 0, 1, 0, \dots), \qquad n > 1. \end{aligned}$$

Then:

(a) *$\{x_n\}$ is a strong basis for ℓ^1.*

(b) *$\{x_n\}$ is a weak* basis for ℓ^1.*

(c) *$\{x_n\}$ is not a Schauder weak* basis for ℓ^1.*

Proof. (a) Define $y_n \in \ell^\infty$ by

$$y_1 = e_1 + \sum_{n=2}^{\infty} (-1)^n e_n = (1, 1, -1, 1, -1, 1, -1, \dots),$$
$$y_n = e_n, \qquad n > 1.$$

The series defining y_1 is only meant in the obvious formal way, and does not converge in the norm of ℓ^∞. It is easy to check that $\langle x_m, y_n \rangle = \delta_{mn}$, so $(\{x_n\}, \{y_n\})$ is a biorthogonal system for ℓ^1. Now, if $x \in \ell^1$ then $\sum |\langle x, e_n \rangle| < \infty$. Therefore, we can write

$$\begin{aligned} \sum_{n=1}^{N} \langle x, y_n \rangle x_n &= \langle x, y_1 \rangle x_1 + \sum_{n=2}^{N} \langle x, y_n \rangle x_n \\ &= \Big(\langle x, e_1 \rangle + \sum_{n=2}^{\infty} \langle x, e_n \rangle \Big) e_1 + \sum_{n=2}^{N} \langle x, e_n \rangle \big((-1)^{n+1} e_1 + e_n \big) \\ &= \sum_{n=1}^{N} \langle x, e_n \rangle e_n + \sum_{n>N} (-1)^n \langle x, e_n \rangle e_1, \end{aligned} \tag{10.4}$$

where the infinite series appearing in (10.4) do converge in the norm of ℓ^1. Therefore,

$$\Big\| \sum_{n=1}^{N} \langle x, e_n \rangle e_n - \sum_{n=1}^{N} \langle x, y_n \rangle x_n \Big\|_{\ell^1} = \Big\| \sum_{n>N} (-1)^n \langle x, e_n \rangle e_1 \Big\|_{\ell^1} \le \sum_{n>N} |\langle x, e_n \rangle|.$$

Choose any $\varepsilon > 0$. Then since $\sum |\langle x, e_n \rangle| < \infty$, there exists an $N > 0$ such that

$$\sum_{n>N} |\langle x, e_n \rangle| < \varepsilon.$$

Since $x = \sum \langle x, e_n \rangle e_n$, it follows that

$$\Big\| x - \sum_{n=1}^{N} \langle x, e_n \rangle e_n \Big\|_{\ell^1} = \Big\| \sum_{n>N} \langle x, e_n \rangle e_n \Big\|_{\ell^1} \le \sum_{n>N} |\langle x, e_n \rangle| < \varepsilon.$$

Therefore,

$$\Big\| x - \sum_{n=1}^{N} \langle x, y_n \rangle x_n \Big\|_{\ell^1} \le \Big\| x - \sum_{n=1}^{N} \langle x, e_n \rangle e_n \Big\|_{\ell^1} + \Big\| \sum_{n=1}^{N} \langle x, e_n \rangle e_n - \sum_{n=1}^{N} \langle x, y_n \rangle x_n \Big\|_{\ell^1} < 2\varepsilon.$$

Hence $x = \sum \langle x, y_n \rangle x_n$, with strong convergence of this series. Since $\{x_n\}$ and $\{y_n\}$ are biorthogonal, it therefore follows from Theorem 7.3 that $\{x_n\}$ is a strong basis for ℓ^1.

(b) Let $x \in \ell^1$. Then it follows from part (a) that $x = \sum \langle x, y_n \rangle\, x_n$, with strong convergence of the series. Since strong convergence implies weak* convergence by Lemma 1.48, it follows that $x = \sum \langle x, y_n \rangle\, x_n$ in the weak* topology. Therefore, we need only show that this representation is unique. Suppose we also had $x = \sum c_n x_n$ for some scalars c_n, with weak* convergence of this series. Then $\sum d_n x_n = 0$ in the weak* topology, where $d_n = c_n - \langle x, y_n \rangle$. In particular, since $e_m \in c_0$, we have for $m > 1$ that

$$\begin{aligned} 0 \;=\; \lim_{N\to\infty} \Big\langle e_m,\, \sum_{n=1}^N d_n x_n \Big\rangle \;&=\; \lim_{N\to\infty} \sum_{n=1}^N d_n \langle e_m, x_n \rangle \\ &=\; \lim_{N\to\infty} \sum_{n=1}^N d_n \Big(\langle e_m, (-1)^{n+1} e_1 \rangle + \langle e_m, e_n \rangle \Big) \;=\; d_m. \end{aligned}$$

Therefore $d_m = 0$ for $m > 1$. Since $\sum d_n x_n = 0$, this implies $d_1 x_1 = 0$ as well. However, $x_1 = e_1 \neq 0$, so $d_1 = 0$. Hence $c_n = \langle x, y_n \rangle$ for every n, as desired.

(c) We showed in part (b) that $\{x_n\}$ is a weak* basis for ℓ^1, and that $\{y_n\}$ is the associated sequence of coefficient functionals. However, $y_1 \notin c_0$, and therefore y_1 cannot be weak* continuous. To see this directly, note that $e_n \to 0$ weak* in ℓ^1, since if $z = (z_n) \in c_0$ then $\langle z, e_n \rangle = z_n \to 0$. However, $\langle e_n, y_1 \rangle = (-1)^n \not\to 0 = \langle 0, y_1 \rangle$, so y_1 is not weak* continuous. Therefore $\{x_n\}$ is not a Schauder weak* basis for ℓ^1. $\square$

We close this chapter by quoting the following example, which shows that a strong basis for X^* need not be a weak* basis for X^*.

Lemma 10.9. [Sin70, p. 150]. *Let $X = c_0$, so that $X^* = \ell^1$. Let $\{e_n\}$ be the standard basis for ℓ^1, and define*

$$\begin{aligned} x_1 \;&=\; e_1, \\ x_n \;&=\; e_{n-1} - e_n \;=\; (0, \dots, 0, 1, -1, 0, \dots), \qquad n > 1. \end{aligned}$$

Then $\{x_n\}$ is a strong basis for ℓ^1, but $\{x_n\}$ is not a weak basis for ℓ^1.*

IV. BASES AND FRAMES IN HILBERT SPACES

11. RIESZ BASES IN HILBERT SPACES

Let H be a Hilbert space. Recall from Definition 4.13 that a basis $\{x_n\}$ for H is *equivalent* to a basis $\{y_n\}$ for H if there exists a topological isomorphism $S\colon H \to H$ such that $Sx_n = y_n$ for all n. In this case, we write $\{x_n\} \sim \{y_n\}$. It is clear that $\sim$ is an equivalence relation on the set of all bases for H. In particular, we saw in Corollary 4.15 that all orthonormal bases in H are equivalent. We will show in this chapter that the class of all bases that are equivalent to orthonormal bases coincides with the class of all bounded unconditional bases for H, and we will discuss some of the properties of such bases.

Definition 11.1. A basis $\{x_n\}$ for a Hilbert space H is a *Riesz basis* for H if it is equivalent to some (and therefore every) orthonormal basis for H. $\diamondsuit$

Clearly, all Riesz bases are equivalent since all orthonormal bases are equivalent.

Remark 11.2. We show in Theorem 11.9 that bounded unconditional bases and Riesz bases are equivalent. Hence a bounded basis is a Riesz basis if and only if it is unconditional. It is very difficult to exhibit a bounded basis for a Hilbert space H that is not a Riesz basis for H. Babenko [Bab48] proved that if $0 < \alpha < 1/2$, then $\{|t|^\alpha e^{2\pi int}\}_{n\in\mathbf{Z}}$ is a bounded basis for $L^2[0,1]$ that is not a Riesz basis. It is easy to see that $\{|t|^\alpha e^{2\pi int}\}_{n\in\mathbf{Z}}$ is minimal in $L^2[0,1]$, since $\{|t|^{-\alpha} e^{2\pi int}\}_{n\in\mathbf{Z}}$ is contained in $L^2[0,1]$ and is biorthogonal to $\{|t|^\alpha e^{2\pi int}\}_{n\in\mathbf{Z}}$. However, the proof that $\{|t|^\alpha e^{2\pi int}\}_{n\in\mathbf{Z}}$ is a conditional basis is difficult. $\diamondsuit$

As with bases or unconditional bases, we can show that Riesz bases are preserved by topological isomorphisms.

Lemma 11.3. *Riesz bases are preserved by topological isomorphisms. That is, if $\{x_n\}$ is a Riesz basis for a Hilbert space H and $S\colon H \to K$ is a topological isomorphism, then $\{Sx_n\}$ is a Riesz basis for K.*

Proof. Since H possesses a basis, it is separable. Therefore K, being isomorphic to H, is separable as well. By Theorem 1.21, all separable Hilbert spaces are *isometrically* isomorphic, so there exists an *isometry* Z that maps H onto K. Further, by definition of Riesz basis, there exists an orthonormal basis $\{e_n\}$ for H and a topological isomorphism $T\colon H \to H$ such that $Te_n = x_n$. Since Z is an isometric isomorphism, the sequence $\{Ze_n\}$ is an orthonormal basis for K. Hence, STZ^{-1} is a topological isomorphism of K onto itself which has the property that $STZ^{-1}(Ze_n) = STe_n = Sx_n$. Hence $\{Sx_n\}$ is equivalent to an orthonormal basis for K, so we conclude that $\{Sx_n\}$ is a Riesz basis for K. $\square$

This yields one half of our characterization of Riesz bases.

