Gabor Schauder bases and the Balian-Low theorem

Christopher Heil[a)]
School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332-0160

Alexander M. Powell[b)]
Department of Mathematics, Vanderbilt University, Nashville, Tennessee 37240

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The Balian-Low Theorem is a strong form of the uncertainty principle for Gabor systems that form orthonormal or Riesz bases for $L^2(\mathbb{R})$. In this paper we investigate the Balian-Low Theorem in the setting of Schauder bases. We prove that new weak versions of the Balian-Low Theorem hold for Gabor Schauder bases, but we constructively demonstrate that several variants of the BLT can fail for Gabor Schauder bases that are not Riesz bases. We characterize a class of Gabor Schauder bases in terms of the Zak transform and product $\mathcal{A}_2$ weights; the Riesz bases correspond to the special case of weights that are bounded away from zero and infinity. © *2006 American Institute of Physics.* [DOI: 10.1063/1.2360041]

I. INTRODUCTION

A *Gabor* or *Weyl-Heisenberg system* consists of a discrete set of time-frequency shifts of a fixed *window function* or *atom* $g \in L^2(\mathbb{R})$. Often, the indexing set is required to possess some structure. For example, we will consider *lattice Gabor systems* of the form

$$\mathcal{G}(g,\alpha,\beta) = \{M_{\beta n}T_{\alpha k}g\}_{k,n\in\mathbb{Z}} = \{e^{2\pi i\beta nt}g(t-\alpha k)\}_{k,n\in\mathbb{Z}}, \tag{1.1}$$

where T_x is the translation operator $T_x g(t)=g(t-x)$, M_ξ is the modulation operator $M_\xi g(t) = e^{2\pi i\xi t}g(t)$, and the compositions $T_x M_\xi$ or $M_\xi T_x$ are called *time-frequency shift operators*. Gabor systems can also be defined in higher dimensions and on general lattices or even completely irregular sets of time-frequency shifts. One may also consider continuous Gabor transforms. For background on the theory and applications of Gabor systems, we refer to Refs. 1–3.

Gabor[4] proposed using the Gaussian function $\varphi(t)=e^{-t^2}$ as a window, with respect to the unit time-frequency shift lattice ($\alpha=\beta=1$). However, while this Gaussian Gabor system $\mathcal{G}(\varphi,1,1)$ is complete in $L^2(\mathbb{R})$ and remains complete if any single element of $\mathcal{G}(\varphi,1,1)$ is removed (cf. Ref. 5, p. 168), it is not an orthonormal basis, a Riesz basis, a frame, or a Schauder basis for $L^2(\mathbb{R})$ (even if the "extra" element is removed).

However, completeness alone is too weak a property to be useful in practice. A *Schauder basis* allows unique representations of elements of $L^2(\mathbb{R})$ in terms of the basis elements, i.e., $\mathcal{G}(g,\alpha,\beta)$ is a Schauder basis if for each $f \in L^2(\mathbb{R})$ there exist unique scalars $c_{kn}(f)$ such that

$$f = \sum_{k,n\in\mathbb{Z}} c_{kn}(f)M_{\beta n}T_{\alpha k}g, \tag{1.2}$$

with convergence of the series in the norm of L^2 with respect to some fixed ordering of the series. However, for a Schauder basis there need not be any direct relation between the size of the coefficients $c_{kn}(f)$ and the norm of f. In contrast, if $\mathcal{G}(g,\alpha,\beta)$ is a *frame* (defined precisely in Sec.

[a)]Electronic mail: heil@math.gatech.edu
[b)]Electronic mail: alexander.m.powell@vanderbilt.edu

II A), then it follows not merely that $\mathcal{G}(g,\alpha,\beta)$ is complete, but that every $f\in L^2(\mathbb{R})$ can be written as in (1.2) for a canonical choice of scalars $c_{kn}(f)$ whose ℓ^2 norm forms an equivalent norm for $L^2(\mathbb{R})$. Moreover, for a frame the series (1.2) converges *unconditionally* in the L^2 norm, i.e., regardless of ordering. However, a frame need not provide unique expansions—the scalars $c_{kn}(f)$ need not be unique in general. The canonical choice of scalars is determined by the canonical dual frame, which for a lattice Gabor frame $\mathcal{G}(g,\alpha,\beta)$ is another lattice Gabor frame $\mathcal{G}(\tilde{g},\alpha,\beta)$ generated by a dual window $\tilde{g}\in L^2(\mathbb{R})$. A *Riesz basis* is both a frame and a Schauder basis, and thus provides unique unconditionally convergent expansions whose coefficients stably encode the norm of f.

The Density Theorem for Gabor systems provides necessary (but not sufficient) conditions for $\mathcal{G}(g,\alpha,\beta)$ to be complete, a frame, a Riesz basis, or a Riesz sequence [a sequence that need not be complete but forms a Riesz basis for its closed span within $L^2(\mathbb{R})$]. Moreover, these conditions are formulated solely in terms of the lattice $\alpha\mathbb{Z}\times\beta\mathbb{Z}$.

Theorem 1.1 (Density Theorem): *Let $g\in L^2(\mathbb{R})$ and α, $\beta>0$ be given.*

(a) *If $\mathcal{G}(g,\alpha,\beta)$ is complete in $L^2(\mathbb{R})$, then $\alpha\beta\leq 1$. In particular, if $\mathcal{G}(g,\alpha,\beta)$ is a frame, then $\alpha\beta\leq 1$.*
(b) *If $\mathcal{G}(g,\alpha,\beta)$ is a Riesz basis in $L^2(\mathbb{R})$, then $\alpha\beta=1$.*
(c) *If $\mathcal{G}(g,\alpha,\beta)$ is a Riesz sequence in $L^2(\mathbb{R})$, then $\alpha\beta\geq 1$.*

The Density Theorem has a long and involved history, including extensions to general lattices, to irregular Gabor frames in higher dimensions, and beyond Gabor frames to the setting of abstract "localized frames." Some of the main references include Refs. 6–12. We refer to Ref. 13 for a detailed survey of the Density Theorem, with extensive references to the original literature.

By the Density Theorem, there is a clear separation between "overcomplete" frames and "undercomplete" Riesz sequences, with the Riesz bases corresponding to the *critical density lattices* that satisfy $\alpha\beta=1$. Moreover, there exists a simple exact characterization, in terms of the Zak transform, of those g such that $\mathcal{G}(g,\alpha,\beta)$ is a Riesz basis for $L^2(\mathbb{R})$ (see Theorem 2.8).

Unfortunately, the *Balian-Low Theorem* (BLT) is a classical result that implies that if $\mathcal{G}(g,\alpha,\beta)$ is a Riesz basis, then the window g must have poor joint localization in the time-frequency plane. Specifically, the localization integrals

$$\int |t|^2|g(t)|^2\,dt \quad \text{and} \quad \int |\xi|^2|\hat{g}(\xi)|^2\,d\xi$$

cannot both be finite. These same localization integrals appear in the Heisenberg uncertainty principle and, indeed, the BLT may be viewed as a strong form of the uncertainty principle for functions that generate a Gabor Riesz basis (we refer to Ref. 14 for a survey of uncertainty principles).

As with the Density Theorem the BLT has a long and involved history. Stated independently by Balian[15] and Low[16] for the case of orthonormal bases, a complete proof, and an extension to Riesz bases was given by Coifman, Daubechies, and Semmes.[7] Battle[17] gave an elegant new proof for the orthonormal basis case, which shows the intimate connection between the BLT and the operator theory associated with the Heisenberg uncertainty principle. This proof was further extended by Daubechies and Janssen in Ref. 18. Related theorems that relate distinct uncertainty principles to Riesz basis properties have been proved in Refs. 19–22, and extensions of the BLT to higher dimensions appear in Refs. 23–25. We collectively refer to these results as Balian-Low Theorems. Surveys of the Balian-Low Theorems appear in Refs. 19 and 26, and we also note the related results in the papers in Refs. 27–30.

In this paper we determine the extent to which the class of Balian-Low Theorems extend to Gabor systems $\mathcal{G}(g,\alpha,\beta)$ that form Schauder bases but that need not be Riesz bases. Despite the large literature on Gabor systems, very little has been known to date about Gabor Schauder bases. The initial work in this direction[31] proved the existence of a particular Gabor Schauder basis that is not a Riesz basis, and also provided partial results suggesting that the Density Theorem applies to Gabor Schauder bases; cf. also Ref. 32 for the setting of windowed exponentials in $L^2(\Omega)$. However, these papers clearly indicate that all results become more delicate and difficult for Schauder bases, with the consequence that open questions abound.

In this paper, we accomplish several goals, as follows.

(a) We prove that the Density Theorem does hold for lattice Gabor Schauder bases. Specifically, if $\mathcal{G}(g,\alpha,\beta)$ is a Schauder basis, then $\alpha\beta=1$. We prove that the dual basis is also a Gabor system of the form $\mathcal{G}(\tilde{g},\alpha,\beta)$ for a unique dual window $\tilde{g}$.
(b) We give a characterization of a class of Gabor Schauder bases $\mathcal{G}(g,\alpha,\beta)$ in terms of the Zak transform of the window g and product $\mathcal{A}_2$ weights.
(c) We prove that several variants of the Balian-Low Theorems fail to hold for Gabor Schauder bases.
(d) We prove new weak versions of the BLT for Gabor Schauder bases, and show that as a consequence, if the window g of a Gabor Schauder basis $\mathcal{G}(g,\alpha,\beta)$ is well concentrated in the time-frequency plane, then the dual window $\tilde{g}$ is poorly concentrated.

Thus, counterintuitively, the window of a Gabor Schauder basis can have much better joint time-frequency localization than the generator of a Riesz basis. At the same time, the weak BLT for Gabor Schauder bases implies a striking difference from the behavior of Gabor frames. To discuss this more precisely, let us quantify time-frequency localization by the behavior of the *short-time Fourier transform* (STFT) of g:

$$V_\varphi g(x,\xi) = \langle g, M_\xi T_x \varphi\rangle = \int g(t)\varphi(t-x)e^{-2\pi i\xi t}\,dt, \quad (x,\xi)\in\mathbb{R}^2.$$

If we normalize as $\varphi(t)=(\pi/2)^{-1/4}e^{-t^2}$, then we have $\|\varphi\|_2=1$ and $\|V_\varphi g\|_2=\|g\|_2$. We define the *modulation space* $M^1(\mathbb{R})$ to be the function space consisting of all $g\in L^2(\mathbb{R})$ whose STFT is integrable, i.e.,

$$M^1(\mathbb{R}) = \left\{ g\in L^2(\mathbb{R}) : \|g\|_{M^1} = \|V_\varphi g\|_1 = \int\int |V_\varphi g(x,\xi)|\,dx\,d\xi < \infty \right\}. \tag{1.3}$$

Thus, a function $g\in M^1(\mathbb{R})$ possesses an L^1-type of joint localization in the time-frequency plane. We refer to Ref. 20, Chaps. 11–13 for a detailed background on the class of modulation spaces (of which M^1 is the prototypical example). In particular, we note that the choice of Gaussian window φ above is only for convenience; any nonzero Schwartz-class window, or indeed any nonzero window in M^1, may be used to define the modulation spaces, with each choice of window defining the same space under an equivalent norm.

