Operator Theory:
Advances and Applications, Vol. 58
I. Gohberg, ed., Birkhäuser, Basel (1992) pp. 1–25.


# UNCERTAINTY PRINCIPLES FOR TIME-FREQUENCY OPERATORS

John Benedetto*, Christopher Heil†, and David Walnut‡

Abstract: This paper explores some of the connections between classical Fourier analysis and time-frequency operators, as related to the role of the uncertainty principle in Gabor and wavelet basis expansions.

## 1. Introduction.

This paper will explore some of the connections between classical Fourier analysis and recent results in the area of time-frequency analysis, specifically Gabor and wavelet theory. Our starting point is the Poisson Summation Formula, which connects Fourier series on the circle with Fourier transforms on the line. This extraordinarily useful formula can be used, for example, to derive sampling formulas for band-limited functions and estimates for aliasing error when an arbitrary function is sampled. Following Section 1A, which contains a review of basic Fourier analysis, we present in Section 1B a proof of this sampling theorem and, as a corollary, the sampling theorem of Shannon and Whittaker. Next we consider the Gabor transform, which separates the time and frequency content of a function by using a windowed Fourier transform. The sampling theory of this transform can be cast in the language of decompositions of arbitrary functions in sets of basis-like functions and leads us to a discussion of frames and unconditional bases for Hilbert spaces. The wavelet transform also separates time and frequency content in a function

*Also Professor of Mathematics, University of Maryland, College Park; acknowledges partial support from NSF Grant DMS-9002420.
†Also Pure Mathematics Instructor, Massachusetts Institute of Technology; acknowledges partial support from NSF Grant DMS-9007212.
‡Also Assistant Professor of Mathematics, George Mason University, Fairfax, Virginia.

but substitutes scale for frequency and does not explicitly use a Fourier transform. The sampling theory here leads to the notion of multiresolution analysis and the construction of wavelet orthonormal bases. Sections 2A, B, and C contain the statements and proofs of theorems related to these two time-frequency operators.

Underlying our entire discussion is the classical uncertainty principle, which limits the degree to which a function can be localized both in time and frequency. The sampling and decomposition formulas outlined in the first two sections of the paper can be seen as the realization of an arbitrary function as combinations of functions well-localized in time and frequency. The uncertainty principle leads us to ask how well-localized those basis-like functions can be. For band-limited functions and the Fourier transform, the sparsest set of sampling which will permit exact reconstruction requires functions that are not well-localized in time, namely $\frac{\sin(\pi x)}{\pi x}$ and its translates. A denser sampling set leads to better localized basis functions. For the Gabor transform, the Balian–Low theorem tells us how well we can do for bases of Gabor functions, namely, that simultaneous quadratic localization is not possible in both time and frequency. In Section 3A we present an operator-theoretic proof of this theorem due to Battle. In this case, as with the previous one, allowing denser sets of sampling and overdetermined systems (frames) enables us to get better localization. For the wavelet transform, we present a generalization of a theorem due to Battle which states that any wavelet basis cannot have simultaneous exponential localization in time and frequency.

**A. Basic Fourier analysis.** Let $L^2[-T/2, T/2)$ denote the Hilbert space of $T$-periodic, square-integrable functions. Given $f \in L^2[-T/2, T/2)$, the Fourier coefficients of $f$ are

$$c_n(f) \;=\; \frac{1}{\sqrt{T}} \int_{-T/2}^{T/2} f(t)\, e^{-2\pi i n t/T}\, dt.$$

Then we have

$$f(t) \;=\; \frac{1}{\sqrt{T}} \sum_n c_n(f)\, e^{2\pi i n t/T}, \tag{1.1}$$

where the sum converges in $L^2[-T/2, T/2)$. The Plancherel formula is

$$\int_{-T/2}^{T/2} |f(t)|^2\, dt \;=\; \sum_n |c_n(f)|^2,$$

and the Parseval formula is

$$\int_{-T/2}^{T/2} f(t)\,\overline{g(t)}\,dt \;=\; \sum_n c_n(f)\,\overline{c_n(g)},$$

for every $f,\, g \in L^2[-T/2, T/2)$.

Let $L^2(\mathbf{R})$ denote the Hilbert space of functions on the line which are square-integrable. Given $f \in L^2(\mathbf{R})$, the Fourier transform of $f$ is

$$\hat{f}(\gamma) \;=\; \int f(t)\, e^{-2\pi i \gamma t}\, dt,$$

where the integral is defined as an $L^2$-limit in the usual way if it does not converge absolutely. Then we have

$$f(t) \;=\; \int \hat{f}(\gamma)\, e^{2\pi i t \gamma}\, d\gamma,$$

where, as before, the integral is defined as an $L^2$-limit if it does not converge absolutely. The Plancherel formula is

$$\int |f(t)|^2\, dt \;=\; \int |\hat{f}(\gamma)|^2\, d\gamma$$

and the Parseval formula is

$$\int f(t)\,\overline{g(t)}\,dt \;=\; \int \hat{f}(\gamma)\,\overline{\hat{g}(\gamma)}\,d\gamma$$

for every $f,\, g \in L^2(\mathbf{R})$.