Corollary 11.4. *All Riesz bases are bounded unconditional bases.*

Proof. Let $\{x_n\}$ be a Riesz basis for a Hilbert space H. Then there exists an orthonormal basis $\{e_n\}$ for H and a topological isomorphism $S\colon H \to H$ such that $Se_n = x_n$ for every n. However, $\{e_n\}$ is a bounded unconditional basis, and bounded unconditional bases are preserved by topological isomorphisms by Lemma 9.3(b), so $\{x_n\}$ must be a bounded unconditional basis for H. □

Before presenting the converse to this result, we require some basic facts about Riesz bases.

Lemma 11.5. *Let $(\{x_n\}, \{a_n\})$ and $(\{y_n\}, \{b_n\})$ be bases for a Hilbert space H. If $\{x_n\} \sim \{y_n\}$, then $\{a_n\} \sim \{b_n\}$.*

Proof. By Corollary 8.3, $(\{a_n\}, \{x_n\})$ and $(\{b_n\}, \{y_n\})$ are both bases for H. Suppose now that $\{x_n\} \sim \{y_n\}$. Then there exists a topological isomorphism $S\colon H \to H$ such that $Sx_n = y_n$ for every n. The adjoint mapping S^* is also a topological isomorphism of H onto itself, and we have

$$\langle x_m, S^* b_n\rangle \;=\; \langle Sx_m, b_n\rangle \;=\; \langle y_m, b_n\rangle \;=\; \delta_{mn} \;=\; \langle x_m, a_n\rangle.$$

Since $\{x_n\}$ is complete, it follows that $S^* b_n = a_n$ for every n, and therefore $\{a_n\} \sim \{b_n\}$. □

We obtain as a corollary a characterization of Riesz bases as those bases which are equivalent to their own biorthogonal systems.

Corollary 11.6. *Let $(\{x_n\}, \{y_n\})$ be a basis for a Hilbert space H. Then the following statements are equivalent.*

(a) *$\{x_n\}$ is a Riesz basis for H.*

(b) *$\{y_n\}$ is a Riesz basis for H.*

(c) *$\{x_n\} \sim \{y_n\}$.*

Proof. (a) $\Rightarrow$ (b), (c). Assume that $\{x_n\}$ is a Riesz basis for H. Then $\{x_n\} \sim \{e_n\}$ for some orthonormal basis $\{e_n\}$ of H. By Lemma 11.5, it follows that $\{x_n\}$ and $\{e_n\}$ have equivalent biorthogonal systems. However, $\{e_n\}$ is biorthogonal to itself, so this implies $\{y_n\} \sim \{e_n\} \sim \{x_n\}$. Hence $\{y_n\}$ is equivalent to $\{x_n\}$, and $\{y_n\}$ is a Riesz basis for H.

(b) $\Rightarrow$ (a), (c). By Corollary 8.3, $(\{y_n\}, \{x_n\})$ is a basis for H. Therefore, this argument follows symmetrically.

(c) $\Rightarrow$ (a), (b). Assume that $\{x_n\} \sim \{y_n\}$. Then there exists a topological isomorphism $S\colon H \to H$ such that $Sx_n = y_n$ for every n. Since $(\{x_n\}, \{y_n\})$ is a basis, it follows that for each $x \in H$,

$$x \;=\; \sum_n \langle x, y_n\rangle\, x_n \;=\; \sum_n \langle x, Sx_n\rangle\, x_n,$$

whence

$$Sx \;=\; \sum_n \langle x, Sx_n\rangle\, Sx_n.$$

Therefore,

$$\langle Sx, x\rangle \;=\; \sum_n |\langle x, Sx_n\rangle|^2 \;\ge\; 0.$$

Thus S is a continuous and positive linear operator on H, and therefore has a continuous and positive square root $S^{1/2}$ [Wei80, Theorem 7.20]. Similarly, S^{-1} is positive and has a positive square root, which must be $S^{-1/2} = (S^{1/2})^{-1}$. Thus $S^{1/2}$ is a topological isomorphism of H onto itself. Moreover, $S^{1/2}$ is self-adjoint, so

$$\langle S^{1/2}x_m, S^{1/2}x_n\rangle \;=\; \langle x_m, S^{1/2}S^{1/2}x_n\rangle \;=\; \langle x_m, Sx_n\rangle \;=\; \langle x_m, y_n\rangle \;=\; \delta_{mn}.$$

Hence $\{S^{1/2}x_n\}$ is an orthonormal sequence in H, and it must be complete since $\{x_n\}$ is complete and $S^{1/2}$ is a topological isomorphism. Therefore $\{x_n\}$ is the image of the orthonormal basis $\{S^{1/2}x_n\}$ under the topological isomorphism $S^{-1/2}$. Hence $\{x_n\}$ is a Riesz basis. By symmetry, $\{y_n\}$ is a Riesz basis as well. $\square$

Definition 11.7. A sequence $\{x_n\}$ in a Hilbert space H is a *Bessel sequence* if

$$\forall\, x \in H, \quad \sum_n |\langle x, x_n\rangle|^2 \;<\; \infty. \quad \diamondsuit$$

Lemma 11.8. *If $\{x_n\}$ is a Bessel sequence, then the* coefficient mapping $Ux = (\langle x, x_n\rangle)$ *is a continuous linear mapping of H into ℓ^2. In other words, there exists a constant $B > 0$ such that*

$$\forall\, x \in H, \quad \sum_n |\langle x, x_n\rangle|^2 \;\le\; B\,\|x\|.$$

Proof. We will use the Closed Graph Theorem (Theorem 1.46) to show that U is continuous. Suppose that $y_N \to y \in H$, and that $Uy_N \to (c_n) \in \ell^2$. Then for each fixed m,

$$\bigl|c_m - \langle y_N, x_m\rangle\bigr| \;\le\; \biggl(\sum_n \bigl|c_n - \langle y_N, x_n\rangle\bigr|\biggr)^{1/2} \;=\; \bigl\|(c_n) - Uy_N\bigr\|_{\ell^2} \;\to\; 0 \quad \text{as } N \to \infty.$$

Therefore $c_m = \lim_{N\to\infty} \langle y_N, x_m\rangle = \langle y, x_m\rangle$ for every m. Hence $(c_n) = (\langle y, x_m\rangle) = Uy$, so U has a closed graph, and therefore is continuous. $\square$

The constant B in Lemma 11.8 is sometimes referred to as a *Bessel bound* or *upper frame bound* for $\{x_n\}$ (compare Definition 12.1).

Now we can prove that Riesz bases and bounded unconditional bases are equivalent.

Theorem 11.9. [GK69, p. 320], [You80, p. 32]. *If $\{x_n\}$ be a sequence in a Hilbert space H, then the following statements are equivalent.*

(a) *$\{x_n\}$ is a Riesz basis for H.*

(b) *$\{x_n\}$ is a bounded unconditional basis for H.*

(c) $\{x_n\}$ *is a basis for* H, *and*

$$\sum_n c_n x_n \text{ converges} \quad \iff \quad \sum_n |c_n|^2 < \infty.$$

(d) $\{x_n\}$ *is complete in* H *and there exist constants* A, $B > 0$ *such that*

$$\forall\, c_1, \dots, c_N, \quad A \sum_{n=1}^{N} |c_n|^2 \;\le\; \Big\| \sum_{n=1}^{N} c_n x_n \Big\|^2 \;\le\; B \sum_{n=1}^{N} |c_n|^2.$$

(e) *There is an equivalent inner product* $(\cdot,\cdot)$ *for* H *such that* $\{x_n\}$ *is an orthonormal basis for* H *with respect to* $(\cdot,\cdot)$.

(f) $\{x_n\}$ *is a complete Bessel sequence and possesses a biorthogonal system* $\{y_n\}$ *that is also a complete Bessel sequence.*

Proof. (a) $\Rightarrow$ (b). This is the content of Corollary 11.4.

(a) $\Leftrightarrow$ (c). Assume that $\{x_n\}$ is a basis for H, and let $\{e_n\}$ be any orthonormal basis for H. Then $\{x_n\}$ is a Riesz basis for H if and only if $\{x_n\} \sim \{e_n\}$. By Theorem 4.14, $\{x_n\} \sim \{e_n\}$ if and only if

$$\sum_n c_n x_n \text{ converges} \quad \iff \quad \sum_n c_n e_n \text{ converges.}$$

However, by Theorem 1.19(a),

$$\sum_n c_n e_n \text{ converges} \quad \iff \quad \sum_n |c_n|^2 < \infty.$$

Hence, statement (a) holds if and only if statement (d) holds.

(a) $\Rightarrow$ (d). Suppose that $\{x_n\}$ is a Riesz basis for H. Then there exists an orthonormal basis $\{e_n\}$ for H and a topological isomorphism $S\colon H \to H$ such that $Se_n = x_n$ for every n. Therefore, for any scalars $c_1, \dots, c_N$ we have

$$\Big\| \sum_{n=1}^{N} c_n x_n \Big\|^2 \;=\; \Big\| S\Big(\sum_{n=1}^{N} c_n e_n \Big) \Big\|^2 \;\le\; \|S\|^2 \, \Big\| \sum_{n=1}^{N} c_n e_n \Big\|^2 \;=\; \|S\|^2 \sum_{n=1}^{N} |c_n|^2,$$

the last equality following from the Plancherel formula (Theorem 1.20). Similarly,

$$\sum_{n=1}^{N} |c_n|^2 \;=\; \Big\| \sum_{n=1}^{N} c_n e_n \Big\|^2 \;=\; \Big\| S^{-1}\Big(\sum_{n=1}^{N} c_n x_n \Big) \Big\|^2 \;\le\; \|S^{-1}\|^2 \, \Big\| \sum_{n=1}^{N} c_n x_n \Big\|^2.$$

Hence statement (d) holds with $A = \|S^{-1}\|^{-2}$ and $B = \|S\|^2$.