In Gabor analysis, it is essential to jointly understand the behavior of a frame $\mathcal{G}(g,\alpha,\beta)$ and its dual frame $\mathcal{G}(\tilde{g},\alpha,\beta)$. For a Riesz basis, the Balian-Low Theorems imply that both the window and the dual window must have poor localization:

$$\text{For a Gabor Riesz basis:} \quad g\notin M^1(\mathbb{R}) \quad \text{and} \quad \tilde{g}\notin M^1(\mathbb{R}).$$

In contrast, Gröchenig and Leinert[63] proved that if the window of a Gabor frame is well localized, then the dual window is as well, and, conversely:

$$\text{For a Gabor frame:} \quad g\in M^1(\mathbb{R}) \Leftrightarrow \tilde{g}\in M^1(\mathbb{R}).$$

The Gröchenig and Leinert proof used deep results on symmetric C^* algebras. A new proof, based on a type of noncommutative Wiener's lemma, that extends to irregular Gabor frames and abstract localized frames, was given in Refs. 33 and 12.

In this paper, we constructively demonstrate the following:

$$\text{For a Gabor Schauder basis:} \quad g \in M^1(\mathbb{R}),\ \tilde{g} \notin M^1(\mathbb{R}) \text{ is possible.}$$

Moreover, we show that whenever the window of a Gabor Schauder basis is well localized, then the dual window must be poorly localized, specifically the following:

$$\text{For a Gabor Schauder basis:} \quad g \in M^1(\mathbb{R}) \Rightarrow \tilde{g} \notin M^1(\mathbb{R}). \tag{1.4}$$

We note that many types of weight functions play important roles in many aspects of time-frequency analysis (see Gröchenig's recent survey[34] for details and references). However, to date the Muckenhoupt, or $\mathcal{A}_p$, weights have had no application within time-frequency analysis. Our results show that the Muckenhoupt weight class is important in time-frequency analysis as a window class via the Zak transform. This is in contrast to the usual use of weights, which normally are used in time-frequency analysis in the quantification of time-frequency concentration.

Overview: Our paper is organized as follows. In Sec. II A we present background information on Schauder bases, Riesz bases, and the Zak transform. In Sec. III we recall the classical Balian-Low Theorem and several of its variants. These results, which are all stated in terms of Riesz bases for $L^2(\mathbb{R})$, form the backdrop for our investigation into the Balian-Low Theorem for Schauder bases. In Sec. IV we investigate some basic properties of Gabor Schauder bases, including the structure of the dual basis, and we present a short proof of a weak Balian-Low Theorem for Gabor Schauder bases. In Sec. V we show how to use product $\mathcal{A}_2$ weights to give a characterization of a class of Gabor Schauder bases for $L^2(\mathbb{R})$. In Sec. VI we show that several versions of the Balian-Low Theorem do not extend to the setting of Schauder bases. In Sec. VII, we prove that weak versions of the M^1 BLT and Amalgam BLT hold for Gabor Schauder bases and exact systems.

II. BACKGROUND AND PRELIMINARIES

A. Schauder bases, frames, and Riesz bases

We recall the definition and basic facts regarding Schauder bases, frames, and Riesz bases in Hilbert spaces. We refer to Refs. 35–38 for additional background.

Definition 2.1: An ordered collection $\mathcal{F}=\{f_n\}_{n=0}^{\infty}$ in a Hilbert space H is a *Schauder basis* for H if for each $f \in H$ there exist unique scalars $c_n(f)$ such that

$$f = \sum_{n=0}^{\infty} c_n(f) f_n, \tag{2.1}$$

where the series converges in the norm of H.

It is important to point out that Schauder basis expansions may converge conditionally, i.e., the order of summation in (2.1) matters.

The linear functionals c_n in (2.1) can be shown to be continuous. Therefore there exist $\tilde{f}_n \in H$ such that $c_n(f)=\langle f, \tilde{f}_n\rangle$. Further, $\tilde{\mathcal{F}}=\{\tilde{f}_n\}_{n=0}^{\infty}$ is the unique sequence in H that is *biorthogonal* to $\mathcal{F}$, i.e.,

$$\langle f_m, \tilde{f}_n\rangle = \delta_{mn}.$$

Every Schauder basis has an associated biorthogonal sequence, but the converse is not true, i.e., the existence of a biorthogonal system does not imply that the original sequence is a Schauder basis. The existence of a biorthogonal sequence is equivalent to the statement that $\mathcal{F}$ is *minimal*, i.e., no element of $\mathcal{F}$ lies in the closed linear span of the remaining elements. A system that is both complete and minimal is said to be *exact*. There exist complete and minimal systems that are not Schauder bases.

Given a sequence $\mathcal{F}=\{f_n\}_{n=0}^{\infty}$ that has a biorthogonal sequence $\tilde{\mathcal{F}}=\{\tilde{f}_n\}_{n=0}^{\infty}$, we define the *partial sum operators* $S_N: H \to H$ by

$$S_N(f) = \sum_{n=0}^{N} \langle f, \tilde{f}_n \rangle f_n.$$

There are several equivalent definitions of a Schauder basis, of which we will need the following.

Theorem 2.2: *Given a collection $\mathcal{F}=\{f_n\}_{n=0}^{\infty}$ in a Hilbert space H, the following statements are equivalent.*

(a) *$\mathcal{F}$ is a Schauder basis.*

(b) *There exists a biorthogonal sequence $\tilde{\mathcal{F}}=\{\tilde{f}_n\}_{n=0}^{\infty}$ such that the partial sum operators S_N converge in the strong operator topology to the identity map, i.e.,*

$$\forall f \in H, \quad f = \sum_{n=0}^{\infty} \langle f, \tilde{f}_n \rangle f_n. \tag{2.2}$$

(c) *There exists a biorthogonal sequence $\tilde{\mathcal{F}}=\{\tilde{f}_n\}_{n=0}^{\infty}$ such that the partial sum operators are uniformly bounded in operator norm, i.e.,*

$$\sup_N \|S_N\| < \infty.$$

The number $C=\sup_N\|S_N\|$ is called the *basis constant* for $\mathcal{F}$. The biorthogonal sequence $\tilde{\mathcal{F}}$ is itself a Schauder basis for H, and is called the *dual basis* to $\mathcal{F}$.

In contrast to Schauder bases, frames are defined in terms of a norm-equivalence criterion.

Definition 2.3: A collection $\mathcal{F}=\{f_n\}_{n=0}^{\infty}$ in a Hilbert space H is a *frame* for H if there exist constants A, $B>0$, called *frame bounds*, such that

$$\forall f \in H, \quad A\|f\|^2 \leq \sum_{i=0}^{\infty} |\langle f, f_i \rangle|^2 \leq B\|f\|^2. \tag{2.3}$$

A sequence for which the upper inequality in (2.3) is satisfied, but not necessarily the lower inequality, is called a *Bessel sequence*.

If $\mathcal{F}$ is a frame, then the *frame operator* $Sf=\sum_{n=0}^{\infty}\langle f, f_n\rangle f_n$ is a bounded, positive, and invertible mapping of H onto itself. The *canonical dual frame* $\tilde{\mathcal{F}}=\{\tilde{f}_n\}_{n=0}^{\infty}$ defined by $\tilde{f}_n=S^{-1}(f_n)$ yields frame expansions exactly of the form in (2.2). Moreover, those series converge unconditionally for each f. However, in general, a frame need not be a Schauder basis. In particular, the scalars $\langle f, \tilde{f}_n\rangle$ in (2.2) need not be unique.

Definition 2.4: A collection $\mathcal{F}=\{f_n\}_{n=0}^{\infty}$ in a Hilbert space H is a *Riesz basis* for H if it is the image of an orthonormal basis for H under a continuous, invertible map of H onto itself.

Among other characterizations, the following theorem shows that a Riesz basis is precisely a sequence that is both a frame and a Schauder basis.

Theorem 2.5: *Given a collection $\mathcal{F}=\{f_n\}_{n=0}^{\infty}$ in a Hilbert space H, the following statements are equivalent.*

(a) *$\mathcal{F}$ is a Riesz basis.*

(b) *$\mathcal{F}$ is a bounded unconditional basis, i.e., $\mathcal{F}$ is a Schauder basis, the basis expansions in (2.1) converge unconditionally for each $f \in H$, and $0<\inf_n\|f_n\|\leq\sup_n\|f_n\|<\infty$.*

(c) *$\mathcal{F}$ is an exact frame, i.e., it is a frame and is biorthogonal to its canonical dual frame.*

(d) *$\mathcal{F}$ is complete and there exist constants A, $B>0$, such that*

$$\forall c_0, \ldots, c_N, \quad A\sum_{n=0}^{N} |c_N|^2 \leq \left\| \sum_{n=0}^{N} c_n f_n \right\|^2 \leq B\sum_{n=0}^{N} |c_n|^2.$$

(e) *$\mathcal{F}$ is a complete Bessel sequence and possesses a biorthogonal sequence that is also a complete Bessel sequence.*

The dual basis of a Riesz basis coincides with its canonical dual frame, and is itself a Riesz basis for H.

We say that a sequence is a *Riesz sequence* if it forms a Riesz basis for its closed span within H.

Example 2.6: Let $\varphi(t)=e^{-t^2}$. von Neumann (Ref. 39, p. 406) claimed (without proof) that $\mathcal{G}(\varphi,1,1)$ is complete in $L^2(\mathbb{R})$. Gabor conjectured in Ref. 4, Eq. 1.29 that every function in $L^2(\mathbb{R})$ can be represented in the form

$$f = \sum_{k,n\in\mathbb{Z}} c_{kn}(f) M_n T_k \varphi, \tag{2.4}$$

for some scalars $c_{kn}(f)$. von Neumann's claim of completeness was proved in Refs. 40 and 41. Janssen proved in Ref. 42 that Gabor's conjecture is true, but he showed that the series in (2.4) converges only in the sense of tempered distributions—not in the norm of L^2. Thus the Gabor system $\mathcal{G}(\varphi,1,1)$ fits none of the definitions given previously in this section.