**B. The Poisson Summation Formula.** We define $PW_\Omega$ to be the space of functions $f \in L^2(\mathbf{R})$ such that $\mathrm{supp}(\hat{f}) \subset [-\Omega, \Omega]$. We call this the space of $\Omega$-band-limited functions.

The Poisson Summation Formula (PSF) is:

THEOREM 1.1. *Let $f \in L^2(\mathbf{R})$ be such that*

$$|f(t)| \;\le\; C(1+|t|)^{-\alpha} \qquad \textit{and} \qquad |\hat{f}(\gamma)| \;\le\; C(1+|\gamma|)^{-\alpha} \tag{1.2}$$

*for some $C > 0$ and $\alpha > 1$. Then $f$ and $\hat{f}$ are continuous functions, and, for any $P > 0$,*

$$\sum_n f(t+nP) \;=\; \frac{1}{P}\sum_n \hat{f}(n/P)\, e^{2\pi i n t/P} \tag{1.3}$$

*and*

$$\sum_n \hat{f}(\gamma + nP) \;=\; \frac{1}{P} \sum_n f(n/P)\, e^{-2\pi i n\gamma/P}. \tag{1.4}$$

*Proof.* By (1.2), the function $F(t) = \sum f(t+nP)$, the $P$-periodization of $f$, converges absolutely and uniformly on compact sets and is in $L^2[-P/2, P/2)$. Moreover,

$$\int_{-P/2}^{P/2} \sum_n f(t+nP)\, e^{-2\pi i n t/P}\, dt \;=\; \int f(t)\, e^{-2\pi i n t/P}\, dt \;=\; \hat{f}(n/P).$$

Therefore, by (1.1),

$$F(t) \;=\; \frac{1}{\sqrt{P}} \sum_n c_n(F)\, e^{2\pi i n t/P} \;=\; \frac{1}{P} \sum_n \hat{f}(n/P)\, e^{2\pi i n t/P},$$

so (1.3) holds. Equation (1.4) follows similarly. $\square$

If $f \in L^2(\mathbf{R})$ is compactly supported then the $P$-periodization of $f$ is an element of $L^2[-P/2, P/2)$. In this case, (1.3) holds as written with the sums on each side converging in $L^2[-P/2, P/2)$, and the assumption (1.2) is not required. The right side of (1.3) is the Fourier series of the function defined by the sum on the left. Similarly, if $f \in PW_\Omega$ then (1.4) holds as written with the sums converging in $L^2[-P/2, P/2)$.

We now state and prove as a corollary to the PSF the sampling formula of Shannon and Whittaker.

THEOREM 1.2. *Let $f \in PW_\Omega$. Then given $T \geq 2\Omega$,*

$$f(t) \;=\; \frac{1}{T} \sum_n f(n/T)\, \frac{\sin \pi(Tt - n)}{\pi(Tt - n)}.$$

*Moreover,*

$$\int |f(t)|^2\, dt \;=\; \frac{1}{T} \sum_n |f(n/T)|^2. \tag{1.5}$$

*Proof.* By (1.3) and the remarks following Theorem 1.1,

$$\sum_n \hat{f}(\gamma + nT) \;=\; \frac{1}{T} \sum_n f(n/T)\, e^{-2\pi i n\gamma/T}.$$

We can write

$$\int_{-T/2}^{T/2} \sum_n \hat{f}(\gamma + nT)\, e^{2\pi i t\gamma}\, d\gamma \;=\; \frac{1}{T} \int_{-T/2}^{T/2} \sum_n f(n/T)\, e^{-2\pi i \gamma(n/T - t)}\, d\gamma. \tag{1.6}$$

Because the left hand side of (1.6) converges in $L^1[-T/2, T/2)$ we have

$$\int_{-T/2}^{T/2} \sum_n \hat{f}(\gamma + nT)\, e^{2\pi i t\gamma}\, d\gamma \;=\; \int_{-T/2}^{T/2} \hat{f}(\gamma)\, e^{2\pi i t\gamma}\, d\gamma \;=\; f(t).$$

Also, since the right hand side of (1.6) is the Fourier series of a function in $L^2[-T/2, T/2)$ we can integrate term-by-term, obtaining

$$\sum_n f(n/T)\, \frac{1}{T} \int_{-T/2}^{T/2} e^{2\pi i\gamma(t - n/T)}\, d\gamma \;=\; \sum_n f(n/T)\, \frac{\sin \pi(Tt - n)}{\pi(Tt - n)}.$$

Finally, (1.5) follows since $\left\{\frac{\sqrt{T}\sin\pi(Tt-n)}{\pi(Tt-n)}\right\}$ is an orthonormal set in $L^2[-T/2, T/2)$. □

We conclude this section with a variant of the sampling formula which is in keeping with the spirit of this paper. $\mathcal{S}(\mathbf{R})$ denotes the Schwartz class of infinitely differentiable, rapidly decreasing functions on the line.