(a) ⇒ (e). Suppose that $\{x_n\}$ is a Riesz basis for H. Then there exists an orthonormal basis $\{e_n\}$ for H and a topological isomorphism $S\colon H \to H$ such that $Se_n = x_n$ for every n. Define

$$(x, y) \;=\; \langle Sx, Sy\rangle \qquad \text{and} \qquad |||x|||^2 \;=\; (x, x) \;=\; \langle Sx, Sx\rangle \;=\; \|Sx\|^2.$$

It is easy to see that $(\cdot,\cdot)$ is an inner product for H, and that $|||\cdot|||$ is the corresponding induced norm. Further,

$$|||x|||^2 \;=\; \|Sx\|^2 \;\le\; \|S\|^2\,\|x\|^2 \qquad \text{and} \qquad \|x\|^2 \;=\; |||S^{-1}x|||^2 \;\le\; |||S^{-1}|||^2\,|||x|||^2, \tag{11.1}$$

where $|||S^{-1}|||$ is the operator norm of S^{-1} with respect to the norm $|||\cdot|||$. In fact, we have

$$|||S^{-1}||| \;=\; \sup_{|||x|||=1} |||S^{-1}x||| \;=\; \sup_{\|Sx\|=1} \|x\| \;=\; \sup_{\|y\|=1} \|S^{-1}y\| \;=\; \|S^{-1}\|,$$

although this equality is not needed for our proof. It follows from (11.1) that $|||\cdot|||$ is an equivalent norm to $\|\cdot\|$. By definition, $(\cdot,\cdot)$ is therefore an equivalent inner product to $\langle\cdot,\cdot\rangle$.

It remains to show that $\{x_n\}$ is an orthonormal basis with respect to the inner product $(\cdot,\cdot)$. By Theorem 1.19, it suffices to show that $\{x_n\}$ is a complete orthonormal sequence with respect to $(\cdot,\cdot)$. The orthonormality follows from the calculation

$$(x_m, x_n) \;=\; \langle Sx_m, Sx_n\rangle \;=\; \langle e_m, e_n\rangle \;=\; \delta_{mn}.$$

For the completeness, suppose that there is an $x \in H$ such that $(x, x_n) = 0$ for every n. Then $0 = (x, x_n) = \langle Sx, Sx_n\rangle = \langle Sx, e_n\rangle$ for every n. Since $\{e_n\}$ is complete with respect to $\langle\cdot,\cdot\rangle$, this implies that $Sx = 0$. Since S is a topological isomorphism, we therefore have $x = 0$. Hence $\{x_n\}$ is complete with respect to $(\cdot,\cdot)$.

(a) ⇒ (f). Suppose that $\{x_n\}$ is a Riesz basis for H. Then, by Corollary 11.6, $\{x_n\}$ possesses a biorthogonal sequence $\{y_n\}$ which is itself a Riesz basis for H. Suppose now that $x \in H$. Then since $(\{x_n\},\{y_n\})$ is a basis, we have that $x = \sum \langle x, y_n\rangle\, x_n$. Since we have already shown that statement (a) implies statement (c), the convergence of this series implies that $\sum |\langle x, y_n\rangle|^2 < \infty$. Therefore $\{y_n\}$ is a Bessel sequence. Further, $\{y_n\}$ is complete since $(\{y_n\},\{x_n\})$ is also a basis for H (Corollary 8.3). A symmetric argument implies that $\{x_n\}$ is a complete Bessel sequence as well.

(b) ⇒ (f). Suppose that $(\{x_n\},\{y_n\})$ is a bounded unconditional basis for H. Then, by Corollary 8.3, $(\{y_n\},\{x_n\})$ is also a bounded unconditional basis for H. Therefore, if $x \in H$ then $x = \sum \langle x, x_n\rangle\, y_n$, with unconditional convergence of this series. By Orlicz's Theorem (Theorem 3.1), this implies that $\sum |\langle x, x_n\rangle|^2\, \|y_n\|^2 < \infty$. However, by definition of bounded basis, there exist constants C_1, C_2 so that $0 < C_1 \le \|y_n\| \le C_2 < \infty$ for all n. Hence $\sum |\langle x, x_n\rangle|^2 < \infty$, so $\{x_n\}$ is a Bessel sequence, and it must be complete since it is a basis. A symmetric argument implies that $\{y_n\}$ is also a complete Bessel sequence.

(d) ⇒ (a). Suppose that statement (d) holds, and let $\{e_n\}$ be any orthonormal basis for H. Choose any $x \in H$. Then, by Theorem 1.20, $x = \sum \langle x, e_n \rangle\, e_n$, and $\sum |\langle x, e_n\rangle|^2 = \|x\|^2 < \infty$. Choose $M < N$, and define $c_1 = \cdots = c_M = 0$ and $c_n = \langle x, e_n\rangle$ for $n = M+1, \dots, N$. Then, by hypothesis (d),

$$\Bigl\| \sum_{n=M+1}^{N} \langle x, e_n \rangle\, x_n \Bigr\|^2 \;=\; \Bigl\| \sum_{n=1}^{N} c_n x_n \Bigr\|^2 \;\le\; B \sum_{n=1}^{N} |c_n|^2 \;=\; B \sum_{n=M+1}^{N} |\langle x, e_n\rangle|^2.$$

Since $\sum |\langle x, e_n\rangle|^2$ is a Cauchy series of real numbers, it follows that $\sum \langle x, e_n\rangle\, x_n$ is a Cauchy series in H and hence must converge in H. Therefore, we can define $Sx = \sum \langle x, e_n \rangle\, x_n$. Clearly S defined in this way is a linear mapping of H into itself, and we claim that S is a topological isomorphism of H onto itself.

By applying hypothesis (d) and taking the limit as $N \to \infty$, we have

$$A\,\|x\|^2 \;=\; A \sum_n |\langle x, e_n\rangle|^2 \;\le\; \|Sx\|^2 \;\le\; B \sum_n |\langle x, e_n\rangle|^2 \;=\; B\,\|x\|^2. \tag{11.2}$$

It follows that S is continuous and injective, and that $S^{-1}\colon \mathrm{Range}(S) \to H$ is continuous as well. Further, $Se_m = \sum \langle e_m, e_n\rangle\, x_n = x_n$ for every n, so $\mathrm{Range}(S)$ contains every x_m, and therefore contains $\mathrm{span}\{x_n\}$, which is dense in H since $\{x_n\}$ is complete. Therefore, if we show that $\mathrm{Range}(S)$ is closed, then it will follow that $\mathrm{Range}(S) = H$ and hence that S is a topological isomorphism of H onto itself.

Suppose then that $y_n \in \mathrm{Range}(S)$ and that $y_n \to y \in H$. Then there exist $x_n \in H$ such that $Sx_n = y_n$. Hence $\{Sx_n\}$ is a Cauchy sequence in H. However, by (11.2) we have $A\,\|x_m - x_n\| \le \|Sx_m - Sx_n\|$, so $\{x_n\}$ is Cauchy as well, and therefore must converge to some $x \in H$. Since S is continuous, it follows that $y_n = Sx_n \to Sx$. Since we also have $y_n \to y$, we must have $y = Sx \in \mathrm{Range}(S)$. Hence $\mathrm{Range}(S)$ is closed.

Thus S is a topological isomorphism of H onto itself. Finally, since S maps $\{e_n\}$ onto $\{x_n\}$, we conclude that $\{x_n\}$ is a Riesz basis for H.

(d) ⇒ (b). Suppose that statement (d) holds. Choose $N > 0$, and define $a_n = \delta_{Nn}$. Then, by hypothesis (d),

$$A \;=\; A \sum_{n=1}^{N} |a_n|^2 \;\le\; \Bigl\| \sum_{n=1}^{N} a_n x_n \Bigr\|^2 \;=\; \|x_N\|^2 \;=\; \Bigl\| \sum_{n=1}^{N} a_n x_n \Bigr\|^2 \;\le\; B \sum_{n=1}^{N} |a_n|^2 \;=\; B.$$

Hence $\{x_n\}$ is norm-bounded above and below. In particular, each x_n is nonzero.

It remains to show that $\{x_n\}$ is an unconditional basis. Therefore, choose any scalars $c_1, \dots, c_N$ and any signs $\varepsilon_1, \dots, \varepsilon_N = \pm 1$. Then by hypothesis (d),

$$\Bigl\| \sum_{n=1}^{N} \varepsilon_n c_n x_n \Bigr\|^2 \;\le\; B \sum_{n=1}^{N} |\varepsilon_n c_n|^2 \;=\; B \sum_{n=1}^{N} |c_n|^2 \;\le\; \frac{B}{A} \Bigl\| \sum_{n=1}^{N} c_n x_n \Bigr\|^2.$$

This, combined with the fact that $\{x_n\}$ is complete and that every x_n is nonzero, implies by Theorem 9.7 that $\{x_n\}$ is an unconditional basis for H.

(d) $\Rightarrow$ (c). Suppose that statement (d) holds. Choose $N > 0$, and define $a_n = \delta_{Nn}$. Then by hypothesis (d),

$$\|x_N\|^2 \;=\; \left\|\sum_{n=1}^{N} a_n x_n\right\|^2 \;\geq\; A \sum_{n=1}^{N} |a_n|^2 \;=\; A.$$

Hence $\{x_n\}$ is norm-bounded below. In particular, each x_n is nonzero.