On the other hand, Lyubarskii[43] and Seip and Wallstén[44,45] proved that $\mathcal{G}(\varphi,\alpha,\beta)$ is a frame for $L^2(\mathbb{R})$ whenever $0<\alpha\beta<1$. Moreover, $\mathcal{G}(\varphi,\alpha,\beta)$ is an incomplete Riesz sequence in $L^2(\mathbb{R})$ whenever $\alpha\beta>1$.

B. The Zak transform

The Zak transform was introduced by Gelfand[46] and goes by several names. It is often called the *Weil-Brezin map* in representation theory and abstract harmonic analysis. Zak rediscovered this transform, which he called the *k-q transform*, in his work on quantum mechanics, e.g., Ref. 47. For more information, we refer to Janssen's influential paper[48] and survey,[49] or to Gröchenig's text.[1]

The Zak transform is an extremely useful tool for analyzing Gabor systems at the critical density $\alpha\beta=1$. Because the unitary dilation $D_\alpha f(t)=\alpha^{1/2}f(\alpha t)$ maps the Gabor system $\mathcal{G}(g,\alpha,1/\alpha)$ to the Gabor system $\mathcal{G}(D_\alpha g,1,1)$, when working at the critical density we always can, by a change of variables, reduce to the case $\alpha=\beta=1$. This is what we will do throughout, i.e., when we are at the critical density we will only consider Gabor systems of the form

$$\mathcal{G}(g,1,1) = \{M_n T_k g\}_{k,n\in\mathbb{Z}}.$$

The Zak transform is the unitary operator Z: $L^2(\mathbb{R})\to L^2(Q)$, where $Q=[0,1)^2$, formally defined for $f\in L^2(\mathbb{R})$ by

$$Zf(t,\xi) = \sum_{k\in\mathbb{Z}} f(t-k)e^{2\pi i k\xi}, \quad (t,\xi)\in[0,1)^2. \tag{2.5}$$

It can be shown that the series above converges in the norm of $L^2(Q)$, and that Z is a unitary map of $L^2(\mathbb{R})$ onto $L^2(Q)$. The utility of the Zak transform is made apparent by the following theorem, where we use the notation

$$E_{n,k}(t,\xi) = e^{2\pi i n t}e^{-2\pi i k\xi}. \tag{2.6}$$

Theorem 2.7: *Let $g\in L^2(\mathbb{R})$ be given. Then*

$$Z(M_nT_kg)(t,\xi) = (E_{n,k}\cdot Zg)(t,\xi) = E_{n,k}(t,\xi)Zg(t,\xi).$$

In other words, the Zak transform diagonalizes time-frequency shifts.

Since $\{E_{n,k}\}_{k,n\in\mathbb{Z}}$ forms an orthonormal basis for $L^2(Q)$ and since Z is unitary, the following characterization follows easily; cf. Ref. 50 Theorem 4.3.3.

Theorem 2.8: *Let $g\in L^2(\mathbb{R})$ be given.*

(a) *$\mathcal{G}(g,1,1)$ is complete if and only if $Zg \neq 0$ a.e.*

(b) *$\mathcal{G}(g,1,1)$ is minimal if and only if $1/Zg \in L^2(Q)$. In this case $\mathcal{G}(g,1,1)$ is also complete, i.e., it is exact.*

(c) *$\mathcal{G}(g,1,1)$ is a frame for $L^2(\mathbb{R})$ if and only if there exist $0 < A \leq B < \infty$ such that $A \leq |Zg|^2 \leq B$ a.e. In this case $\mathcal{G}(g,1,1)$ is a Riesz basis for $L^2(\mathbb{R})$, and A, B are frame bounds for $\mathcal{G}(g,1,1)$.*

(d) *$\mathcal{G}(g,1,1)$ is an orthonormal basis for $L^2(\mathbb{R})$ if and only if $|Zg|^2 = 1$ a.e.*

C. Modulation spaces

In addition to the modulation space $M^1(\mathbb{R})$ defined in (1.3), we will need the following particular modulation spaces. We refer to Ref. 1 for details.

Definition 2.9: Let $\varphi(t) = (\pi/2)^{-1/4} e^{-t^2}$. For $1 \leq p < \infty$ and $s \geq 0$, the modulation space $M_s^p(\mathbb{R})$ consists of all tempered distributions $g \in S'(\mathbb{R})$ for which the norm

$$\|g\|_{M_s^p} = \|V_\varphi g\|_{L_s^p} = \left(\int \int |V_\varphi g(x,\xi)|^p (1 + |x| + |\xi|)^{ps} \, dx \, d\xi \right)^{1/p}$$

is finite.

For any α, β with $0 < \alpha\beta < 1$, the following is an equivalent norm for $M_s^p(\mathbb{R})$:

$$|||g|||_{M_s^p} = \left(\sum_{k \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} |\langle g, M_{\beta n} T_{\alpha k} \varphi \rangle|^p (1 + |\alpha k| + |\beta n|)^{ps} \right)^{1/p}. \tag{2.7}$$

For $s \geq 0$, we have $M_s^2(\mathbb{R}) = L_s^2(\mathbb{R}) \cap H^s(\mathbb{R})$, where

$$L_s^2(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}) : \int |f(t)|^2 (1 + |t|)^{2s} \, dt < \infty \right\}$$

and

$$H^s(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}) : \int |\hat{f}(\xi)|^2 (1 + |\xi|)^{2s} \, d\xi \right\}.$$

It follows from the discrete-type norm given in (2.7) that $M_s^2(\mathbb{R}) \subseteq M^1(\mathbb{R})$ if $s > 1$. However, $M_1^2(\mathbb{R})$ does not embed into $M^1(\mathbb{R})$, nor conversely.

III. THE BALIAN-LOW THEOREMS

In this section we recall the precise statement of several variants of the Balian-Low Theorem. We use the Fourier transform normalized by $\hat{f}(\xi) = \int f(t) e^{-2\pi i \xi t} \, dt$. Recall that, by the Density Theorem, if a lattice Gabor system is a Riesz basis, then necessarily we are at the critical density, and hence it suffices to consider $\alpha = \beta = 1$. The first part of the following theorem is the classical wording of the BLT; we also give an equivalent wording in terms of modulation spaces.

Theorem 3.1 (Classical Balian-Low Theorem): *If $g \in L^2(\mathbb{R})$ and*

$$\int |t|^2 |g(t)|^2 \, dt < \infty \quad \text{and} \quad \int |\xi|^2 |\hat{g}(\xi)|^2 \, d\xi < \infty,$$

then $\mathcal{G}(g,1,1)$ is not a Riesz basis for $L^2(\mathbb{R})$. Equivalently, if $g \in M_1^2(\mathbb{R})$, then $\mathcal{G}(g,1,1)$ is not a Riesz basis for $L^2(\mathbb{R})$.

The following theorem summarizes four variations of the Balian-Low Theorem that involve different quantifications of time-frequency localization. Part (a) is formulated in terms of the modulation space $M^1(\mathbb{R})$ that was defined in (1.3). Part (b) is formulated in terms of the *Wiener amalgam space*,

$$W(\mathcal{C},\ell^1)=\left\{f:\ \mathbb{R}\to\mathbb{C}: f \text{ is continuous and } \sum_{n\in\mathbb{Z}}\|f\cdot\chi_{[n,n+1)}\|_\infty<\infty\right\}.$$

For details on the amalgam spaces, with extensive references to the original literature, we refer to the survey in Ref. 62.

Theorem 3.2: *Fix $g\in L^2(\mathbb{R})$: If any of the following hypotheses hold:*

(a) $g\in M^1(\mathbb{R})$;
(b) $g\in W(\mathcal{C},\ell^1)$;
(c) $1<q<2<p<\infty$, $\frac{1}{p}+\frac{1}{q}=1$, $0<\varepsilon<2-q$,

$$\int|t|^{p+\varepsilon}|g(t)|^2\,dt<\infty,\quad\text{and}\quad\int|\xi|^{q+\varepsilon}|\hat{g}(\xi)|^2\,d\xi<\infty;$$

(d) $\sup_{N>0}\int|t|^N|g(t)|^2\,dt<\infty$ *and* $\int|\xi||\hat{g}(\xi)|^2\,d\xi<\infty$,

then $\mathcal{G}(g,1,1)$ is not a Riesz basis for $L^2(\mathbb{R})$.

The theorem corresponding to the hypothesis in part (a) of Theorem 3.2 was proved in Ref. 23, and it extends to Gabor systems on arbitrary lattices in higher dimensions. We will refer to this theorem as the M^1 BLT. Since M^1 does not embed into $M_1^2(\mathbb{R})$ nor conversely, this result is distinct from Theorem 3.1.

The theorem corresponding to part (b) is known as the *Amalgam BLT*.[51,19] It extends to rectangular lattices of the form $\alpha\mathbb{Z}^d\times\beta\mathbb{Z}^d$ in higher dimensions. For rectangular lattices, the Amalgam BLT implies the M^1 BLT. However, it is not known if the Amalgam BLT extends to arbitrary lattices, so in that setting these two results are distinct. Furthermore, it was shown in Ref. 19 that the Amalgam BLT and the Classical BLT are distinct.

Part (c) follows from part (a) and Ref. 52, Theorem 1; see Ref. 30, Eq. (2.6).

Part (d) was proved in Ref. 22. It is worth mentioning that part (c) actually holds for all $\varepsilon>0$, but we shall only deal with the case where ε is sufficiently small. We refer to parts (c) and (d) as *nonsymmetric* (p,q) *BLTs*.

IV. SCHAUDER BASES AND A WEAK BALIAN-LOW THEOREM

In this section we will prove some basic facts about Gabor Schauder bases, and establish that a weak version of the BLT holds for Schauder bases.

Gabor systems are naturally indexed by $\mathbb{Z}\times\mathbb{Z}$, but do not come equipped with a standard enumeration. Since Schauder basis expansions may depend critically on ordering any discussion of Gabor Schauder bases must specify a particular enumeration of $\mathbb{Z}\times\mathbb{Z}$. Therefore, whenever we say that $G(g,\alpha,\beta)$ is a Gabor Schauder basis, we shall implicitly mean that there exists an enumeration of the Gabor system $G(g,\alpha,\beta)$, which is a Schauder basis. When necessary we shall explicitly define the enumeration.

A. The Density Theorem and the Dual Basis

We first observe that the existing results for Gabor Schauder bases implicitly contain a Density Theorem. We make this explicit as follows (we refer to Ref. 11 for the definition of Beurling density). As a consequence, for lattice Gabor Schauder bases, we will always be able to reduce to the case $\alpha=\beta=1$.