Theorem 1.3. *Let $f \in PW_\Omega$ and let $T > 2\Omega$ be given. Then there exists a function $s \in \mathcal{S}(\mathbf{R})$ such that*

$$f(t) \;=\; \frac{1}{T} \sum_n f(n/T)\, s(t - n/T).$$

*Proof.* Let $s$ be be any function such that $\hat{s} \in C^\infty(\mathbf{R})$, $\hat{s} = 1$ on $[-\Omega, \Omega]$, and $\hat{s} = 0$ outside $[-T/2, T/2]$. Then clearly $s \in \mathcal{S}(\mathbf{R})$ and $\hat{f}(\gamma)\, \hat{s}(\gamma) = \hat{f}(\gamma)$. The PSF implies that

$$\int \sum_n \hat{f}(\gamma + nT)\, \hat{s}(\gamma)\, e^{2\pi i t\gamma}\, d\gamma \;=\; \frac{1}{T} \int \sum_n f(n/T)\, e^{-2\pi i\gamma(n/T - t)}\, \hat{s}(\gamma)\, d\gamma. \tag{1.7}$$

Using an argument similar to one used in the proof of Theorem 1.2, the left hand side of (1.7) is

$$\int \hat{f}(\gamma)\, \hat{s}(\gamma)\, e^{2\pi i t\gamma}\, d\gamma \;=\; \int \hat{f}(\gamma)\, e^{2\pi i t\gamma}\, d\gamma \;=\; f(t),$$

and the right hand side of (1.7) is

$$\sum_n f(n/T)\, \frac{1}{T} \int \hat{s}(\gamma)\, e^{2\pi i\gamma(t - n/T)}\, d\gamma \;=\; \frac{1}{T} \sum_n f(n/T)\, s(t - n/T). \quad \square$$

## 2. Sampling Results for Time-Frequency Transformations.

**A. Time-frequency operators and frames.** In this section we investigate some results concerning two time-frequency transformations, the continuous Gabor transform and the continuous wavelet transform. These operators are closely related to the Fourier transform and display simultaneously the time and frequency content of a signal.

Definition 2.1. Let $g \in L^2(\mathbf{R})$ be fixed. The *continuous Gabor transform* (*CGT*) with analyzing function $g$, denoted $\Psi_g$, is defined by

$$\Psi_g f(t,\omega) \;=\; \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i \omega x}\, \overline{g(x-t)}\, dx,$$

where $f \in L^2(\mathbf{R})$.

In the above definition, $t$ is the time variable and $\omega$ the frequency variable. The transform $\Psi_g f(t,\omega)$ is formed by shifting the window function $g$ so that it is centered at $t$, then taking the Fourier transform. In this way, $\Psi_g f(t,\omega)$ displays the frequency content of $f$ near time $t$.

The definition of the continuous wavelet transform uses the same idea of a sliding window but replaces the frequency variable by a scale variable.

Definition 2.2. Let $\psi \in L^2(\mathbf{R})$ be fixed. The *continuous wavelet transform* (*CWT*) with analyzing function $\psi$, denoted $\Phi_\psi$, is defined by

$$\Phi_\psi f(a,b) \;=\; \frac{1}{a} \int_{-\infty}^{\infty} f(x)\, \overline{\psi\Big(\frac{x-b}{a}\Big)}\, dx,$$

where $f \in L^2(\mathbf{R})$.

Note that $\Phi_\psi$ is a time-frequency localization operator where $a$ is the scale (frequency) variable and $b$ the time variable.

We now present some elementary properties of the CGT and CWT. Proofs of these results may be found in numerous places, e.g., [D1; HW]. We point out here that the results are immediate consequences of the Plancherel and Parseval formulas for the Fourier transform.

THEOREM 2.3. *Let $g \in L^2(\mathbf{R})$. Then for all $f \in L^2(\mathbf{R})$,*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\Psi_g f(t,\omega)|^2 \, dt \, d\omega \;=\; \int_{-\infty}^{\infty} |g(x)|^2 \, dx \int_{-\infty}^{\infty} |f(x)|^2 \, dx.$$

THEOREM 2.4. *Let $\psi \in L^2(\mathbf{R})$ and suppose that*

$$\int_{-\infty}^{\infty} \frac{|\hat{\psi}(\gamma)|^2}{|\gamma|} \, d\gamma \;<\; \infty.$$

*Then*

$$\int_0^{\infty}\int_{-\infty}^{\infty} |\Phi_\psi f(a,b)|^2 \, db \, \frac{da}{a}$$
$$= \int_0^{\infty} |\hat{f}(\omega)|^2 \, d\omega \int_0^{\infty} \frac{|\hat{\psi}(\gamma)|^2}{|\gamma|} \, d\gamma \;+\; \int_{-\infty}^{0} |\hat{f}(\omega)|^2 \, d\omega \int_{-\infty}^{0} \frac{|\hat{\psi}(\gamma)|^2}{|\gamma|} \, d\gamma.$$

The Poisson Summation Formula can be seen as a sampling result which characterizes the recovery of a function from regular samples of its Fourier transform. In some cases the recovery is exact, for example when $f$ is supported in the interval $[0,1]$. Such a sampling result can be cast in the language of Fourier series, i.e., in the representation of a function as a superposition of complex exponentials. In the case of the time-frequency transformations considered here, there is a well-developed theory of the recovery of a function from samples of its CGT or CWT. Here recovery is exact in all of $L^2(\mathbf{R})$, and, as in the Fourier series case, sampling results can be cast in the language of the representation of a function as a superposition of a fixed collection of basic functions. The most convenient setting for such representations is that of a *frame* in a Hilbert space, first described by Duffin and Schaeffer [DS] in relation to non-harmonic Fourier series.