We will show now that $\{x_n\}$ is a basis for H. To do this, choose any $M < N$, and any scalars $c_1, \dots, c_N$. Then, by hypothesis (d),

$$\left\|\sum_{n=1}^{M} c_n x_n\right\|^2 \;\leq\; B \sum_{n=1}^{M} |c_n|^2 \;\leq\; B \sum_{n=1}^{N} |c_n|^2 \;\leq\; \frac{B}{A} \left\|\sum_{n=1}^{N} c_n x_n\right\|^2.$$

This, combined with the fact that $\{x_n\}$ is complete and that every x_n is nonzero, implies by Theorem 7.3 that $\{x_n\}$ is a basis.

It therefore only remains to show that $\sum c_n x_n$ converges if and only if $\sum |c_n|^2 < \infty$. To do this, let (c_n) be any sequence of scalars. Choose any $M < N$, and define $a_1 = \cdots = a_M = 0$ and $a_n = c_n$ for $n = M+1, \dots, N$. Then, by hypothesis (d),

$$A \sum_{n=1}^{N} |a_n|^2 \;\leq\; \left\|\sum_{n=1}^{N} a_n x_n\right\|^2 \;\leq\; B \sum_{n=1}^{N} |a_n|^2.$$

However, by the definition of a_n, this simply states that

$$A \sum_{n=M+1}^{N} |c_n|^2 \;\leq\; \left\|\sum_{n=M+1}^{N} c_n x_n\right\|^2 \;\leq\; B \sum_{n=M+1}^{N} |c_n|^2.$$

Therefore, $\sum c_n x_n$ is a Cauchy series in H if and only if $\sum |c_n|^2$ is a Cauchy series of real numbers. Hence one series converges if and only if the other series converges.

(e) $\Rightarrow$ (d). Suppose that $(\cdot,\cdot)$ is an equivalent inner product for H such that $\{x_n\}$ is an orthonormal basis with respect to $(\cdot,\cdot)$. Let $|||\cdot|||$ denote the norm induced by $(\cdot,\cdot)$. Then, by definition of equivalent inner product, $|||\cdot|||$ and $\|\cdot\|$ are equivalent norms, i.e., there exist constants A, $B > 0$ such that

$$\forall\, x \in H, \quad A\, |||x|||^2 \;\leq\; \|x\|^2 \leq B\, |||x|||^2. \tag{11.3}$$

Since $\{x_n\}$ is complete in the norm $|||\cdot|||$ and since $\|\cdot\|$ is equivalent to $|||\cdot|||$, we must have that $\{x_n\}$ is complete in H with respect to $\|\cdot\|$. To see this explicitly, suppose that $x \in H$ and that $\varepsilon > 0$ is given. Then since $\operatorname{span}\{x_n\}$ is dense in H with respect to the norm $|||\cdot|||$, there must exist $y \in \operatorname{span}\{x_n\}$ such that $|||x - y||| < \varepsilon$. By (11.3), we therefore have $\|x - y\| < B^{1/2}\varepsilon$. Hence $\operatorname{span}\{x_n\}$ is also dense in H with respect to $\|\cdot\|$, and therefore $\{x_n\}$ is complete with respect to this norm.

Now choose any scalars $c_1, \dots, c_N$. Since $\{x_n\}$ is orthonormal with respect to $(\cdot,\cdot)$, we have by the Plancherel formula (Theorem 1.20) that $|||\sum_{n=1}^{N} c_n x_n|||^2 = \sum_{n=1}^{N} |c_n|^2$. Combined with (11.3), this implies that

$$A \sum_{n=1}^{N} |c_n|^2 \;\leq\; \left\|\sum_{n=1}^{N} c_n x_n\right\|^2 \;\leq\; B \sum_{n=1}^{N} |c_n|^2.$$

Hence statement (d) holds.

(f) ⇒ (c). Suppose that statement (f) holds. Since $\{x_n\}$ and $\{y_n\}$ are both Bessel sequences, it follows from Lemma 11.8 that there exist constants C, $D > 0$ such that

$$\forall\, x \in H, \quad \sum_n |\langle x, x_n\rangle|^2 \;\le\; C\,\|x\|^2 \qquad \text{and} \qquad \sum_n |\langle x, y_n\rangle|^2 \;\le\; D\,\|x\|^2. \tag{11.4}$$

We will show now that $(\{x_n\},\{y_n\})$ is a basis for H. Since $\{x_n\}$ is assumed to be complete and since $\{y_n\}$ is biorthogonal to $\{x_n\}$, it suffices by Theorem 7.3 to show that $\sup \|S_N\| < \infty$, where S_N is the partial sum operator $S_N x = \sum_{n=1}^N \langle x, y_n\rangle\, x_n$. We compute:

$$\begin{aligned}
\|S_N x\|^2 &= \sup_{\|y\|=1} |\langle S_N x, y\rangle|^2 && \text{by Theorem 1.16(b)}\\
&= \sup_{\|y\|=1} \left|\sum_{n=1}^N \langle x, y_n\rangle\, \langle x_n, y\rangle\right|^2\\
&\le \sup_{\|y\|=1} \left(\sum_{n=1}^N |\langle x, y_n\rangle|^2\right)\left(\sum_{n=1}^N |\langle x_n, y\rangle|^2\right) && \text{by Cauchy–Schwarz}\\
&\le \sup_{\|y\|=1} D\,\|x\|^2\, C\,\|y\|^2 && \text{by (11.4)}\\
&= CD\,\|x\|^2.
\end{aligned}$$

Hence $\sup \|S_N\|^2 \le CD < \infty$, as desired.

Finally, we must show that $\sum c_n x_n$ converges if and only if $\sum |c_n|^2 < \infty$. Suppose first that $x = \sum c_n x_n$ converges. Then we must have $c_n = \langle x, y_n\rangle$ since $(\{x_n\},\{y_n\})$ is a basis for H. It therefore follows from (11.4) that $\sum |c_n|^2 = \sum |\langle x, y_n\rangle|^2 \le D\,\|x\|^2 < \infty$.

Conversely, suppose that $\sum |c_n|^2 < \infty$. Then for any $M < N$,

$$\begin{aligned}
\left\|\sum_{n=M+1}^N c_n x_n\right\|^2 &= \sup_{\|y\|=1} \left|\left\langle \sum_{n=M+1}^N c_n x_n,\; y\right\rangle\right|^2 && \text{by Theorem 1.16(b)}\\
&= \sup_{\|y\|=1} \left|\sum_{n=M+1}^N c_n\, \langle x_n, y\rangle\right|^2\\
&\le \sup_{\|y\|=1} \left(\sum_{n=M+1}^N |c_n|^2\right)\left(\sum_{n=M+1}^N |\langle x_n, y\rangle|^2\right) && \text{by Cauchy–Schwarz}\\
&\le \sup_{\|y\|=1} \left(\sum_{n=M+1}^N |c_n|^2\right) C\,\|y\|^2 && \text{by (11.4)}\\
&= C \sum_{n=M+1}^N |c_n|^2.
\end{aligned}$$

Hence $\sum c_n x_n$ is a Cauchy series in H, and therefore must converge. □

12. FRAMES IN HILBERT SPACES

Frames were introduced by Duffin and Schaeffer in the context of nonharmonic Fourier series [DS52]. They were intended as an alternative to orthonormal or Riesz bases in Hilbert spaces. Much of the abstract theory of frames is elegantly laid out in that paper. Frames for $L^2(\mathbf{R})$ based on time-frequency or time-scale translates of functions were later constructed by Daubechies, Grossmann, and Meyer in [DGM86]. Such frames play an important role in Gabor and wavelet analysis. Expository discussions of these connections can be found in [Dau92] and [HW89]. Gröchenig has given the nontrivial extension of frames to Banach spaces [Grö91].

This chapter is an essentially expository review of basic results on frames in Hilbert spaces. We have combined results from many sources, including [Dau90], [DGM86], [DS52], [You80] and others, with remarks, examples, and minor results of our own. This chapter is based on [Hei90] and [HW89].

Definition 12.1. A sequence $\{x_n\}$ in a Hilbert space H is a *frame* for H if there exist constants A, $B > 0$ such that the following *pseudo–Plancherel formula* holds:

$$\forall\, x \in H, \quad A\,\|x\|^2 \;\le\; \sum_n |\langle x, x_n\rangle|^2 \;\le\; B\,\|x\|^2. \tag{12.1}$$

The constants A, B are *frame bounds*; A is the *lower* bound and B is the *upper* bound. The frame is *tight* if $A = B$. The frame is *exact* if it ceases to be a frame whenever any single element is deleted from the sequence. ◇

If $\{x_n\}$ is a frame then $\sum |\langle x, x_n\rangle|^2$ is an absolutely convergent series of nonnegative real numbers. It therefore converges unconditionally by Lemma 2.4. Hence $\sum |\langle x, x_{\sigma(n)}\rangle|^2 = \sum |\langle x, x_n\rangle|^2 < \infty$ for any permutation σ of $\mathbf{N}$. As a consequence, every rearrangement of a frame is also a frame, and therefore we could use any countable set to index a frame if we wished.

Example 12.2. By the Plancherel formula (Theorem 1.20), every orthonormal basis $\{e_n\}$ is a tight frame with $A = B = 1$. Moreover, $\{e_n\}$ is an exact frame since if we delete any element e_m, then $\sum_{n\neq m} |\langle e_m, e_n\rangle|^2 = 0$, and therefore $\{e_n\}_{n\neq m}$ cannot be a frame. ◇

We will see in Theorem 12.21 that the class of exact frames for H coincides with the class of Riesz bases for H. Further, we shall see in Proposition 12.10 that even though an inexact frame is not a basis, the pseudo-Plancherel formula (12.1) implies that every element $x \in H$ can be expressed as $x = \sum c_n x_n$ with specified c_n. However, in this case the scalars c_n will not be unique.