Theorem 4.1 (Density Theorem for Lattice Gabor Schauder Bases): *Let $g\in L^2(\mathbb{R})$ and $\alpha,\beta>0$ be given. If $\mathcal{G}(g,\alpha,\beta)$ is a Schauder basis for $L^2(\mathbb{R})$, then $\alpha\beta=1$.*

Proof: First, if $\mathcal{G}(g,\alpha,\beta)$ is a Schauder basis, then it is complete, so Theorem 1.1 implies that $\alpha\beta\le 1$. On the other hand, see Ref. 31, Corollary 4.6, which gives partial necessary conditions for an irregular Gabor system to form a Schauder basis, and implies that the lower and upper Beurling densities of the index set $\alpha\mathbb{Z}\times\beta\mathbb{Z}$ must satisfy $D^{\pm}(\alpha\mathbb{Z}\times\beta\mathbb{Z})\le 1$. The Beurling density of the lattice $\alpha\mathbb{Z}\times\beta\mathbb{Z}$ is $D^{\pm}(\alpha\mathbb{Z}\times\beta\mathbb{Z})=1/(\alpha\beta)$, so this implies that $\alpha\beta\ge 1$. □

It was conjectured in Ref. 31 that if $\mathcal{G}(g,\Lambda)=\{M_\xi T_x g\}_{(x,\xi)\in\Lambda}$ is a Gabor Schauder basis with respect to an arbitrary sequence of time-frequency shifts Λ, then we must have $D^{\pm}(\Lambda)=1$. This conjecture remains open for non-lattice Gabor Schauder bases.

It is well known that the dual basis to a Gabor Riesz basis $\mathcal{G}(g,\alpha,\beta)$ is itself a Gabor Riesz basis $\mathcal{G}(\tilde{g},\alpha,\beta)$. This follows from the fact that if $\mathcal{G}(g,\alpha,\beta)$ is a frame, then the frame operator S commutes with the time-frequency shifts $M_{\beta n}T_{\alpha k}$. For a Gabor Schauder basis, the frame operator need not be a bounded or even a well-defined map. Nonetheless, we show next that the dual of a lattice Gabor Schauder basis is also a lattice Gabor Schauder basis. By Theorem 4.1, it suffices to consider $\alpha=\beta=1$.

Theorem 4.2: *If $g\in L^2(\mathbb{R})$ and $\mathcal{G}(g,1,1)$ is an exact system in $L^2(\mathbb{R})$, then the biorthogonal system has the form $\mathcal{G}(\tilde{g},1,1)$, where the dual window $\tilde{g}\in L^2(\mathbb{R})$ is defined by the condition*

$$Z\tilde{g} = 1/\overline{Zg}.$$

In particular, this determines the dual basis for a Gabor Schauder basis.

Proof: By Theorem 2.8 we have that the function $G=1/\overline{Zg}$ belongs to $L^2(\mathbb{R})$. Therefore, by the unitarity of the Zak transform we may define $\tilde{g}=Z^{-1}G\in L^2(\mathbb{R})$.

If (k,n), $(j,m)\in\mathbb{Z}^2$ then by Theorem 2.7, we have that

$$\langle M_nT_kg,M_mT_j\tilde{g}\rangle = \langle ZM_nT_kg,ZM_mT_j\tilde{g}\rangle = \int_0^1\int_0^1 E_{n,k}(t,\xi)Zg(t,\xi)\overline{E_{m,j}(t,\xi)Z\tilde{g}(t,\xi)}dt\,d\xi$$

$$= \int_0^1\int_0^1 E_{n,k}(t,\xi)\overline{E_{m,j}(t,\xi)}dt\,d\xi = \langle E_{n,k},E_{m,n}\rangle = \delta_{jk}\delta_{mn}.$$

Thus, $\mathcal{G}(\tilde{g},1,1)$ is biorthogonal to $\mathcal{G}(g,1,1)$. Since an exact system has a unique biorthogonal system, it follows that $\mathcal{G}(\tilde{g},1,1)$ is the biorthogonal system. □

The following elementary lemma will be used later.

Lemma 4.3: Let $g\in L^2(\mathbb{R})$. Let $\{(k_j,n_j)\}_{j=1}^\infty$ be an enumeration of $\mathbb{Z}\times\mathbb{Z}$. If $\mathcal{G}(g,1,1)$ is a Gabor Schauder basis with respect to this enumeration, then it is also a Gabor Schauder basis with respect to the enumeration $\{(-k_j,-n_j)\}_{j=1}^\infty$.

Equivalently, if $\{M_{n_j}T_{k_j}g\}_{j=1}^\infty$ is a Schauder basis for $L^2(\mathbb{R})$ with dual basis $\{M_{n_j}T_{k_j}\tilde{g}\}_{j=1}^\infty$, then $\{M_{-n_j}T_{-k_j}g\}_{j=1}^\infty$ is a Schauder basis for $L^2(\mathbb{R})$ with dual basis $\{M_{-n_j}T_{-k_j}\tilde{g}\}_{j=1}^\infty$.

B. A weak classical Balian-Low Theorem for Gabor Schauder bases

Daubechies and Janssen proved in Ref. 18 that a weak version of the BLT holds for Gabor systems $\mathcal{G}(g,1,1)$ that are exact in $L^2(\mathbb{R})$. In particular, all Schauder bases are exact. As the proof is simpler for Gabor Schauder bases than general exact systems, we include it here. It will be convenient to state this result in terms of the unbounded operators P and X on $L^2(\mathbb{R})$, defined by

$$Pf(t) = tf(t) \quad \text{and} \quad Xf(t) = (P\hat{f})^\vee(t),$$

where $h^\vee$ denotes the inverse Fourier transform.

Theorem 4.4 (Weak BLT): *Let $g\in L^2(\mathbb{R})$. If $\mathcal{G}(g,1,1)$ is a Schauder basis for $L^2(\mathbb{R})$ with dual basis $\mathcal{G}(\tilde{g},1,1)$, then at least one of the functions,*

$$Pg, \quad Xg, \quad P\tilde{g}, \quad X\tilde{g},$$

does not belong to $L^2(\mathbb{R})$. Equivalently, if g, $\tilde{g}\in M_1^2(\mathbb{R})$, then $\mathcal{G}(g,1,1)$ is not a Schauder basis for $L^2(\mathbb{R})$.

Proof: Although the proof is formally similar to that in Ref. 17, extra care must be taken since Schauder basis expansions converge only conditionally in general. To this end, suppose that $\mathcal{G}(g,1,1)$ is a Schauder basis for $L^2(\mathbb{R})$ with respect to a specific enumeration $\{(k_j,n_j)\}_{j=1}^\infty$ of $\mathbb{Z}\times\mathbb{Z}$.

We proceed by contradiction. Assume that Pg, Xg, $P\tilde{g}$, $X\tilde{g}$ all belong to $L^2(\mathbb{R})$. By direct calculation, for each $k, n \in \mathbb{Z}$ we have

$$\langle Xg, M_n T_k \tilde{g}\rangle = \langle M_{-n}T_{-k}g, X\tilde{g}\rangle \quad \text{and} \quad \langle M_n T_k g, P\tilde{g}\rangle = \langle Pg, M_{-n}T_{-k}\tilde{g}\rangle. \tag{4.1}$$

Next, using (4.1) and Lemma 4.3, we compute that

$$\begin{aligned}\langle Xg, P\tilde{g}\rangle &= \left\langle \lim_{N\to\infty} \sum_{j=1}^{N} \langle Xg, M_{n_j}T_{k_j}\tilde{g}\rangle M_{n_j}T_{k_j}g, P\tilde{g} \right\rangle = \lim_{N\to\infty} \sum_{j=1}^{N} \langle M_{-n_j}T_{-k_j}g, X\tilde{g}\rangle \langle Pg, M_{-n_j}T_{-k_j}\tilde{g}\rangle \\ &= \left\langle Pg, \lim_{N\to\infty} \sum_{j=1}^{N} \langle X\tilde{g}, M_{-n_j}T_{-k_j}g\rangle M_{-n_j}T_{-k_j}\tilde{g} \right\rangle = \langle Pg, X\tilde{g}\rangle. \end{aligned} \tag{4.2}$$

Because the commutator $[X,P]=XP-PX=-1/(2\pi i)I$ is a multiple of the identity operator I (on the domain of $[X,P]$), we have by Ref. 19, Lemma 7.2 that

$$\frac{1}{2\pi i}\langle g, \tilde{g}\rangle = \langle Pg, X\tilde{g}\rangle - \langle Xg, P\tilde{g}\rangle. \tag{4.3}$$

Therefore, $\langle g, \tilde{g}\rangle = 0$ by (4.2). But this is a contradiction, because the definition of biorthogonality implies that $\langle g, \tilde{g}\rangle = 1$. □

V. $\mathcal{A}_2$ WEIGHTS AND CHARACTERIZATIONS OF GABOR SCHAUDER BASES

Theorem 2.8 shows that the Zak transform can be used to characterize many properties of a Gabor system. In this section we address this issue of whether Gabor Schauder bases also admit a simple characterization in the Zak transform domain.

First, let us give a basic equivalent reformulation of the definition of Gabor Schauder bases. This reformulation, which is an immediate consequence of Theorems 2.2 and 2.7, reveals the issues that must be addressed in order to find the desired characterization.

Lemma 5.1: Let $g \in L^2(\mathbb{R})$ be given, and let $\{(k_j, n_j)\}_{j=1}^{\infty}$ be an enumeration of $\mathbb{Z}\times\mathbb{Z}$. Then the following statements are equivalent.

(a) *$\mathcal{G}(g,1,1)$ is a Schauder basis for $L^2(\mathbb{R})$ with respect to the enumeration $\{(k_j, n_j)\}_{j=1}^{\infty}$.*
(b) *$1/|Zg| \in L^2(Q)$, and the partial sum operators $S_N: L^2(Q) \to L^2(Q)$, defined by*

$$S_N F = \sum_{j=1}^{N} \langle F, E_{n_j,k_j}/\overline{Zg}\rangle (E_{n_j,k_j}\cdot Zg), \tag{5.1}$$

are uniformly bounded in operator norm, i.e., $\sup_N \|S_N\| < \infty$.

The boundedness of partial sum operators in higher dimensions is, in general, a delicate issue, e.g., Ref. 53. However, for rectangular partial sums the problem is well understood, using $\mathcal{A}_2$ weights as a tool.

A. Background on $\mathcal{A}_2$ weights

We give here some background and basic results on $\mathcal{A}_2$ weights. We let $L^1(\mathbb{T})$ denote the space of 1-periodic functions on $\mathbb{R}$ that are integrable on $[0,1]$. We use the convention that $0\cdot\infty = 0$, and the symbol $\mathbf{1}_S$ will denote the characteristic function of a set $S \subseteq \mathbb{R}$. We use the notation $A \lesssim B$ to mean that there exists an absolute constant C, such that $A \leq CB$.