DEFINITION 2.5. Let $H$ be a separable Hilbert space. A collection $\{x_n\} \subset H$ is a *frame* if there exist constants $A$, $B > 0$ such that for all $x \in H$,

$$A\,\|x\|^2 \;\leq\; \sum_n |\langle x, x_n\rangle|^2 \;\leq\; B\,\|x\|^2.$$

A frame is said to be *tight* if $A = B$ and *exact* if $\{x_n\}$ ceases to be a frame upon the removal of any element.

We say $\{x_n\}$ is a *Riesz basis* if there exist constants $A$, $B > 0$ such that $A \leq \|x_n\| \leq B$ for all $n$ and if every $x \in H$ can be written $x = \sum c_n(x)\, x_n$ for a unique choice of scalars $\{c_n(x)\}$ and this sum converges unconditionally (every rearrangement of the sum also converges, and to the same value).

A frame is a generalization of an orthonormal basis or a Riesz basis. With a frame, one has representations of elements in the Hilbert space in terms of the frame, and one can recover elements in a constructive way from the frame coefficients $\{\langle x, x_n\rangle\}$. The difference is that frames may be overcomplete in the sense that each element of a frame is in the closed linear span of the remaining elements. The following theorem collects some standard results about frames, proofs of which can be found in, e.g., [DS; HW; D1].

THEOREM 2.6. *Let $H$ be a separable Hilbert space and $\{x_n\}$ a collection of vectors in $H$. Then*

(a) *$\{x_n\}$ is a frame for $H$ if and only if the operator $S$ defined by $Sx = \sum \langle x, x_n\rangle\, x_n$ converges in $H$ for each $x$ and is a topological isomorphism of $H$ onto itself. In this case, we have the identities $x = \sum \langle x, x_n\rangle\, S^{-1}x_n = \sum \langle x, S^{-1}x_n\rangle\, x_n$, and $\{S^{-1}x_n\}$ is also a frame for $H$, called the dual frame of $\{x_n\}$.*

(b) *$\{x_n\}$ is an exact frame if and only if $\{x_n\}$ is a Riesz basis for $H$.*

(c) *If $\{x_n\}$ is such that $\|x_n\| = 1$ for all $n$, and if $\{x_n\}$ forms a tight frame with $A = B = 1$, then $\{x_n\}$ is an orthonormal basis for $H$.*

*Proof.* For a proof of (a), see, e.g., [HW, Theorem 2.1.3]. For (b), see [HW, Theorem 2.2.2]. To see (c), observe that for $n$ fixed,

$$\|x_n\|^2 \;=\; \sum_k |\langle x_n, x_k\rangle|^2 \;=\; \|x_n\|^4 \;+\; \sum_{k\neq n} |\langle x_n, x_k\rangle|^2.$$

Since $\|x_n\| = 1$, $\langle x_n, x_k\rangle = 0$ for all $n \neq k$. Thus $\{x_n\}$ is an orthonormal system. Because of (a), the span of $\{x_n\}$ is dense in $H$, i.e., $\{x_n\}$ is complete in $H$. A complete orthonormal sequence is an orthonormal basis. □

**B. Frames and sampling for the CGT.** In this section we investigate the connections between sampling results for the CGT and the existence of frames in the Hilbert space $L^2(\mathbf{R})$.

DEFINITION 2.7. Let $a, b > 0$ and $g \in L^2(\mathbf{R})$ be fixed. Then if the collection $\{e^{2\pi imbx}g(x-na)\}$ is a frame for $L^2(\mathbf{R})$, it is called a *Gabor frame*. We will denote the elements of such a frame by $\{E_{mb}T_{na}g\}$, where

$$T_x g(t) \;=\; g(t-x) \quad \text{and} \quad E_y g(t) \;=\; e^{2\pi i y t} g(t).$$

In his fundamental paper [G], Gabor investigated the case when $a = b = 1$ and $g(x) = e^{-\pi x^2}$ (cf., Section 3A). In this case, the collection $\{E_m T_n g\}$ is complete in $L^2(\mathbf{R})$ but does not form a frame. The idea of applying the notion of a frame to sets of the form $\{E_{mb}T_{na}g\}$ was first described in [DGM].

The following result can be viewed as a sampling theorem for the CGT. Its proof is immediate from the definition of a Gabor frame.

THEOREM 2.8. *The collection* $\{E_{mb}T_{na}g\}$ *is a Gabor frame if and only if there exist constants* $0 < A < B$ *such that for all* $f \in L^2(\mathbf{R})$,

$$A\,\|f\|_2^2 \;\le\; \sum_n \sum_m |\Psi_g f(na, mb)|^2 \;\le\; B\,\|f\|_2^2.$$

We now introduce the Zak transform, which we will use to partially characterize Gabor frames (see [AT; D; HW; J]). We let $Q$ denote the unit square, i.e., $Q = [0,1)^2$.