The following example shows that tightness and exactness are distinct concepts.

Example 12.3. Let $\{e_n\}$ be an orthonormal basis for a separable Hilbert space H.

(a) $\{e_n\}$ is a tight exact frame for H with frame bounds $A = B = 1$.

(b) $\{e_1, e_1, e_2, e_2, e_3, e_3, \dots\}$ is a tight inexact frame with bounds $A = B = 2$, but it is not orthogonal and it is not a basis, although it does contain an orthonormal basis. Similarly, if $\{f_n\}$ is another orthonormal basis for H then $\{e_n\} \cup \{f_n\}$ is a tight inexact frame.

(c) $\{e_1, e_2/2, e_3/3, \dots\}$ is a complete orthogonal sequence and it is a basis for H, but it does not possess a lower frame bound and hence is not a frame.

(d) $\{e_1, e_2/\sqrt{2}, e_2/\sqrt{2}, e_3/\sqrt{3}, e_3/\sqrt{3}, e_3/\sqrt{3}, \dots\}$ is a tight inexact frame with bounds $A = B = 1$, and no nonredundant subsequence is a frame.

(e) $\{2e_1, e_2, e_3, \dots\}$ is a nontight exact frame with bounds $A = 1$, $B = 2$. $\diamondsuit$

We show now that all frames must be complete, although part (c) of the preceding example shows that there exist complete sequences which are not frames.

Lemma 12.4. *If $\{x_n\}$ is a frame for a Hilbert space H, then $\{x_n\}$ is complete in H.*

Proof. If $x \in H$ satisfies $\langle x, x_n\rangle = 0$ for all n, then $A\,\|x\|^2 \le \sum |\langle x, x_n\rangle|^2 = 0$. $\square$

As a consequence of this result, if H possesses a frame $\{x_n\}$ then it must be separable, since the set of all finite linear combinations $\sum_{n=1}^{N} c_n x_n$ with rational c_n (or rational real and imaginary parts if the c_n are complex) will form a countable, dense subset of H. Conversely, every separable Hilbert space does possess a frame since it possesses an orthonormal basis, which is a tight exact frames.

Example 12.5. Let $a, b > 0$ be fixed. If the collection $\{e^{2\pi i m b x} g(x - na)\}_{m,n\in\mathbf{Z}}$ of time-frequency translates of a single $g \in L^2(\mathbf{R})$ forms a frame for $L^2(\mathbf{R})$ then it is called a *Gabor frame*. Similarly, if the collection $\{a^{n/2} g(a^n x - mb)\}_{m,n\in\mathbf{Z}}$ of time-scale translates of $g \in L^2(\mathbf{R})$ forms a frame then it is called a *wavelet frame*. We refer to [Dau92], [HW89] for expository treatments of these types of frames. $\diamondsuit$

Recall from Definition 11.7 that $\{x_n\}$ is a *Bessel sequence* if $\sum |\langle x, x_n\rangle|^2 < \infty$ for every $x \in H$. By Lemma 11.8, or directly from the Uniform Boundedness Principle, every Bessel sequence must possess an upper frame bound $B > 0$, i.e.,

$$\forall\, x \in H, \quad \sum |\langle x, x_n\rangle|^2 \le B\|x\|^2.$$

The number B is sometimes called the *Bessel bound*, or simply the *upper frame bound* for $\{x_n\}$ (even though an arbitrary Bessel sequence need not satisfy a lower frame bound and therefore need not be a frame). In applications, a sequence which is a frame is often easily shown to be a Bessel sequence, while the lower frame bound is often more difficult to establish.

We now prove some basic properties of Bessel sequences and frames. Part (a) of the following lemma is proved in [DS52].

Lemma 12.6. *Let $\{x_n\}$ be a Bessel sequence with Bessel bound B.*

(a) *If $(c_n) \in \ell^2$ then $\sum c_n x_n$ converges unconditionally in H, and*

$$\Bigl\|\sum_n c_n x_n\Bigr\|^2 \;\le\; B \sum_n |c_n|^2.$$

(b) *$Ux = (\langle x, x_n\rangle)$ is a continuous mapping of H into ℓ^2, with $\|U\| \le B^{1/2}$. Its adjoint is the continuous mapping $U^*\colon \ell^2 \to H$ given by $U^*(c_n) = \sum c_n x_n$.*

(c) *If $\{x_n\}$ is a frame then U is injective and U^* is surjective.*

Proof. (a) Let F be any finite subset of $\mathbf{N}$. Then,

$$\begin{aligned}
\Bigl\|\sum_{n\in F} c_n x_n\Bigr\|^2 &= \sup_{\|y\|=1} \Bigl|\Bigl\langle \sum_{n\in F} c_n x_n,\; y\Bigr\rangle\Bigr|^2 && \text{by Theorem 1.16(b)}\\
&= \sup_{\|y\|=1} \Bigl|\sum_{n\in F} c_n \langle x_n, y\rangle\Bigr|^2 \\
&\le \sup_{\|y\|=1} \Bigl(\sum_{n\in F} |c_n|^2\Bigr)\Bigl(\sum_{n\in F} |\langle x_n, y\rangle|^2\Bigr) && \text{by Cauchy–Schwarz}\\
&\le \sup_{\|y\|=1} \Bigl(\sum_{n\in F} |c_n|^2\Bigr) B\|y\|^2 && \text{by definition of frame}\\
&= B \sum_{n\in F} |c_n|^2. && (12.2)
\end{aligned}$$

Since $\sum |c_n|^2$ is an absolutely and unconditionally convergent series of real numbers, it therefore follows from (12.2) and Theorem 2.8 that $\sum c_n x_n$ converges unconditionally in H. Setting $F = \{1, \dots, N\}$ in (12.2) and taking the limit as $N \to \infty$ then yields the desired inequality $\|\sum c_n x_n\|^2 \le B \sum |c_n|^2$.

(b) By definition of Bessel bound, we have $\|Ux\|_{\ell^2}^2 = \sum |\langle x, x_n\rangle|^2 \le B\,\|x\|^2$. Hence U is a continuous mapping of H into ℓ^2, and $\|U\| \le B^{1/2}$.

The adjoint $U^*\colon \ell^2 \to H$ of H is therefore well-defined and continuous, so we need only verify that it has the correct form. Now, if $(c_n) \in \ell^2$ then we know by part (a) that $\sum c_n x_n$ converges to an element of H. Therefore, given $x \in H$ we can compute

$$\langle x, U^*(c_n)\rangle \;=\; \langle Ux, (c_n)\rangle_{\ell^2} \;=\; \bigl\langle (\langle x, x_n\rangle),\; (c_n)\bigr\rangle_{\ell^2} \;=\; \sum_n \langle x, x_n\rangle\, \bar{c}_n \;=\; \Bigl\langle x,\; \sum_n c_n x_n\Bigr\rangle.$$

Hence $U^*(c_n) = \sum c_n x_n$.

(b) The fact that U is injective follows immediately from the fact that frames are complete. The fact that U^* is surjective follows from the fact that U is injective. $\square$

Definition 12.7. Let $\{x_n\}$ be a frame for a Hilbert space H.

(a) The *coefficient mapping* for $\{x_n\}$ is the continuous mapping $U\colon H \to \ell^2$ defined by $Ux = (\langle x, x_n\rangle)$ for $x \in H$.

(b) The *synthesis mapping* for $\{x_n\}$ is the continuous mapping $U^*\colon \ell^2 \to H$ defined by $U^*(c_n) = \sum c_n x_n$ for $(c_n) \in \ell^2$.

(c) The *frame operator* for $\{x_n\}$ is the continuous mapping $S\colon H \to H$ defined by

$$Sx \;=\; U^*Ux \;=\; \sum_n \langle x, x_n\rangle\, x_n, \qquad x \in H. \quad \diamondsuit$$

Proposition 12.8. [DS52]. *Given a sequence $\{x_n\}$ in a Hilbert space H, the following statements are equivalent.*

(a) *$\{x_n\}$ is a frame with frame bounds A, B.*

(b) *$Sx = \sum \langle x, x_n\rangle\, x_n$ is a positive, bounded, linear mapping of H into H which satisfies $AI \le S \le BI$.*

Proof. (a) $\Rightarrow$ (b). Assume that $\{x_n\}$ is a frame. Then $S = U^*U$ is continuous by Lemma 12.6. In fact, we have $\|S\| \le \|U^*\|\,\|U\| \le B$. Note that

$$\langle AIx, x\rangle \;=\; A\,\|x\|^2, \qquad \langle Sx, x\rangle \;=\; \sum_n |\langle x, x_n\rangle|^2, \qquad \langle BIx, x\rangle \;=\; B\,\|x\|^2. \tag{12.3}$$

Therefore, $\langle AIx, x\rangle \le \langle Sx, x\rangle \le \langle BIx, x\rangle$ by definition of frame, so $AI \le S \le BI$. Additionally, $\langle Sx, x\rangle \ge 0$ for every x, so S is a positive operator.