Definition 5.2 ($\mathcal{A}_2$ weight): A non-negative function $w \in L^1(\mathbb{T})$ is an $\mathcal{A}_2(\mathbb{T})$ *weight*, denoted $w \in \mathcal{A}_2(\mathbb{T})$, if there exists a constant $C>0$ such that for every interval $I \subset \mathbb{R}$ we have

$$\left(\frac{1}{|I|}\int_I w(t)dt\right)\left(\frac{1}{|I|}\int_I \frac{1}{w(t)}dt\right) \leq C.$$

We refer to the smallest such C as the $\mathcal{A}_2(\mathbb{T})$ characteristic of w, denoted $\|w\|_{\mathcal{A}_2(\mathbb{T})}$. Note that every $w \in \mathcal{A}_2(\mathbb{T})$ satisfies $1 \leq \|w\|_{\mathcal{A}_2(\mathbb{T})}$.

Now define

$$e_n(t) = e^{2\pi i n t}.$$

The following fundamental theorem of Hunt, Muckenhoupt, and Wheeden addresses the uniform boundedness of the Fourier partial sum operators,

$$T_N f = \sum_{n=-N}^{N} \langle f, e_n \rangle e_n, \tag{5.2}$$

in weighted L^2 spaces. In particular, the equivalence of statements (a) and (b) in the following theorem is a special case of Ref. 54, Theorem 8. We let $L^2_w(D)$ be the space of all complex-valued functions f on D for which

$$\|f\|_{2,w} = \left(\int_D |f(x)|^2 w(x)dx\right)^{1/2} < \infty.$$

Theorem 5.3 (Hunt, Muckenhoupt, Wheeden): *Let w be a non-negative function in $L^1(\mathbb{T})$. For $N \geq 0$, let $T_N: L^2_w(\mathbb{T}) \to L^2_w(\mathbb{T})$ be the operator formally defined by (5.2), and let $\|T_N\|_{2,w}$ denote its operator norm. Then the following statements are equivalent.*

(a) $\sup_N \|T_N\|_{2,w} < \infty$.
(b) $w \in \mathcal{A}_2(\mathbb{T})$.

Furthermore,

$$\|w\|_{\mathcal{A}_2(\mathbb{T})} \lesssim \sup_N \|T_N\|^2_{2,w} \lesssim \|w\|^2_{\mathcal{A}_2(\mathbb{T})}. \tag{5.3}$$

Along with the explicit equivalence of (a) and (b), the first inequality in (5.3) is implicit in the proof of Ref. 54, Theorem 8. For the second inequality in (5.3), the proof of Ref. 54, Thm. 8, along with the fact that $1 \leq \|w\|_{\mathcal{A}_2(\mathbb{T})}$, shows that

$$\sup_N \|T_N\|^2_{2,w} \lesssim (C_w)^2 + \|w\|_{\mathcal{A}_2(\mathbb{T})} \lesssim (C_w)^2 + \|w\|^2_{\mathcal{A}_2(\mathbb{T})},$$

where C_w is the norm bound for the conjugate function on $L^2_w(\mathbb{T})$ (see Ref. 54, Theorem 1). It therefore suffices to know that C_w is controlled from above by $\|w\|_{\mathcal{A}_2(\mathbb{T})}$. This is a consequence of the corresponding sharp result for the Hilbert transform proved in Ref. 55. For our purposes, the full strength of (5.3) is not needed, and any reasonable relation between $\sup_N \|T_N\|^2_{2,w}$ and $\|w\|_{\mathcal{A}_2(\mathbb{T})}$ is enough for our subsequent results. In this regard, the weaker estimates given by Refs. 56 and 57 would also suffice.

Rewriting Theorem 5.3 in terms of the partial sum operators for windowed systems of exponentials gives the following. Here $\langle \cdot, \cdot \rangle$ denotes the usual inner product on $L^2(\mathbb{T})$.

Corollary 5.4: Let $w \in L^2(\mathbb{T})$ be nonzero a.e., and define $\tilde{w} = 1/\bar{w}$. Let $\{n_j\}_{j=1}^{\infty}$ be the enumeration $\{0, 1, -1, 2, -2, 3, -3, \ldots\}$ of $\mathbb{Z}$. For $N \geq 1$, let $S_N: L^2(\mathbb{T}) \to L^2(\mathbb{T})$ be the operator, formally defined by

$$S_N f = \sum_{j=1}^{N} \langle f, e_{n_j} \cdot \tilde{w} \rangle (e_{n_j} \cdot w),$$

and let $\|S_N\|$ *denote its operator norm. Then the following statements are equivalent.*

(a) $\sup_N \|S_N\| < \infty$.
(b) $|w|^2 \in \mathcal{A}_2(\mathbb{T})$.

Proof: Let $T_N : L^2_{|w|^2}(\mathbb{T}) \to L^2_{|w|^2}(\mathbb{T})$ be defined by (5.2). By Theorem 5.3, it suffices to show that $\sup_N \|T_N\|_{2,|w|^2} < \infty$ if and only if $\sup_N \|S_N\| < \infty$.

Suppose that $C = \sup_N \|T_N\|_{2,|w|^2} < \infty$. Then for $f \in L^2(\mathbb{T})$, we have

$$\|S_{2N+1}f\|_2 = \|T_N(f/w)\|_{2,|w|^2} \le C\|f/w\|_{2,|w|^2} = C\|f\|_2.$$

Since $|w|^2 \in \mathcal{A}_2(\mathbb{T})$, we have $A = \|\tilde{w}\|_2 \|w\|_2 < \infty$. Therefore

$$\|S_{2N}f\|_2 \le \|S_{2N-1}f\|_2 + |\langle f, e_N \tilde{w}\rangle| \|e_N w\|_2 \le C\|f\|_2 + \|f\|_2 \|\tilde{w}\|_2 \|w\|_2 = (C+A)\|f\|_2.$$

Hence $\sup_N \|S_N\| \le C + A < \infty$.

For the converse, suppose that $\sup_N \|S_N\| < \infty$. If $f \in L^2_{|w|^2}(\mathbb{T})$ then

$$\|T_N(f)\|_{2,|w|^2} = \|S_{2N+1}(fw)\|_2 \le C\|fw\|_2 = C\|f\|_{2,|w|^2}.$$

Hence $\sup_N \|T_N\|_{2,|w|^2} < \infty$. □

B. Product domains and enumerations of $\mathbb{Z}\times\mathbb{Z}$

The results of the preceding section readily extend to product domains. In fact, many related results are true more generally in the setting of singular integral operators and maximal functions on product domains, e.g., Refs. 58 and 59.

We let $L^1(\mathbb{T}\times\mathbb{T})$ denote the space of functions on $\mathbb{R}^2$ that are 1-periodic in each variable and integrable on $[0,1]^2$.

Definition 5.5 (product $\mathcal{A}_2$ weight): A non-negative function $w \in L^1(\mathbb{T}\times\mathbb{T})$ is an $\mathcal{A}_2(\mathbb{T}\times\mathbb{T})$ weight if there exists $C > 0$ such that for all intervals $I, J \subset \mathbb{R}$, we have

$$\left(\frac{1}{|I||J|}\int_J\int_I w(x,y)\,dx\,dy\right)\left(\frac{1}{|I||J|}\int_I\int_J \frac{1}{w(x,y)}\,dx\,dy\right) \le C.$$

The above definition of product $\mathcal{A}_2$ weights is equivalent to requiring that w satisfies a uniform $\mathcal{A}_2$ condition in each variable, e.g., see Ref. 59, p. 15. More precisely, $w \in \mathcal{A}_2(\mathbb{T}\times\mathbb{T})$ if and only if for almost every $x \in \mathbb{T}$ and for every interval $J \subset \mathbb{R}$,

$$\left(\frac{1}{|J|}\int_J w(x,y)\,dy\right)\left(\frac{1}{|J|}\int_J \frac{1}{w(x,y)}\,dy\right) \le C,$$

and likewise for the y variable. For perspective, recall that the product weights in $\mathcal{A}_2(\mathbb{T}\times\mathbb{T})$ differ from classical $\mathcal{A}_2(\mathbb{T}^2)$ weights in that they involve averages over rectangles instead of just squares.

For our purposes, formally define

$$T_{M,N}F = \sum_{m=-M}^{M}\sum_{n=-N}^{N} \langle F, E_{n,k}\rangle E_{n,k},$$

where $E_{n,k}(t,\xi) = e^{2\pi i n t} e^{-2\pi i k \xi}$ is as defined in (2.6).

Theorem 5.6: *Let* $w \in L^2(\mathbb{T}\times\mathbb{T})$ *be non-negative. Then the following statements are equivalent.*

(a) $\sup_{M,N} \|T_{M,N}\|_{2,w} < \infty$.
(b) $w \in \mathcal{A}_2(\mathbb{T}\times\mathbb{T})$.

Proof (b)$\Rightarrow$(a): This direction follows Ref. 58, p. 128. Assume that $w \in \mathcal{A}_2(\mathbb{T}\times\mathbb{T})$. Let

$T_N f = \sum_{n=-N}^{N} \langle f, e_n \rangle e_n$ be as defined in (5.2). Then since $w_t = w(t,\cdot)$ and $w^\xi = w(\cdot,\xi)$ satisfy uniform $\mathcal{A}_2$ conditions, we have by Theorem 5.3 that there exists a constant C such that for a.e. t and ξ we have

$$\int_0^1 |T_N f(\xi)|^2 w_t(\xi) d\xi \leq C \int_0^1 |f(\xi)|^2 w_t(\xi) d\xi, \quad f \in L^2_{w_t}(\mathbb{T}),$$

$$\int_0^1 |T_N f(t)|^2 w^\xi(t) dt \leq C \int_0^1 |f(t)|^2 w^\xi(t) dt, \quad f \in L^2_{w^\xi}(\mathbb{T}).$$

Choose any $F \in L^2_w(\mathbb{T}\times\mathbb{T})$. By Fubini's Theorem, $F_t = F(t,\cdot) \in L^2_{w_t}(\mathbb{T})$ and $F^\xi = F(\cdot,\xi) \in L^2_{w^\xi}(\mathbb{T})$ for a.e. t and ξ. Define

$$T_N^1 F(t,\xi) = T_N F^\xi(t) = \sum_{-N}^{N} \langle F^\xi, e_n \rangle e_n(t), \quad T_M^2 F(t,\xi) = T_M F_t(\xi) = \sum_{-N}^{N} \langle F_t, e_n \rangle e_n(\xi).$$

Then $T_{M,N} F = T_N^1 T_M^2 F$, so

$$\|T_{M,N}F\|_2^2 = \int_0^1 \int_0^1 |T_N^1 T_M^2 F(t,\xi)|^2 w(t,\xi) dt\, d\xi \leq C \int_0^1 \int_0^1 |T_M^2 F(t,\xi)|^2 w(t,\xi) dt\, d\xi$$

$$\leq C^2 \int_0^1 \int_0^1 |F(t,\xi)|^2 w(t,\xi) d\xi\, dt = C^2 \|F\|_{2,w}^2.$$

(a)$\Rightarrow$(b): This direction is similar to the proof of Theorem 8 in Ref. 54. Suppose that $C = \sup_{M,N} \|T_{M,N}\|_{2,w} < \infty$. Write

$$T_{M,N} F(t,\xi) = \int_0^1 \int_0^1 F(u,\eta) D_N(t-u) D_M(\xi-\eta) du\, d\eta,$$

where

$$D_N(t) = \frac{\sin 2\pi\left(N+\frac{1}{2}\right)t}{\sin \pi t}.$$

Choose any rectangle $I \times J$. Without loss of generality, we may assume that $|I|$, $|J|$ are small, e.g., $|I|, |J| \leq \frac{1}{16}$. Choose any integer N such that $1/(32N) \leq |I| \leq 1/(16N)$. If $t, u \in I$ then $t-u \in I - I \subset [-1/(8N), 1/(8N)]$, so

$$D_N(t-u) \geq D_N(1/(8N)) \gtrsim N.$$

Similarly, choose M so that $1/(32M) \leq |J| \leq 1/(16M)$.