DEFINITION 2.9. Given $a > 0$, the **Zak transform**, or **Weil-Brezin map**, denoted $Z_a$, is the unitary mapping from $L^2(\mathbf{R})$ onto $L^2(Q)$ given by

$$Z_a f(t,\omega) \;=\; a^{1/2} \sum_k e^{2\pi i k \omega}\, f(a(t-k)).$$

$Z_a f$ satisfies the following *quasi-periodicity* relations:

$$Z_a f(t+1,\omega) \;=\; e^{2\pi i \omega} Z_a f(t,\omega)$$

$$Z_a f(t,\omega+1) \;=\; Z_a f(t,\omega).$$

LEMMA 2.10. *Suppose that* $ab = 1/N$ *for some integer* $N \ge 1$. *Then*

$$Z_a(E_{mb}T_{na}g)(t,\omega) \;=\; e^{2\pi i mn/N}\, e^{2\pi i m t/N}\, e^{-2\pi i n \omega}\, Z_a g(t, \omega - m/N).$$

THEOREM 2.11. *Let* $g \in L^2(\mathbf{R})$, *and suppose that* $ab = 1/N$ *for some* $a$, $b > 0$, $N \in \mathbf{Z}$. *Then in order that* $\{E_{mb}T_{na}g\}$ *be a Gabor frame, it is necessary and sufficient that there exist constants* $A$, $B > 0$ *such that*

$$A \;\le\; \sum_{j=0}^{N-1} |Z_a g(t, \omega - j/N)|^2 \;\le\; B. \tag{2.1}$$

*Proof.* Given $f \in L^2(\mathbf{R})$, we have by the unitarity of $Z_a$, Lemma 2.10, and Plancherel's formula for Fourier series,

$$\begin{aligned}
&\sum_n \sum_m |\langle f, E_{mb}T_{na}g\rangle|^2 \\
&= \sum_n \sum_m \left| \int_0^1 \int_0^1 Z_a f(t,\omega)\, \overline{Z_a g(t,\omega - m/N)}\, e^{-2\pi i m t/N}\, e^{2\pi i n \omega}\, dt\, d\omega \right|^2 \\
&= \sum_{j=0}^{N-1} \sum_n \sum_k \left| \int_0^1 \int_0^1 Z_a f(t,\omega)\, \overline{Z_a g(t,\omega - j/N)}\, e^{-2\pi i j t/N}\, e^{-2\pi i k t}\, e^{2\pi i n \omega}\, dt\, d\omega \right|^2 \\
&= \int_0^1 \int_0^1 |Z_a f(t,\omega)|^2 \sum_{j=0}^{N-1} |Z_a g(t,\omega - j/N)|\, dt\, d\omega.
\end{aligned}$$

Therefore, if (2.1) holds then

$$A\,\|Z_a f\|^2_{L^2(Q)} \;\le\; \int_0^1 \int_0^1 |Z_a f(t,\omega)|^2 \sum_{j=0}^{N-1} |Z_a g(t,\omega - j/N)|\, dt\, d\omega \;\le\; B\,\|Z_a f\|^2_{L^2(Q)},$$

which by the unitarity of $Z_a$ is

$$A\|f\|_2^2 \;\le\; \sum_n \sum_m |\langle f, E_{mb}T_{na}g\rangle|^2 \;\le\; B\|f\|_2^2.$$

Conversely, suppose (2.1) does not hold, say that

$$\operatorname*{ess\,inf}_{(t,\omega)\in Q} \sum_{j=0}^{N-1} |Z_a g(t,\omega - j/N)|^2 \;=\; 0.$$

Then given $\epsilon > 0$, there is a set $E \subset Q$ of positive measure such that

$$\sum_{j=0}^{N-1} |Z_a g(t,\omega - j/N)|^2 \;<\; \epsilon$$

for $(t,\omega) \in E$. Let $f \in L^2(\mathbf{R})$ be such that

$$Z_a f(t,\omega) \;=\; \frac{1}{|E|^{1/2}}\, \chi_E(t,\omega)$$

for $(t,\omega) \in Q$. Then $\|f\|_2 = \|Z_a f\|_{L^2(Q)} = 1$, but

$$\int_0^1 \int_0^1 |Z_a f(t,\omega)|^2 \sum_{j=0}^{N-1} |Z_a g(t,\omega - j/N)|\, dt\, d\omega \;=\; \sum_n \sum_m |\langle f, E_{mb}T_{na}g\rangle|^2 \;<\; \epsilon.$$

That is, $\{E_{mb}T_{na}g\}$ fails to have a lower frame bound. A similar argument shows that $\{E_{mb}T_{na}g\}$ fails to have an upper frame bound if

$$\operatorname*{ess\,sup}_{(t,\omega)\in Q} \sum_{j=0}^{N-1} |Z_a g(t,\omega - j/N)|^2 \;=\; +\infty. \quad \square$$

**C. Frames and sampling for the CWT.** In this section we investigate the connections between sampling results for the CWT and the existence of orthonormal bases of wavelets in $L^2(\mathbf{R})$.