(b) $\Rightarrow$ (a). Assume that statement (b) holds. Then $\langle AIx, x\rangle \le \langle Sx, x\rangle \le \langle BIx, x\rangle$ for every $x \in H$. By (12.3), this implies that $\{x_n\}$ is a frame for H. $\square$

Our next goal is to show that the frame operator S is a topological isomorphism of H onto itself. We will require the following lemma. To motivate this lemma, note that if $T\colon H \to H$ is a positive definite operator, i.e., $\langle Tx, x\rangle > 0$ for all $x \ne 0$, then $(x, y) = \langle Tx, y\rangle$ defines an inner product on H that is equivalent to the original inner product. If we let $|||x||| = (x, x)^{1/2}$ denote the corresponding induced norm, then the Cauchy–Schwarz inequality applied to $(\cdot,\cdot)$ states that

$$|\langle Tx, y\rangle|^2 \;=\; |(x, y)|^2 \;\le\; |||x|||^2\;|||y|||^2 \;=\; (x, x)\,(y, y) \;=\; \langle Tx, x\rangle\,\langle Ty, y\rangle.$$

The following lemma states that this inequality remains valid even if T is only assumed to be a positive operator, rather than positive definite. In this case, $(x, x) = \langle Tx, x\rangle \ge 0$ for all x, but we may have $(x, x) = 0$ when $x \ne 0$. Hence $(\cdot,\cdot)$ need not be an inner product in this case. However, the proof of the Cauchy–Schwarz inequality does adapt to this more general situation.

Lemma 12.9 (Generalized Cauchy–Schwarz). *Let H be a Hilbert space. If $T: H \to H$ is a positive operator, then*

$$\forall\, x, y \in H, \quad |\langle Tx, y\rangle|^2 \;\le\; \langle Tx, x\rangle\, \langle Ty, y\rangle. \quad \diamondsuit$$

Proposition 12.10. [DS52]. *If $\{x_n\}$ is a frame for a Hilbert space H, then the following statements hold.*

(a) *The frame operator S is a topological isomorphism of H onto itself. Moreover, S^{-1} satisfies $B^{-1}I \le S^{-1} \le A^{-1}I$.*

(b) *$\{S^{-1}x_n\}$ is a frame for H, with frame bounds B^{-1}, A^{-1}.*

(c) *The following series converge unconditionally for each $x \in H$:*

$$x \;=\; \sum_n \langle x, S^{-1}x_n\rangle\, x_n \;=\; \sum_n \langle x, x_n\rangle\, S^{-1}x_n. \tag{12.4}$$

(d) *If the frame is tight, i.e., $A = B$, then $S = AI$, $S^{-1} = A^{-1}I$, and*

$$\forall\, x \in H, \quad x \;=\; A^{-1} \sum \langle x, x_n\rangle x_n.$$

Proof. (a) We know that S is continuous since $S = U^*U$ and U is continuous. Further, it follows from $AI \le S \le BI$ that $A\,\|x\|^2 = \langle AIx, x\rangle \le \langle Sx, x\rangle \le \|Sx\|\,\|x\|$. Hence,

$$\forall\, x \in H, \quad A\,\|x\| \;\le\; \|Sx\|. \tag{12.5}$$

This implies immediately that S is injective, and that $S^{-1}\colon \mathrm{Range}(S) \to H$ is continuous. Hence, if we show that S is surjective then it follows that S is a topological isomorphism.

Before showing that $\mathrm{Range}(S) = H$, we will show that $\mathrm{Range}(S)$ is closed. Suppose that $y_n \in \mathrm{Range}(S)$ and that $y_n \to y \in H$. Then $y_n = Sx_n$ for some $x_n \in H$. Hence $\{Sx_n\}$ is a Cauchy sequence in H. However, by (12.5), we have $A\,\|x_m - x_n\| \le \|Sx_m - Sx_n\|$, so $\{x_n\}$ is Cauchy as well. Therefore $x_n \to x$ for some $x \in H$. Since S is continuous, we therefore have $y_n = Sx_n \to Sx$. Since we also have $y_n \to y$, we conclude that $y = Sx \in \mathrm{Range}(S)$, so $\mathrm{Range}(S)$ is closed.

Now we will show that $\mathrm{Range}(S) = H$. Suppose that $y \in H$ was orthogonal to $\mathrm{Range}(S)$, i.e., $\langle y, Sx\rangle = 0$ for every $x \in H$. Then $A\,\|y\|^2 = \langle AIy, y\rangle \le \langle Sy, y\rangle = 0$, so $y = 0$. Since $\mathrm{Range}(S)$ is a closed subspace of H, it follows that $\mathrm{Range}(S) = H$. Thus S is surjective, and therefore is a topological isomorphism.

Finally, we will show that S^{-1} satisfies $B^{-1}I \le S^{-1} \le A^{-1}I$. First, note that S^{-1} is positive since S is positive. This also follows from the computation

$$0 \;\le\; A\,\|S^{-1}x\|^2 \;=\; \langle AI(S^{-1}x), S^{-1}x\rangle \;\le\; \langle S(S^{-1}x), S^{-1}x\rangle \;=\; \langle x, S^{-1}x\rangle \;\le\; \|x\|\,\|S^{-1}x\|.$$

As a consequence, $\|S^{-1}\| \le A^{-1}$. Hence $\langle S^{-1}x, x\rangle \le \|S^{-1}x\|\,\|x\| \le A^{-1}\,\|x\|^2 = \langle A^{-1}Ix, x\rangle$, so $S^{-1} \le A^{-1}I$. Lastly, by Lemma 12.9,

$$\begin{aligned} \|x\|^4 \;=\; \langle x, x\rangle^2 \;=\; \langle S^{-1}(Sx), x\rangle^2 \;&\le\; \langle S^{-1}(Sx), Sx\rangle\, \langle S^{-1}x, x\rangle \\ &=\; \langle x, Sx\rangle\, \langle S^{-1}x, x\rangle \\ &\le\; B\,\|x\|^2\, \langle S^{-1}x, x\rangle. \end{aligned}$$

Therefore $\langle S^{-1}x, x\rangle \ge B^{-1}\,\|x\|^2 = \langle B^{-1}Ix, x\rangle$, so $S^{-1} \ge B^{-1}I$.

(b) The operator S^{-1} is self-adjoint since it is positive. Therefore,

$$\begin{aligned}\sum_n \langle x, S^{-1}x_n\rangle\, S^{-1}x_n &= \sum_n \langle S^{-1}x, x_n\rangle\, S^{-1}x_n \\ &= S^{-1}\Bigl(\sum_n \langle S^{-1}x, x_n\rangle\, x_n\Bigr) = S^{-1}S(S^{-1}x) = S^{-1}x.\end{aligned}$$

Since we also have $B^{-1}I \le S^{-1} \le A^{-1}I$, it therefore follows from Proposition 12.8 that $\{S^{-1}x_n\}$ is a frame.

(c) We compute

$$x = S(S^{-1}x) = \sum_n \langle S^{-1}x, x_n\rangle\, x_n = \sum_n \langle x, S^{-1}x_n\rangle\, x_n$$

and

$$x = S^{-1}(Sx) = S^{-1}\Bigl(\sum_n \langle x, x_n\rangle\, x_n\Bigr) = \sum_n \langle x, x_n\rangle\, S^{-1}x_n.$$

The unconditionality of the convergence follows from the fact that $\{x_n\}$ and $\{S^{-1}x_n\}$ are both frames.

(d) Follows immediately from parts (a)–(c). □

Definition 12.11. Let $\{x_n\}$ be a frame with frame operator S. Then $\{S^{-1}x_n\}$ is the *dual frame* of $\{x_n\}$.

We now prove some results relating to the uniqueness of the series expressions in (12.4). The following proposition shows that among all choices of scalars (c_n) for which $x = \sum c_n x_n$, the scalars $c_n = \langle x, S^{-1}x_n\rangle$ have the minimal ℓ^2-norm.

Proposition 12.12. [DS52]. *Let $\{x_n\}$ be a frame for a Hilbert space H, and let $x \in H$. If $x = \sum c_n x_n$ for some scalars (c_n), then*

$$\sum_n |c_n|^2 = \sum_n |\langle x, S^{-1}x_n\rangle|^2 + \sum_n |\langle x, S^{-1}x_n\rangle - c_n|^2.$$

In particular, the sequence $(\langle x, S^{-1}x_n\rangle)$ has the minimal ℓ^2-norm among all such sequences (c_n).

Proof. By (12.4), we have $x = \sum a_n x_n$, where $a_n = \langle x, S^{-1}x_n\rangle$. Let (c_n) be any sequence of scalars such that $x = \sum c_n x_n$. Since $\sum |a_n|^2 < \infty$, we may assume without loss of generality that $\sum |c_n|^2 < \infty$. Then $(c_n) \in \ell^2$, and we have

$$\langle x, S^{-1}x\rangle = \Bigl\langle \sum_n a_n x_n,\ S^{-1}x\Bigr\rangle = \sum_n a_n\, \langle S^{-1}x_n, x\rangle = \sum_n a_n \bar{a}_n = \bigl\langle (a_n),\ (a_n)\bigr\rangle_{\ell^2}$$

and

$$\langle x, S^{-1}x\rangle = \Bigl\langle \sum_n c_n x_n,\ S^{-1}x\Bigr\rangle = \sum_n c_n\, \langle S^{-1}x_n, x\rangle = \sum_n c_n \bar{a}_n = \bigl\langle (c_n),\ (a_n)\bigr\rangle_{\ell^2}.$$

Therefore $(c_n - a_n)$ is orthogonal to (a_n) in ℓ^2, whence

$$\|(c_n)\|_{\ell^2}^2 \;=\; \|(c_n - a_n) + (a_n)\|_{\ell^2}^2 \;=\; \|(c_n - a_n)\|_{\ell^2}^2 \;+\; \|(a_n)\|_{\ell^2}^2. \quad \square$$

The following result will be play an important role in characterizing the class of exact frames.