Let $H \geq 0$ be any non-negative 1-periodic function that is supported in the 1-periodic extension of $I \times J$. Then, for $(t,\xi) \in I \times J$ we have

$$|T_{M,N}(t,\xi)| = \int\int_{I\times J} H(u,\eta) D_N(t-u) D_M(\xi-\eta) du\, d\eta \gtrsim MN \int\int_{I\times J} H(u,\eta) du\, d\eta$$

$$\gtrsim \frac{1}{|I||J|} \int\int_{I\times J} H(u,\eta) du\, d\eta.$$

Consequently,

$$\frac{1}{|I|^2|J|^2}\left(\int\int_{I\times J} H\right)^2 \int\int_{I\times J} w(t,\xi)dt\,d\xi \lesssim \int\int_{I\times J} |T_{M,N}(t,\xi)|w(t,\xi)dt\,d\xi = \|T_{M,N}H\|_{2,w}^2 \lesssim \|H\|_{2,w}^2. \tag{5.4}$$

In particular, if H is the 1-periodic extension of $(1/w)\mathbf{1}_{I\times J}$, then $\|H\|_{2,w}^2 = \int\int_{I\times J}(1/w)$, so

$$\left(\frac{1}{|I||J|}\int\int_{I\times J}\frac{1}{w}\right)\left(\frac{1}{|I||J|}\int\int_{I\times J} w\right)\left(\int\int_{I\times J}\frac{1}{w}\right) \lesssim \left(\int\int_{I\times J}\frac{1}{w}\right).$$

Consequently, if $0<\int\int_{I\times J}(1/w)<\infty$, then we conclude that

$$\left(\frac{1}{|I||J|}\int\int_{I\times J}\frac{1}{w}\right)\left(\frac{1}{|I||J|}\int\int_{I\times J} w\right) \lesssim 1. \tag{5.5}$$

This estimate also follows trivially if $\int\int_{I\times J}(1/w)=0$. So, it only remains to consider the case $\int\int_{I\times J}(1/w)=\infty$. In this case there must be some $F\in L^2(I\times J)$ such that $F/w^{1/2}\notin L^1(I\times J)$. Set $H=|F|/w^{1/2}$. Then $\int\int_{I\times J}H=\infty$, while $\|H\|_{2,w}=\|F\|_2$ is finite. Equation (5.4) therefore implies that $\int\int_{I\times J}w=0$, and hence (5.5) holds trivially in this case as well. Therefore $w\in\mathcal{A}_2(\mathbb{T}\times\mathbb{T})$. □

Corollary 5.7: Let $W\in L^2(\mathbb{T}\times\mathbb{T})$ be nonzero a.e., and define $\widetilde{W}=1/\overline{W}$. For M, $N\geqslant 0$ let $S_{M,N}:L^2(\mathbb{T}\times\mathbb{T})\to L^2(\mathbb{T}\times\mathbb{T})$ be the operator formally defined by

$$S_{M,N}F = \sum_{n=-N}^{N}\sum_{m=-M}^{M}\langle F, E_{n,k}\cdot\widetilde{W}\rangle(E_{n,k}\cdot W), \tag{5.6}$$

and let $\|S_{M,N}\|$ denote its operator norm. Then the following statements are equivalent.

(a) $\sup_{M,N}\|S_{M,N}\|<\infty$.
(b) $|W|^2\in\mathcal{A}_2(\mathbb{T}\times\mathbb{T})$.

Proof: Using Theorem 5.6, the proof is similar to that of Corollary 5.4. □

It will be convenient to work with the following set of enumerations of $\mathbb{Z}\times\mathbb{Z}$, which are well suited for dealing with rectangular partial sums.

Definition 5.8: Let Λ be the set containing all enumerations $\{(k_j,n_j)\}_{j=1}^{\infty}$ of $\mathbb{Z}\times\mathbb{Z}$ constructed in the following recursive manner.

(a) The initial terms $(k_1,n_1),\ldots,(k_{J_1},n_{J_1})$ are either

$$(0,0),(1,0),(-1,0),\ldots,(A_1,0),(-A_1,0)$$

or

$$(0,0),(0,1),(0,-1),\ldots,(0,B_1),(0,-B_1),$$

for some positive integers A_1 or B_1.

(b) If $\{(k_j,n_j)\}_{j=1}^{J_k}$ has been constructed to be of the product form $\{-A_k,\ldots,A_k\}\times\{-B_k,\ldots,B_k\}$ for some non-negative integers A_k,B_k, then terms are added to either the top and bottom or the left and right sides to obtain either the rectangle

$$\{-A_k,\ldots,A_k\}\times\{-(B_k+1),\ldots,B_k+1\}$$

or

$$\{-(A_k+1),\ldots,A_k+1\}\times\{-B_k,\ldots,B_k\}.$$

For example, terms would first be added to the top ordered as

$$(0,B_k+1),(1,B_k+1),(-1,B_k+1), \ldots ,(A_k,B_k+1),(-A_k,B_k+1),$$

and then likewise for the bottom. The left and right sides proceed analogously.

Corollary 5.9: Suppose $W \in L^2(\mathbb{T}\times\mathbb{T})$ *is such that* $\widetilde{W}=1/\overline{W} \in L^2(\mathbb{T}\times\mathbb{T})$. *For each enumeration* $\sigma=\{(k_j,n_j)\}_{j=1}^{\infty} \in \Lambda$, *let* $S_N^{\sigma}: L^2(\mathbb{T}\times\mathbb{T}) \to L^2(\mathbb{T}\times\mathbb{T})$ *be the operator formally defined by*

$$S_N^{\sigma}F = \sum_{j=1}^{N} \langle F, E_{n_j,k_j}\cdot \widetilde{W}\rangle (E_{n_j,k_j}\cdot W),$$

and let $\|S_N^{\sigma}\|$ *denote its operator norm. Then the following statements are equivalent.*

(a) $\sup_{N,\sigma}\|S_N^{\sigma}\| < \infty$.
(b) $|W|^2 \in \mathcal{A}_2(\mathbb{T}\times\mathbb{T})$.

Proof (a)$\Rightarrow$(b): Assume that $\sup_{N,\sigma}\|S_N^{\sigma}\|<\infty$. By the definition of Λ, it follows that $\sup_{M,N}\|S_{M,N}\|<\infty$, with $S_{M,N}$ as in (5.6). By Corollary 5.7 we conclude that $|W|^2 \in \mathcal{A}_2(\mathbb{T}\times\mathbb{T})$.

(b)$\Rightarrow$(a): Assume that $|W|^2 \in \mathcal{A}_2(\mathbb{T}\times\mathbb{T})$. Then, by definition, we have that $A=\|W\|_2\|\widetilde{W}\|_2 <\infty$, and by Corollary 5.7 we have that $C=\sup_{M,N}\|S_{M,N}\|<\infty$.

Choose any enumeration $\sigma\in\Lambda$ and any positive integer N. Let M_N be the largest integer $M_N<N$ for which $S_{M_N}^{\sigma}F=S_{J,K}F$ for some integers J, K. Note that

$$\|S_N^{\sigma}F\|_2 \le \|S_{J,K}F\|_2 + \|(S_N^{\sigma}-S_{M_N}^{\sigma})F\|_2 \le C\|F\|_2 + \|(S_N^{\sigma}-S_{M_N}^{\sigma})F\|_2.$$

Thus we must estimate the norm of

$$(S_N^{\sigma}-S_{M_N}^{\sigma})F = \sum_{j=M_N+1}^{N} \langle F, E_{n_j,k_j}\cdot\widetilde{W}\rangle(E_{n_j,k_j}\cdot W). \tag{5.7}$$

This series corresponds to terms that have been added to a rectangle according to the algorithm given in Definition 5.8. For example, if terms have been added to the top of the rectangle, then (5.7) has one of the following two forms:

$$\sum_{n=-L}^{L}\langle F,E_{n,K+1}\cdot\widetilde{W}\rangle(E_{n,K+1}\cdot W) \quad \text{or} \quad \sum_{n=-L}^{L+1}\langle F,E_{n,K+1}\cdot\widetilde{W}\rangle(E_{n,K+1}\cdot W). \tag{5.8}$$

The first sum is bounded by

$$\begin{aligned}\left\|\sum_{n=-L}^{L}\langle F,E_{n,K+1}\cdot\widetilde{W}\rangle(E_{n,K+1}\cdot W)\right\|_2 &= \left\|\sum_{n=-L}^{L}\langle F\cdot E_{0,-K-1},E_{n,0}\cdot\widetilde{W}\rangle(E_{n,0}\cdot W)\right\|_2 = \|S_{L,0}(F\cdot E_{0,-K-1})\|_2 \\ &\le C\|F\cdot E_{0,-K-1}\|_2 = C\|F\|_2. \end{aligned}\tag{5.9}$$

The second sum in (5.8) is bounded by

$$\begin{aligned}\left\|\sum_{n=-L}^{L+1}\langle F,E_{n,K+1}\cdot\widetilde{W}\rangle(E_{n,K+1}\cdot W)\right\|_2 &\le \left\|\sum_{n=-L}^{L}\langle F,E_{n,K+1}\cdot\widetilde{W}\rangle(E_{n,K+1}\cdot W)\right\|_2 + |\langle F,E_{L+1,K+1}\cdot\widetilde{W}\rangle| \\ &\quad \times\|E_{L+1,K+1}\cdot W\|_2 \le C\|F\|_2 + A\|F\|_2. \end{aligned}\tag{5.10}$$

Substituting each of (5.9) and (5.10) into (5.7), we conclude that for this case we have $\|S_N^{\sigma}F\|_2 \le (2C+A)\|F\|_2$. The same estimate applies if terms have been added only to the bottom, left, or right of the rectangle. If terms have been added either to both the top and bottom, or to both the left and right sides of the rectangle, then we end with the estimate $\|S_N^{\sigma}F\|_2 \le (3C+2A)\|F\|_2$. In any case, we conclude that $\sup_{N,\sigma}\|S_N^{\sigma}\| \le (3C+2A) <\infty$. □

C. A Zak transform characterization of Gabor Schauder bases

Using the machinery that we have developed in the previous sections, we can now give a simple Zak transform characterization of lattice Gabor Schauder bases for $L^2(\mathbb{R})$. We then construct examples of Gabor Schauder bases that have interesting properties.