Definition 2.12. Let $\psi \in L^2(\mathbf{R})$ be fixed. Then if the collection $\{2^{j/2}\,\psi(2^j x - k)\}$ is an orthonormal basis for $L^2(\mathbf{R})$, it is called a *wavelet orthonormal basis.*

For simplicity, we will write $\psi_{jk}(x) = 2^{j/2}\,\psi(2^j x - k)$. The existence of wavelet orthonormal bases has been known since Haar, who showed that if $\psi = \chi_{[0,1/2)} - \chi_{[1/2,1)}$ then $\{\psi_{jk}\}$ is a wavelet orthonormal basis, known as the *Haar system.* More recently, Meyer has shown that there exists $\psi$, infinitely differentiable and rapidly decreasing such that $\{\psi_{jk}\}$ is a wavelet orthonormal basis (see Section 3B) and Daubechies has shown that for any $r > 0$, there exist $\psi$, compactly supported and $r$–times continuously differentiable such that $\{\psi_{jk}\}$ is a wavelet orthonormal basis. Wavelet *frames*, especially those which are exact, are discussed in Section 3B.

The following result can be viewed as a sampling theorem for the CWT. Its proof is immediate from the definition of a wavelet orthonormal basis.

Theorem 2.13. *If $\psi \in L^2(\mathbf{R})$ is such that $\|\psi\|_2 = 1$, then $\{\psi_{jk}\}$ is a wavelet orthonormal basis if and only if*

$$\|f\|_2^2 \;=\; \sum_j \sum_k |\Phi_\psi f(2^{-j}, 2^{-j}k)|^2.$$

In order to partially characterize wavelet orthonormal bases, we introduce the notion of a multiresolution analysis due to S. Mallat (see [M; Ma; D2]).

Definition 2.14. A *multiresolution analysis* (MRA) of $L^2(\mathbf{R})$ is a collection of closed subspaces $\{V_j\}_{j\in\mathbf{Z}}$ such that the following conditions hold.

(a) $V_j \subset V_{j+1}$ for all $j$.
(b) $\cup V_j$ is dense in $L^2(\mathbf{R})$.
(c) $\cap V_j = \{0\}$.
(d) $f(x) \in V_j$ if and only if $f(2x) \in V_{j+1}$ for all $j$.
(e) There exists a function $\varphi \in L^2(\mathbf{R})$ such that the collection $\{\varphi(x-k)\}_{k\in\mathbf{Z}}$ is an orthonormal basis for $V_0$.

Given a MRA, one can construct a wavelet orthonormal basis.

LEMMA 2.15. *Let $g \in L^2(\mathbf{R})$. Then $\{g(x-k)\}_{k\in\mathbf{Z}}$ is an orthonormal set if and only if $\sum_{n\in\mathbf{Z}} |\hat{g}(\gamma+n)|^2 \equiv 1$.*

*Proof.* $\{g(x-k)\}_{k\in\mathbf{Z}}$ is an orthonormal set if and only if $g(x)$ is orthogonal to $g(x-k)$ for all $k \neq 0$ and $\|g\|_2 = 1$. That is, if and only if

$$\int g(x)\,\overline{g(x-k)}\,dx \;=\; \int |\hat{g}(\gamma)|^2\, e^{-2\pi i k\gamma}\,d\gamma \;=\; \int_0^1 \sum_{n\in\mathbf{Z}} |\hat{g}(\gamma+n)|^2\, e^{-2\pi i k\gamma}\,d\gamma \;=\; \delta_k.$$

This is true if and only if $\sum_{n\in\mathbf{Z}} |\hat{g}(\gamma+n)|^2 \equiv 1$. □

The following result is due to Y. Meyer.

THEOREM 2.16. *Let $\{V_j\}$ be a MRA. Then there exists $\psi \in L^2(\mathbf{R})$ such that $\{\psi_{jk}\}$ is a wavelet orthonormal basis.*

*Proof.* Since $V_j \subset V_{j+1}$, we may define $W_j$ as the orthogonal complement of $V_j$ in $V_{j+1}$, so that $V_{j+1} = V_j \oplus W_j$ for all $j$. Then $\{W_j\}$ is a collection of mutually orthogonal closed subspaces of $L^2(\mathbf{R})$. Let $P_j$ and $Q_j$ be the orthogonal projectors onto $V_j$ and $W_j$, respectively. Then $P_{j+1} = P_j + Q_j$ for all $j$.

Now suppose that $\psi \in L^2(\mathbf{R})$ is given so that $\{\psi_{0k}\}$ is an orthonormal basis for $W_0$. Then we claim that $\{\psi_{jk}\}$ is a wavelet orthonormal basis for $L^2(\mathbf{R})$. To see this, note that $f(x) \in W_0$ if and only if $f(2^j x) \in W_j$. Therefore for each fixed $j$, $\{\psi_{jk}\}_{k\in\mathbf{Z}}$ is an orthonormal basis for $W_j$. In particular, $\{\psi_{jk}\}_{j,k\in\mathbf{Z}}$ is an orthonormal system. To see that $\{\psi_{jk}\}$ is complete, note that for each $f \in L^2(\mathbf{R})$, $P_j f \to f$ and $P_{-j} f \to 0$ as $j \to \infty$. Thus, $\|f - (P_j - P_{-j})f\|_2 \to 0$ as $j \to \infty$. However,

$$P_j - P_{-j} \;=\; \sum_{k=-j}^{j-1} (P_{k+1} - P_k) \;=\; \sum_{k=-j}^{j-1} Q_k.$$

Since $Q_j f$ lies within the closed span of $\{\psi_{jk}\}_{k\in\mathbf{Z}}$ for each $j$, we conclude that the span of $\{\psi_{jk}\}_{j,k\in\mathbf{Z}}$ is dense in all of $L^2(\mathbf{R})$ and that $\{\psi_{jk}\}_{j,k\in\mathbf{Z}}$ is a wavelet orthonormal basis.