Proposition 12.13. [DS52]. *Let $\{x_n\}$ be a frame for a Hilbert space H.*

(a) *For each m,*

$$\sum_{n \neq m} |\langle x_m, S^{-1}x_n\rangle|^2 \;=\; \frac{1 \;-\; |\langle x_m, S^{-1}x_m\rangle|^2 \;-\; |1 - \langle x_m, S^{-1}x_m\rangle|^2}{2}. \tag{12.6}$$

(b) *If $\langle x_m, S^{-1}x_m\rangle = 1$, then $\langle x_m, S^{-1}x_n\rangle = 0$ for $n \neq m$.*

(c) *The removal of a vector from a frame leaves either a frame or an incomplete set. In fact,*

$$\langle x_m, S^{-1}x_m\rangle \neq 1 \quad \Longrightarrow \quad \{x_n\}_{n\neq m} \text{ is a frame},$$

$$\langle x_m, S^{-1}x_m\rangle = 1 \quad \Longrightarrow \quad \{x_n\}_{n\neq m} \text{ is incomplete}.$$

Proof. (a) Fix any m, and let $a_n = \langle x_m, S^{-1}x_n\rangle$. Then $x_m = \sum a_n x_n$ by (12.4). However, we also have $x_m = \sum \delta_{mn} x_n$, so Proposition 12.12 implies that

$$\begin{aligned} 1 \;=\; \sum_n |\delta_{mn}|^2 \;&=\; \sum_n |a_n|^2 \;+\; \sum_n |a_n - \delta_{mn}|^2 \\ &=\; |a_m|^2 \;+\; \sum_{n\neq m} |a_n|^2 \;+\; |a_m - 1|^2 \;+\; \sum_{n \neq m} |a_n|^2. \end{aligned}$$

Therefore,

$$\sum_{n\neq m} |a_n|^2 \;=\; \frac{1 - |a_m|^2 - |a_m - 1|^2}{2}.$$

(b) Suppose that $\langle x_m, S^{-1}x_m\rangle = 1$. Then we have $\sum_{n\neq m} |\langle x_m, S^{-1}x_n\rangle|^2 = 0$ by (12.6). Hence $\langle S^{-1}x_m, x_n\rangle = 0$ for $n \neq m$.

(c) Suppose that $\langle x_m, S^{-1}x_m\rangle = 1$. Then by part (b), $S^{-1}x_m$ is orthogonal to x_n for every $n \neq m$. However, $S^{-1}x_m \neq 0$ since $\langle S^{-1}x_m, x_m\rangle = 1 \neq 0$. Therefore $\{x_n\}_{n\neq m}$ is incomplete in this case.

On the other hand, suppose that $\langle x_m, S^{-1}x_m\rangle \neq 1$, and set $a_n = \langle x_m, S^{-1}x_n\rangle$. We have $x_m = \sum a_n x_n$ by (12.4). Since $a_m \neq 1$, we therefore have $x_m = \frac{1}{1-a_m} \sum_{n\neq m} a_n x_n$. Hence, for each $x \in H$,

$$|\langle x, x_m\rangle|^2 \;=\; \left|\frac{1}{1 - a_m} \sum_{n\neq m} a_n \,\langle x, x_n\rangle\right|^2 \;\leq\; C \sum_{n\neq m} |\langle x, x_n\rangle|^2,$$

where $C = |1-a_m|^{-2} \sum_{n\neq m} |a_n|^2$. Therefore,

$$\sum_n |\langle x, x_n\rangle|^2 \;=\; |\langle x, x_m\rangle|^2 \;+\; \sum_{n\neq m} |\langle x, x_n\rangle|^2 \;\le\; (1+C) \sum_{n\neq m} |\langle x, x_n\rangle|^2.$$

Hence,

$$\frac{A}{1+C}\,\|x\|^2 \;\le\; \frac{1}{1+C} \sum_n |\langle x, x_n\rangle|^2 \;\le\; \sum_{n\neq m} |\langle x, x_n\rangle|^2 \;\le\; B\,\|x\|^2.$$

Thus $\{x_n\}_{n\neq m}$ is a frame with bounds $A/(1+C)$, B. □

As a consequence, we find that a frame is exact if and only if it is biorthogonal to its dual frame.

Corollary 12.14. *If $\{x_n\}$ is a frame for a Hilbert space H, then the following statements are equivalent.*

(a) *$\{x_n\}$ is an exact frame.*

(b) *$\{x_n\}$ and $\{S^{-1}x_n\}$ are biorthogonal.*

(c) *$\langle x_n, S^{-1}x_n\rangle = 1$ for all n.*

As a consequence, if the frame is tight, i.e., $A = B$, then the following statements are equivalent.

(a') *$\{x_n\}$ is an exact frame.*

(b') *$\{x_n\}$ is an orthogonal sequence.*

(c') *$\|x_n\|^2 = A$ for all n.*

Proof. (a) ⇒ (c). If $\{x_n\}$ is an exact frame, then, by definition, $\{x_n\}_{n\neq m}$ is not a frame for any m. It therefore follows from Proposition 12.13 that $\langle x_m, S^{-1}x_m\rangle = 1$ for every m.

(c) ⇒ (a). Suppose that $\langle x_m, S^{-1}x_m\rangle = 1$ for every m. Proposition 12.13 then implies that $\{x_n\}_{n\neq m}$ is not complete, and hence is not a frame. Therefore $\{x_n\}$ is exact by definition.

(b) ⇒ (c). This follows immediately from the definition of biorthogonality.

(c) ⇒ (b). This follows immediately from Proposition 12.13(b). □

In Example 12.3(d), we constructed a frame that is not norm-bounded below. The following result shows that all frames are norm-bounded above, and that only inexact frames can be unbounded below.

Proposition 12.15. *Let $\{x_n\}$ be a frame for a Hilbert space H. Then the following statements hold.*

(a) *$\{x_n\}$ is norm-bounded above, and $\sup \|x_n\|^2 \le B$.*

(b) *If $\{x_n\}$ is exact then it is norm-bounded below, and $A \le \inf \|x_n\|^2$.*

Proof. (a) With m fixed, we have

$$\|x_m\|^4 \;=\; |\langle x_m, x_m\rangle|^2 \;\le\; \sum_n |\langle x_m, x_n\rangle|^2 \;\le\; B\,\|x_m\|^2.$$

(b) If $\{x_n\}$ is an exact, then $\{x_n\}$ and $\{S^{-1}x_n\}$ are biorthogonal by Corollary 12.15. Therefore, for each fixed m,

$$A\,\|S^{-1}x_m\|^2 \;\le\; \sum_n |\langle S^{-1}x_m, x_n\rangle|^2 \;=\; |\langle S^{-1}x_m, x_m\rangle|^2 \;\le\; \|S^{-1}x_m\|^2\,\|x_m\|^2.$$

Since $\{x_n\}$ is exact we must have $x_m \ne 0$. Since S is a topological isomorphism, we therefore have $S^{-1}x_m \ne 0$ as well, so we can divide by $\|S^{-1}x_m\|^2$ to obtain the desired inequality. □

We collect now some remarks on the convergence of $\sum c_n x_n$ for arbitrary sequences of scalars.

Recall that if $\{x_n\}$ is a frame and $\sum |c_n|^2 < \infty$, then $\sum c_n x_n$ converges (Lemma 12.6). The following example shows that the converse is not true in general.

Example 12.16. [Hei90]. *$\sum c_n x_n$ converges $\not\Rightarrow$ $\sum |c_n|^2 < \infty$.*

Let $\{x_n\}$ be any frame which includes infinitely many zero elements. Let $c_n = 1$ whenever $x_n = 0$, and let $c_n = 0$ when $x_n \ne 0$. Then $\sum c_n x_n = 0$, even though $\sum |c_n|^2 = \infty$.

Less trivially, let $\{e_n\}$ be an orthonormal basis for a Hilbert space H. Define $f_n = n^{-1}e_n$ and $g_n = (1 - n^{-2})^{1/2}e_n$. Then $\{f_n\} \cup \{g_n\}$ is a tight frame with $A = B = 1$. Let $x = \sum n^{-1}e_n$. This is an element of H since $\sum n^{-2} < \infty$. However, in terms of the frame $\{f_n\} \cup \{g_n\}$ we have $x = \sum (1 \cdot f_n + 0 \cdot g_n)$, although $\sum (1^2 + 0^2) = \infty$. ◊

By Lemma 12.6, if $(c_n) \in \ell^2$ then $\sum c_n x_n$ converges unconditionally. The preceding example shows that $\sum c_n x_n$ may converge even though $(c_n) \notin \ell^2$. However, we show next that if $\{x_n\}$ is norm-bounded below, then $\sum c_n x_n$ converges unconditionally exactly for $(c_n) \in \ell^2$.

Proposition 12.17. [Hei90]. *If $\{x_n\}$ be a frame that is norm-bounded below, then*

$$\sum_n |c_n|^2 < \infty \quad\Longleftrightarrow\quad \sum_n c_n x_n \text{ converges unconditionally.}$$

Proof. $\Rightarrow$. This is the content of Lemma 12.6.

$\Leftarrow$. Assume that $\sum c_n x_n$ converges unconditionally. Then Orlicz's Theorem (Theorem 3.1) implies that $\sum |c_n|^2\,\|x_n\|^2 = \sum \|c_n x_n\|^2 < \infty$. Since $\{x_n\}$ is norm-bounded below, it therefore follows that $\sum |c_n|^2 < \infty$. □

We shall see in Example 12.23 that, for an exact frame, $\sum c_n x_n$ converges if and only if it converges unconditionally. The next example shows that, for an inexact frame, $\sum c_n x_n$ may converge conditionally, even if the frame is norm-bounded below.

Example 12.18. [Hei90]. We will construct a frame $\{x_n\}$ which is norm-bounded below and a sequence of scalars (c_n) such that $\sum c_n x_n$ converges but $\sum |c_n|^2 = \infty$.