Note that while Z_g has a natural quasiperiodic extension from $[0,1)^2$ to $\mathbb{R}^2$, the definition of $\mathcal{A}_2$ weight only depends on the absolute value of the weight, and $|Zg|$ is 1-periodic in each variable.

Theorem 5.10: *Let $g \in L^2(\mathbb{R})$ be given. Then the following statements are equivalent.*

(a) *The Gabor system $\mathcal{G}(g,1,1)$ is a Schauder basis for $L^2(\mathbb{R})$ with respect to every enumeration $\sigma=\{(k_j,n_j)\}_{j=1}^{\infty} \in \Lambda$. Further, if C_σ is the basis constant associated to the enumeration σ, then $\sup_{\sigma\in\Lambda} C_\sigma < \infty$.*

(b) $|Zg|^2 \in \mathcal{A}_2(\mathbb{T}\times\mathbb{T})$.

Proof (a)$\Rightarrow$(b): Suppose that statement (a) holds. Then by Lemma 5.1 we have that $1/Zg \in L^2(\mathbb{T}\times\mathbb{T})$. Further, by hypothesis, $\sup_{N,\sigma}\|S_N^\sigma\| \leq \sup_{\sigma\in\Lambda} C_\sigma < \infty$, so Corollary 5.9 implies that $|Zg|^2 \in \mathcal{A}_2(\mathbb{T}\times\mathbb{T})$.

(b)$\Rightarrow$(a): Assume $|Zg|^2 \in \mathcal{A}_2(\mathbb{T}\times\mathbb{T})$. Then, by definition of $\mathcal{A}_2(\mathbb{T}\times\mathbb{T})$, we have that $1/Zg \in L^2(\mathbb{T}\times\mathbb{T})$. Further, we have by Corollary 5.9 that $\sup_{N,\sigma}\|S_N^\sigma\| < \infty$. Lemma 5.1 therefore implies that $\mathcal{G}(g,1,1)$ is a Schauder basis with respect to each $\sigma \in \Lambda$. □

While Theorem 5.10 implies that if $|Zg|^2 \in \mathcal{A}_2(\mathbb{T}\times\mathbb{T})$ then $\mathcal{G}(g,1,1)$ is a Schauder basis with respect to every enumeration $\sigma\in\Lambda$, this does not imply that it is a Schauder basis with respect to every possible enumeration of $\mathbb{Z}\times\mathbb{Z}$, since that would imply that $\mathcal{G}(g,1,1)$ is a Riesz basis. In fact, by Theorem 2.8, the Riesz bases correspond to the subclass of weights in $\mathcal{A}_2(\mathbb{T}\times\mathbb{T})$ that are essentially bounded, i.e., bounded away from zero and infinity. The following example is inspired by Babenko's example from Ref. 60; cf. Ref. 36, Example 11.2, pp. 351–354, and gives examples of Gabor Schauder bases that are not Riesz bases.

Example 5.11. Fix $0<\alpha<1/2$. Assume that $g=g_\alpha \in L^2(\mathbb{R})$ satisfies the following:

(a) g is real-valued;
(b) $\mathrm{supp}(g) \subseteq [0,1]$;
(c) g is infinitely differentiable on every subinterval $(\delta, 1-\delta)$, $0<\delta<1/2$;
(d) $g(t)=t^\alpha$ on $[0,1/4]$;
(e) $g(t)=(1-t)^\alpha$ on $[3/4,1]$;
(f) $g(t-1/2)$ is even;
(g) $g(t) \geq C>0$ for $1/4<t<3/4$.

Since g is supported in [0,1], a direct calculation shows that $|Zg(t,\xi)|^2=|g(t)|^2 \in \mathcal{A}_2(\mathbb{T}\times\mathbb{T})$. Therefore, Theorem 5.10 implies that $\mathcal{G}(g,1,1)$ is a Schauder basis for $L^2(\mathbb{R})$ with respect to every enumeration $\sigma\in\Lambda$. However, since $|Zg|$ is not bounded away from zero, Theorem 2.8 implies that $\mathcal{G}(g,1,1)$ is not a Riesz basis for $L^2(\mathbb{R})$.

We make some additional remarks concerning the windows constructed in Example 5.11. Note that since Zg is bounded, $\mathcal{G}(g,1,1)$ is a Bessel sequence, i.e., it has an upper frame bound. Since g is supported in [0,1], the dual window is easily seen to be the function that is $\tilde{g}=1/\bar{g}$ on [0,1] and zero elsewhere. The dual basis $\mathcal{G}(\tilde{g},1,1)$ possesses a lower frame bound but not an upper frame bound (compare Theorem 7.1).

Theorem 5.10 allows one to generate other interesting examples. Building on Example 5.11, the following example provides a Gabor Schauder basis that has neither an upper frame bound nor a lower frame bound. In other words, e.g., Ref. 36, this Gabor Schauder basis is neither Besselian nor Hilbertian.

Example 5.12: Fix $0<\alpha<1/2$ and let $g=g_\alpha$ be the function from Example 5.11. Define

$$f(t) = \begin{cases} g(2t), & \text{if } 0 \leqslant t \leqslant 1/2, \\ \dfrac{1}{g(2t-1)}, & \text{if } 1/2 < t < 1, \\ 0, & \text{if } t \notin [0,1). \end{cases}$$

As in Example 5.11, we have that $Zf(t,\xi)=f(t)$ for $(t,\xi) \in Q$. Further, Zf, $1/Zf \in L^2(\mathbb{R})$, so we at least have that $\mathcal{G}(f,1,1)$ is complete and minimal. However, one can check that $|Zf|^2 \in \mathcal{A}_2(\mathbb{T}\times\mathbb{T})$, so, in fact, we have that $\mathcal{G}(f,1,1)$ is a Schauder basis with respect to every enumeration $\sigma \in \Lambda$. However, since $|Zg|$ is not bounded away from zero or infinity, we conclude that $\mathcal{G}(f,1,1)$ has neither an upper frame nor a lower frame bound.

For perspective, note that $f \notin M^1(\mathbb{R})$ since any Gabor system generated by a window in $M^1(\mathbb{R})$ is a Bessel sequence, Ref. 1, Theorem 12.2.3. By contrast, we shall later prove (see Theorem 6.1) that the example g from Example 5.11 does belong to $M^1(\mathbb{R})$.

Example 5.13: The two preceding examples were both compactly supported. However, we can easily create noncompactly supported examples. For example, since the Fourier transform is unitary, if g is one of the functions constructed in Example 5.11 or 5.12, then $\mathcal{G}(\hat{g},1,1)$ is a Gabor Schauder basis with respect to every enumeration in Λ, but it is not a Riesz basis, and $\hat{g}$ is not compactly supported.

We can also work directly in the Zak transform domain. Again, let g be any function constructed in Example 5.11 or 5.12, and let h be any function supported in [0,1] such that $|h|^2 \in \mathcal{A}_2(\mathbb{T})$. Then $G(t,\xi)=g(t)h(\xi)$ is such that $|G|^2 \in \mathcal{A}_2(\mathbb{T}\times\mathbb{T})$. Hence $f=Z^{-1}G \in L^2(\mathbb{R})$, and, for every enumeration in Λ, $\mathcal{G}(f,1,1)$ is a Gabor Schauder basis that is not a Riesz basis. By definition of the Zak transform, if h is not a trigonometric polynomial, then f cannot be compactly supported.

In general, while every Schauder basis is complete and minimal, the converse need not be true. If $\varphi(t)=e^{-t^2}$, then we know that a proper subset of $\mathcal{G}(\varphi,1,1)$ is complete and minimal but is not a Schauder basis. The following example shows that it is possible for a lattice Gabor system $\mathcal{G}(g,1,1)$ (not just a subset) to be complete and minimal yet not form a Schauder basis for $L^2(\mathbb{R})$.

Example 5.14: For $n=1,2,3,\ldots$, set $A_n=(3/2)^{n/2}$ and $B_n=(2/3)^{n/2}$. Define disjoint intervals,

$$L_n = [1-2^{-n+1}, 1-2^{-n}-2^{-n-1}], \quad R_n = [1-2^{-n}-2^{-n-1}, 1-2^{-n}],$$

and set $I_n=L_n \cup R_n$. Then $|I_n|=2^{-n}=2|R_n|=2|L_n|$ and $\cup_{n=1}^{\infty} I_n=[0,1)$. For $t \in \mathbb{R}$, let g be the following non-negative function supported in [0,1]:

$$g = \sum_{n=1}^{\infty} (A_n \mathbf{1}_{L_n} + B_n \mathbf{1}_{R_n}).$$

We compute that

$$\int_0^1 |g(t)|^2\, dt = \frac{1}{2}\sum_{n=1}^{\infty} |I_n|(A_n^2 + B_n^2) = \frac{1}{2}\sum_{n=1}^{\infty} 2^{-n}\left(\left(\frac{3}{2}\right)^n + \left(\frac{2}{3}\right)^n\right) < \infty,$$

and

$$\int_0^1 \frac{1}{|g(t)|^2} dt = \frac{1}{2}\sum_{n=1}^{\infty} |I_n|\left(\frac{1}{A_n^2} + \frac{1}{B_n^2}\right) = \int_0^1 |g(t)|^2\, dt < \infty.$$

Further, since g is supported in [0,1], we have that $|Zg(t,\xi)|=g(t)$ for $(t,\xi) \in Q$. Therefore Zg, $1/Zg \in L^2(Q)$, so by Theorem 2.8, $\mathcal{G}(g,1,1)$ is complete and minimal in $L^2(\mathbb{R})$.