In order to construct such a $\psi$, note that since $\varphi \in V_0 \subset V_1$ and since $\{\varphi_{1k}\}_{k\in\mathbf{Z}}$ is an orthonormal basis for $V_1$, $\varphi$ satisfies the *dilation equation*

$$\varphi(x) \;=\; \sum_k h_k\, \varphi(2x-k),$$

where $h_k = 2^{1/2}\,\langle \varphi, \varphi_{1k}\rangle$. Now define

$$\psi(x) \;=\; \sum_k (-1)^k\, h_{1-k}\, \varphi(2x-k).$$

We claim that $\{\psi_{0k}\}$ is an orthonormal basis for $W_0$. In order to see this, define

$$m_0(\gamma) \;=\; \tfrac{1}{2} \sum_k h_k\, e^{-2\pi i k\gamma} \qquad \text{and} \qquad m_1(\gamma) \;=\; \tfrac{1}{2} \sum_k (-1)^k\, h_{1-k}\, e^{-2\pi i k\gamma}. \tag{2.2}$$

Then

$$\begin{aligned} \hat{\varphi}(\gamma) &= m_0(\gamma/2)\, \hat{\varphi}(\gamma/2), \\ \hat{\psi}(\gamma) &= m_1(\gamma/2)\, \hat{\varphi}(\gamma/2), \\ m_1(\gamma) &= e^{2\pi i(\gamma+1/2)}\, \overline{m_0(\gamma+1/2)}. \end{aligned}$$

By Lemma 2.15, since $\{\varphi_{0n}\}$ is an orthonormal set,

$$\begin{aligned} &\sum_n |\hat{\varphi}(\gamma+n)|^2 \\ &= \sum_n |m_0(\gamma/2+n/2)|^2\, |\hat{\varphi}(\gamma/2+n/2)|^2 \\ &= |m_0(\gamma/2)|^2 \sum_{n \text{ even}} |\hat{\varphi}(\gamma/2+n/2)|^2 \;+\; |m_0(\gamma/2+1/2)|^2 \sum_{n \text{ odd}} |\hat{\varphi}(\gamma/2+n/2)|^2 \\ &= |m_0(\gamma/2)|^2 \sum_j |\hat{\varphi}(\gamma/2+j)|^2 \;+\; |m_0(\gamma/2+1/2)|^2 \sum_j |\hat{\varphi}(\gamma/2+j+1/2)|^2 \\ &= |m_0(\gamma/2)|^2 \;+\; |m_0(\gamma/2+1/2)|^2. \end{aligned}$$

Therefore, by Lemma 2.15,

$$|m_0(\gamma/2)|^2 \;+\; |m_0(\gamma/2+1/2)|^2 \;\equiv\; 1.$$

And, since $|m_1(\gamma)| = |m_0(\gamma+1/2)|$,

$$|m_1(\gamma/2)|^2 \;+\; |m_1(\gamma/2+1/2)|^2 \;\equiv\; 1.$$

Then, as above,

$$\sum_n |\hat{\psi}(\gamma+n)|^2 \;=\; |m_1(\gamma/2)|^2 \;+\; |m_1(\gamma/2+1/2)|^2 \;\equiv\; 1,$$

so that $\{\psi_{0k}\}$ is an orthonormal set by Lemma 2.15.

It remains to show that $W_0 = \overline{\text{span}}\{\psi_{0k}\}$. Now, for any $f \in L^2(\mathbf{R})$,

$$P_0 f \;=\; \sum_k c_k\, \varphi_{0k}$$

where $c_k = \langle f, \varphi_{0k}\rangle$, and

$$(P_0 f)^\wedge(\gamma) \;=\; c(\gamma)\,\hat{\varphi}(\gamma) \;=\; c(\gamma)\, m_0(\gamma/2)\,\hat{\varphi}(\gamma/2), \tag{2.3}$$

where $c(\gamma) = \sum c_k\, e^{-2\pi i k\gamma} \in L^2[0,1]$. Similarly,

$$(P_1 f)^\wedge(\gamma) \;=\; a(\gamma/2)\,\hat{\varphi}(\gamma/2), \tag{2.4}$$

where $a(\gamma) = 2^{-1/2} \sum a_k\, e^{-2\pi i k\gamma} \in L^2[0,1]$ and $a_k = \langle f, \varphi_{1k}\rangle$.