Let $\{e_n\}$ be any orthonormal basis for a separable Hilbert space H. Then $\{e_1, e_1, e_2, e_2, \dots\}$ is a frame that is norm-bounded below. The series

$$e_1 \;-\; e_1 \;+\; \frac{e_2}{\sqrt{2}} \;-\; \frac{e_2}{\sqrt{2}} \;+\; \frac{e_3}{\sqrt{3}} \;-\; \frac{e_3}{\sqrt{3}} \;+\; \cdots \tag{12.7}$$

converges strongly in H to 0. However, the series

$$e_1 + e_1 + \frac{e_2}{\sqrt{2}} + \frac{e_2}{\sqrt{2}} + \frac{e_3}{\sqrt{3}} + \frac{e_3}{\sqrt{3}} + \cdots$$

does not converge. Therefore, the series in (12.7) converges conditionally by Theorem 2.8. Since $(n^{-1/2}) \notin \ell^2$, the conditionality of the convergence also follows from Proposition 12.17. $\diamondsuit$

In the remainder of this chapter, we will determine the exact relationship between frames and bases.

Proposition 12.19. *An inexact frame is not a basis.*

Proof. Assume that $\{x_n\}$ is an inexact frame. Then, by definition, $\{x_n\}_{n \neq m}$ is a frame for some m, and is therefore complete. However, no proper subset of a basis can be complete, so $\{x_n\}$ cannot be a basis. Additionally, we have $x_m = \sum \langle x_m, S^{-1}x_n \rangle \, x_n$ by (12.4), and also $x_m = \sum \delta_{mn} x_n$. By Proposition 12.13, the fact that $\{x_n\}_{n \neq m}$ is a frame implies that $\langle x_m, S^{-1}x_n \rangle \neq 1$. Hence these are two distinct representations of x_m in terms of the frame elements, so $\{x_n\}$ cannot be a basis. $\square$

We show now that frames are preserved by topological isomorphisms (compare Lemmas 4.12, 9.3, and 11.3 for bases, unconditional bases, or Riesz bases, respectively).

Lemma 12.20. *Frames are preserved by topological isomorphisms. That is, if $\{x_n\}$ is a frame for a Hilbert space H and $T\colon H \to K$ is a topological isomorphism, then $\{Tx_n\}$ is a frame for K. In this case, we have the following additional statements:*

(a) *If $\{x_n\}$ has frame bounds A, B, then $\{Tx_n\}$ has frame bounds $A\,\|T^{-1}\|^{-2}$, $B\,\|T\|^2$.*

(b) *If $\{x_n\}$ has frame operator S, then $\{Tx_n\}$ has frame operator TST^*.*

(c) *$\{x_n\}$ is exact if and only if $\{Tx_n\}$ is exact.*

Proof. Note that for each $y \in K$,

$$TST^*y = T\left(\sum_n \langle T^*y, x_n \rangle \, x_n\right) = \sum_n \langle y, Tx_n \rangle \, Tx_n.$$

Therefore, the fact that $\{Tx_n\}$ is a frame and both statements (a) and (b) will follow from Proposition 12.8 if we show that $A\,\|T^{-1}\|^{-2} I \leq TST^* \leq B\,\|T\|^2 I$. Now, if $y \in H$ then $\langle TST^*y, y \rangle = \langle S(T^*y), (T^*y) \rangle$, so it follows from $AI \leq S \leq BI$ that

$$A\,\|T^*y\|^2 \leq \langle TST^*y, y \rangle \leq B\,\|T^*y\|^2. \tag{12.8}$$

Further, since T is a topological isomorphism, we have

$$\frac{\|y\|}{\|T^{-1}\|} = \frac{\|y\|}{\|{T^*}^{-1}\|} \leq \|T^*y\| \leq \|T^*\|\,\|y\| = \|T\|\,\|y\|. \tag{12.9}$$

Combining (12.8) and (12.9), we find that

$$\frac{A\,\|y\|^2}{\|T^{-1}\|^2} \;\le\; \langle TST^*y, y\rangle \;\le\; B\,\|T\|^2\,\|y\|^2,$$

which is equivalent to the desired statement $A\,\|T^{-1}\|^{-2}I \le TST^* \le B\,\|T\|^2 I$.

Finally, statement (c) regarding exactness follows from the fact that topological isomorphisms preserve complete and incomplete sequences. □

We can now show that the class of exact frames for H coincides with the class of bounded unconditional bases for H. By Theorem 11.9, this further coincides with the class of Riesz bases for H. The statement and a different proof of the following result can be found in [You80].

Theorem 12.21. *Let $\{x_n\}$ be a sequence in a Hilbert space H. Then $\{x_n\}$ is an exact frame for H if and only if it is a bounded unconditional basis for H.*

Proof. $\Rightarrow$. Assume that $\{x_n\}$ is an exact frame for H. Then $\{x_n\}$ is norm-bounded both above and below by Proposition 12.15. We have from (12.4) that $x = \sum \langle x, S^{-1}x_n\rangle\, x_n$ for all x, with unconditional convergence of this series. To see that this representation is unique, suppose that we also had $x = \sum c_n x_n$. By Corollary 12.5, $\{x_n\}$ and $\{S^{-1}x_n\}$ are biorthogonal sequences. Therefore,

$$\langle x, S^{-1}x_m\rangle \;=\; \Big\langle \sum_n c_n x_n,\; S^{-1}x_m \Big\rangle \;=\; \sum_n c_n\,\langle x_n, S^{-1}x_m\rangle \;=\; \sum_n c_n \delta_{nm} \;=\; c_m.$$

Hence the representation $x = \sum \langle x, S^{-1}x_n\rangle\, x_n$ is unique, so $\{x_n\}$ is a bounded unconditional basis for H.

$\Leftarrow$. Assume that $\{x_n\}$ is a bounded unconditional basis for H. Then $\{x_n\}$ is a Riesz basis by Theorem 11.9. Therefore, by definition of Riesz basis, there exists an orthonormal basis $\{e_n\}$ for H and a topological isomorphism $T\colon H \to H$ such that $Te_n = x_n$ for all n. However, $\{e_n\}$ is an exact frame and exact frames are preserved by topological isomorphisms (Lemma 12.20), so $\{x_n\}$ must be an exact frame for H. □

We can exhibit directly the topological isomorphism T used in the proof of Theorem 12.21. Since S is a positive operator that is a topological isomorphism of H onto itself, it has a square root $S^{1/2}$ that is a positive topological isomorphism of H onto itself [Wei80, Theorem 7.20]. Similarly, S^{-1} has a square root $S^{-1/2}$, and it is easy to verify that $(S^{1/2})^{-1} = S^{-1/2}$. Since $\{x_n\}$ is exact, $\{x_n\}$ and $\{S^{-1}x_n\}$ are biorthogonal by Corollary 12.15. Therefore,

$$\langle S^{-1/2}x_m, S^{-1/2}x_n\rangle \;=\; \langle x_m, S^{-1/2}S^{-1/2}x_n\rangle \;=\; \langle x_m, S^{-1}x_n\rangle \;=\; \delta_{mn}.$$

Thus $\{S^{-1/2}x_n\}$ is an orthonormal sequence. Moreover, it is complete since topological isomorphisms preserve complete sequences. Therefore, $\{S^{-1/2}x_n\}$ is an orthonormal basis for H by Theorem 1.20, and the topological isomorphism $T = S^{1/2}$ maps this orthonormal basis onto the frame $\{x_n\}$.

We can consider the sequence $\{S^{-1/2}x_n\}$ for any frame, not just exact frames. If $\{x_n\}$ is inexact then $\{S^{-1/2}x_n\}$ will not be an orthonormal basis for H, but we show next that it will be a tight frame for H.

Corollary 12.22. *Every frame is equivalent to a tight frame. That is, if $\{x_n\}$ is a frame with frame operator S then $S^{-1/2}$ is a positive topological isomorphism of H onto itself, and $\{S^{-1/2}x_n\}$ is a tight frame with bounds $A = B = 1$.*

Proof. Since $S^{-1/2}$ is a topological isomorphism, it follows from Lemma 12.20 that $\{S^{-1/2}x_n\}$ is a frame for H. Note that for each $x \in H$ we have

$$\sum_n \langle x, S^{-1/2}x_n\rangle\, S^{-1/2}x_n \;=\; S^{-1/2}SS^{-1/2}x \;=\; x \;=\; Ix.$$

Proposition 12.8 therefore implies that the frame is tight and has frame bounds $A = B = 1$. □

Example 12.23. If $\{x_n\}$ is an exact frame then it is a Riesz basis for H. Hence by Theorem 11.9 and Proposition 12.17, we have that

$$\sum_n |c_n|^2 < \infty \quad\Longleftrightarrow\quad \sum_n c_n x_n \text{ converges} \quad\Longleftrightarrow\quad \sum_n c_n x_n \text{ converges unconditionally.}$$

By Example 12.18, these equivalences may fail if the frame is inexact. However, we can construct an inexact frame for which these equivalences remain valid.

Let $\{e_n\}$ be any orthonormal basis for a separable Hilbert space H, and consider the frame $\{x_n\} = \{e_1, e_1, e_2, e_3, \dots\}$. Then the series $\sum c_n x_n$ converges if and only if $\sum |c_n|^2 < \infty$ since $\{x_n\}$ is obtained from an orthonormal basis by the addition of a single element. Further, since $\{x_n\}$ is norm-bounded below, it follows from Proposition 12.17 that $\sum |c_n|^2 < \infty$ if and only if $\sum c_n x_n$ converges unconditionally. ♢

The frame $\{e_1, e_1, e_2, e_3, \dots\}$ considered in Example 12.23 consists of an orthonormal basis plus one additional element. Holub [Hol94] has characterized those frames $\{x_n\}$ which consist of a Riesz basis plus finitely many elements.

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