Next, note that

$$\left(\frac{1}{|I_n|}\int_{I_n}|g(t)|^2\,dt\right)\left(\frac{1}{|I_n|}\int_{I_n}\frac{1}{|g(t)|^2}dt\right)=\frac{1}{4}(A_n^2+B_n^2)\left(\frac{1}{A_n^2}+\frac{1}{B_n^2}\right)=\frac{1}{4}\left(2+\left(\frac{9}{4}\right)^n+\left(\frac{4}{9}\right)^n\right).$$

Letting $n\to\infty$, we see that $|Zg|^2\notin\mathcal{A}_2(\mathbb{T}\times\mathbb{T})$. Therefore, by Theorem 5.10 there exists at least one enumeration $\sigma\in\Lambda$ with respect to which the Gabor system $\mathcal{G}(g,1,1)$ is not a Schauder basis for $L^2(\mathbb{R})$.

VI. COUNTEREXAMPLES TO THE BALIAN-LOW THEOREMS FOR GABOR SCHAUDER BASES

In this section we investigate the extent to which the variants of the Balian-Low Theorem hold in the setting of Schauder bases. It was conjectured in Ref. 31 that if hypothesis (a) of Theorem 3.2 holds, then the conclusion of that theorem remains true if the words "Riesz basis" are replaced by "Schauder basis." In other words, it was conjectured that there do not exist any Gabor Schauder bases whose window function belongs to $M^1(\mathbb{R})$. We will provide counterexamples to that conjecture in this section. Moreover, our counterexamples apply to each of the four hypotheses appearing in Theorem 3.2.

Theorem 6.1: *The conclusion of Theorem 3.2 is false if the words* "Riesz basis" *are replaced by* "Schauder basis."

Specifically, if $g=g_\alpha$ with $0<\alpha<1/2$ is one of the functions constructed in Example 5.11, then $\mathcal{G}(g,1,1)$ is a Schauder basis for $L^2(\mathbb{R})$ with respect to any enumeration $\sigma\in\Lambda$, but it is not a Riesz basis for $L^2(\mathbb{R})$. Further, the following statements hold.

(a) $g\in M^1(\mathbb{R})$.
(b) $g,\ \hat{g}\in W(\mathcal{C},\ell^1)$.
(c) *If* $1<q<2<p<\infty$, $1/p+1/q=1$, $0<\varepsilon<2-q$, *and* $(q+\varepsilon-1)/2<\alpha$, *then*

$$\int|t|^{p+\varepsilon}|g(t)|^2\,dt<\infty\quad\textit{and}\quad\int|\xi|^{q+\varepsilon}|\hat{g}(\xi)|^2\,d\xi<\infty.$$

(d) $\sup_{N>0}\int|t|^N|g(t)|^2dt<\infty$ *and* $\int|\xi||\hat{g}(\xi)|^2\,d\xi<\infty$.

Proof: (c) Assume that p, q, ε, and α satisfy the given conditions, and define

$$I_{p+\varepsilon}(g)=\int|t|^{p+\varepsilon}|g(t)|^2\,dt\quad\text{and}\quad I_{q+\varepsilon}(\hat{g})=\int|\xi|^{q+\varepsilon}|\hat{g}(\xi)|^2\,d\xi. \tag{6.1}$$

Since g is continuous and compactly supported, we certainly have that $I_{p+\varepsilon}(g)<\infty$. Therefore, we just have to show that $I_{q+\varepsilon}(\hat{g})<\infty$.

Let φ be a $C^\infty(\mathbb{R})$ function that equals 1 on $[-2\nu,2\nu]$ and is supported in $[-3\nu,3\nu]$ for some sufficiently small fixed $\nu>0$. Let ψ be a $C^\infty(\mathbb{R})$ function that equals 1 on $[3\nu,1-3\nu]$ and is compactly supported in $[2\nu,1-2\nu]$. Finally, suppose that the partition of unity property $\varphi(t)+\psi(t)+\varphi(t-1)=1$ holds for all $t\in[0,1]$. Since $\psi(t)g(t)$ is $C^\infty(\mathbb{R})$ and is compactly supported, it suffices to show that $I_{q+\varepsilon}(\hat{h}_1)<\infty$ and that $I_{q+\varepsilon}(\hat{h}_2)<\infty$, where $h_1(t)=\varphi(t)g(t)$ and $h_2(t)=\varphi(t-1)g(t)$. Both estimates are similar, so we only prove the estimate for $h=h_1$.

To estimate $I_q(h)$, we use Ref. 61, Proposition 4, p. 139, which implies that for any $0<s<1$ there exists $0<C_s$ such that

$$\int|\xi|^{2s}|\hat{h}(\xi)|^2\,d\xi=C_s\int\int\frac{|h(x+t)-h(x)|^2}{|t|^{1+2s}}dx\,dt.$$

Begin by noting that

$$\int_{\mathbb{R}\backslash[-\nu,\nu]}\int \frac{|h(x+t)-h(x)|^2}{|t|^{1+q+\varepsilon}}dx\,dt \lesssim \int_{\mathbb{R}\backslash[-\nu,\nu]}\frac{\|h\|_2^2}{|t|^{1+q+\varepsilon}}dt<\infty.$$

It remains to estimate the analogous two integrals over the domains $[0,\nu]\times\mathbb{R}$ and $[-\nu,0]\times\mathbb{R}$. Since both proceed similarly, we shall only show the estimates for the first of these, which we, in turn, break up into the following integrals:

$$J_1=\int_0^\nu\int_{-\infty}^{-t}, \quad J_2=\int_0^\nu\int_{-t}^{t}, \quad J_3=\int_0^\nu\int_{t}^{\nu}, \quad J_4=\int_0^\nu\int_{\nu}^{3\nu}, \quad J_5=\int_0^\nu\int_{3\nu}^{\infty}.$$

Since h is supported on $[0,3\nu]$, it follows that $J_1=J_5=0$.

Note that since $q+\varepsilon-2\alpha<1$, we have

$$J_2\lesssim \int_0^\nu \frac{1}{t^{q+\varepsilon+1}}\int_0^{2t}|h(x)|^2\,dx\,dt=\int_0^\nu\int_0^{2t}\frac{x^{2\alpha}}{t^{q+\varepsilon+1}}dx\,dt\lesssim\int_0^\nu\frac{t^{2\alpha+1}}{t^{q+\varepsilon+1}}dt<\infty.$$

Next, since $|(x+t)^\alpha-x^\alpha|\le\alpha t x^{\alpha-1}$ for all $0<x$, t, we have that

$$J_3\lesssim\int_0^\nu\int_t^\nu\frac{t^2x^{2\alpha-2}}{t^{q+\varepsilon+1}}dx\,dt\lesssim\int_0^\nu\frac{t^{2\alpha+1}}{t^{q+\varepsilon+1}}dt<\infty.$$

Finally, the smoothness properties of h on $[\nu,\infty)$ and the Mean Value Theorem imply that if $x+t$ and x are both greater than ν then $|h(x+t)-h(x)|\lesssim|t|$. Therefore, since $q+\varepsilon<2$,

$$J_4\lesssim\int_0^\nu\int_\nu^{3\nu}\frac{t^2}{|t|^{q+\varepsilon+1}}dx\,dt=\int_0^\nu\frac{1}{t^{q+\varepsilon-1}}dt<\infty,$$

which completes the proof.

(d) The estimates in the proof of part (c) together with the fact that g is compactly supported in $[0,1]$ yields the result.

(a) This follows from part (c) and the following modulation space embedding of Gröchenig:[52] If $1/p+1/q=1$, $1<p,q,<\infty$, and $0<\varepsilon$, then

$$\|g\|_{M^1}\lesssim\left(\int|t|^{p+\varepsilon}|g(t)|^2\,dt\right)^{1/2}+\left(\int|\xi|^{q+\varepsilon}|\hat{g}(\xi)|^2\,d\xi\right)^{1/2}.$$

(b) By part (a), we have $g\in M^1(\mathbb{R})\subset W(\mathcal{C},\ell^1)$. Since $M^1(\mathbb{R})$ is invariant under the Fourier transform, we also have $\hat{g}\in M^1(\mathbb{R})\subset W(\mathcal{C},\ell^1)$.

Note that the integrals in (6.1) are also finite when $\varepsilon=0$.

VII. WEAK BLT's FOR GABOR SCHAUDER BASES

We close by proving some new Weak BLTs for exact Gabor systems, including Gabor Schauder bases, in particular. Parts (a) and (b) of the following theorem can be, respectively, viewed as weak versions of the M^1 BLT and Amalgam BLT in the setting of exact Gabor systems.

Theorem 7.1 (weak BLTs for exact Gabor systems): *Let $g\in L^2(\mathbb{R})$ be such that $\mathcal{G}(g,1,1)$ is exact in $L^2(\mathbb{R})$ and let $\tilde{g}=Z^{-1}(1/\overline{Zg})$ be the dual window.*

(a) *If $g\in M^1(\mathbb{R})$ then $\tilde{g}\notin M^1(\mathbb{R})$.*
(b) *If $g\in W(\mathcal{C},\ell^1)$ then $\tilde{g}\notin W(\mathcal{C},\ell^1)$.*
(c) *If $\mathcal{G}(g,1,1)$ is a Bessel sequence, then $\mathcal{G}(\tilde{g},1,1)$ is Bessel if and only if $\mathcal{G}(g,1,1)$ is a Riesz basis.*

Proof: (b) If $g\in W(\mathcal{C},\ell^1)$, then Zg has a continuous extension to all of $\mathbb{R}^2$ by Ref. 19, Theorem 3.2. But the quasiperiodicity of Zg then forces Zg to have a zero.[1] Hence $Z\tilde{g}=1/\overline{Zg}$ is not continuous, so $\tilde{g}\notin W(\mathcal{C},\ell^1)$.

(a) This follows immediately from part (b) and the fact that $M^1(\mathbb{R}) \subset W(\mathcal{C}, \ell^1)$.

(c) This follows from Theorem 2.5(e). □

Remark 7.2: In Ref. 18, Daubechies and Janssen proved that (in equivalent modulation space terminology) if $g \in M_2^2(\mathbb{R})$, then $\mathcal{G}(g,1,1)$ is not exact. Hence, if $\mathcal{G}(g,1,1)$ is to be a Schauder basis and $g \in M_s^2(\mathbb{R})$, then we must have $0 < s < 2$. For $1 < s < 2$ we will have $g \in M_s^2(\mathbb{R}) \subset M^1(\mathbb{R})$, and hence $\tilde{g} \notin M^1(\mathbb{R})$.

In particular, the function $g = g_\alpha$ constructed in Example 5.11 can be shown to lie in $M_s^2(\mathbb{R})$ for $0 < s < 1$. While this does not imply that $g \in M^1(\mathbb{R})$, the stronger embeddings of Gröchenig[52] imply that $g \in M^1(\mathbb{R})$.

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