Consider the system of equations

$$\begin{pmatrix} m_0(\gamma) & m_1(\gamma) \\ m_0(\gamma+1/2) & m_1(\gamma+1/2) \end{pmatrix} \begin{pmatrix} c(\gamma) \\ d(\gamma) \end{pmatrix} \;=\; \begin{pmatrix} a(\gamma) \\ a(\gamma+1/2) \end{pmatrix}, \tag{2.5}$$

where $a(\gamma) \in L^2[0,1]$ is given. The determinant of the matrix on the left reduces to

$$e^{2\pi i(\gamma+1/2)} \left(|m_0(\gamma)|^2 + |m_0(\gamma+1/2)|^2\right) \;=\; e^{2\pi i(\gamma+1/2)},$$

which is non-zero for all $\gamma$. The solution of the system is therefore

$$c(\gamma) \;=\; e^{-2\pi i(\gamma+1/2)} \big(m_1(\gamma+1/2)a(\gamma) - m_1(\gamma)a(\gamma+1/2)\big),$$
$$d(\gamma) \;=\; e^{-2\pi i(\gamma+1/2)} \big(m_0(\gamma)a(\gamma+1/2) - m_0(\gamma+1/2)a(\gamma)\big).$$

Since $m_0$ and $m_1$ are bounded, $c,\, d \in L^2[0,1]$.

Now, for any $f \in L^2(\mathbf{R})$, $Q_0 f = P_1 f - P_0 f$, and so by (2.3) and (2.4),

$$(Q_0 f)^\wedge(\gamma) \;=\; \big(a(\gamma/2) - c(\gamma)\, m_0(\gamma/2)\big)\,\hat{\varphi}(\gamma/2).$$

However, by (2.5), $a(\gamma/2) - c(\gamma)\, m_0(\gamma/2) = d(\gamma)\, m_1(\gamma/2)$, so that

$$(Q_0 f)^\wedge(\gamma) \;=\; d(\gamma)\, m_1(\gamma/2)\,\hat{\varphi}(\gamma/2) \;=\; d(\gamma)\,\hat{\psi}(\gamma)$$

for some $d(\gamma) = \sum d_k\, e^{-2\pi i k\gamma} \in L^2[0,1]$. Thus, $Q_0 f = \sum d_k\, \psi_{0k}$, which implies that the span of $\{\psi_{0k}\}$ is dense in $W_0$, and hence that $\{\psi_{jk}\}$ is a wavelet orthonormal basis for $L^2(\mathbf{R})$. □

In some cases it is possible to define a MRA starting with the auxiliary function $m_0$ defined by (2.2). The following result of Cohen [C], which we state without proof, characterizes when this is possible.

Theorem 2.17. *Let $h_k$ be a real-valued sequence and suppose that*

(a) $\sum |h_k|\, k^\epsilon < \infty$ *for some* $\epsilon > 0$,

(b) $m_0(0) = 1$,

(c) $|m_0(\gamma)|^2 + |m_0(\gamma + 1/2)|^2 = 1$ *for all* $\gamma$.

*Let $\varphi$ be defined by*

$$\hat{\varphi}(\gamma) \;=\; \prod_{j=1}^{\infty} m_0(\gamma/2^j).$$

*Then $\{\varphi(x-k)\}$ is an orthonormal set if and only if there exists a compact set $K$ such that*

(d) *$K$ is a finite union of closed intervals of total measure 1.*

(e) *for every $x \in [-1/2, 1/2]$ there is an integer $k$ such that $x + k \in K$,*

(f) *$K$ contains a neighborhood of zero,*

(g) *there exists $\epsilon > 0$ such that $|m_0(2^{-k}\gamma)| \geq \epsilon$ for all $k \geq 1$ and $\gamma \in K$.*

*In this case, taking $V_0 = \overline{\mathrm{span}}\{\varphi(x-k)\}$ and defining $V_j = \{f(2^{-j}x) : f \in V_0\}$, the $\{V_j\}$ are a MRA, and therefore a wavelet orthonormal basis exists for $L^2(\mathbf{R})$.*

In the special case obtained by putting $K = [-1/2, 1/2]$, conditions (d)–(g) are satisfied if and only if $|m_0(\gamma)| > 0$ for $|\gamma| < 1/4$. This fact was proved by Mallat [Ma].

## 3. Uncertainty Principles for Exact Gabor and Wavelet Frames.

The previous sections described several time-frequency analysis techniques, including Gabor and wavelet frames. In applications, exact frames (especially those which are actually orthonormal bases) are often appealing, as they can often be implemented with fast discrete algorithms. We focus on such exact systems in this section, establishing some limitations on the joint time-frequency concentration of the elements of Gabor and wavelet systems which form exact frames. These results may thus be considered "uncertainty principles" for such nonredundant systems.

First, however, we review some general results on exact frames and the classical uncertainty principle in Hilbert spaces. By Theorem 2.6(a), any exact frame $\{x_n\}$ for a Hilbert space $H$ is a *Riesz basis* for $H$. Moreover, $c_n(x) = \langle x, \tilde{x}_n\rangle$ where $\{\tilde{x}_n\}$ is the dual frame to $\{x_n\}$, i.e., $\tilde{x}_n = S^{-1}x_n$. Therefore, $\{x_n\}$ and $\{\tilde{x}_n\}$ are *biorthonormal*, i.e., $\langle x_m, \tilde{x}_n\rangle = \delta_{mn}$.

The following inequality is the *classical uncertainty principle*. Its proof, which is quite elementary, was recorded by N. Wiener and H. Weyl.

THEOREM 3.1. *If* $f \in L^2(\mathbf{R})$ *then* $\|f\|_2^2 \leq 4\pi \,\|tf(t)\|_2 \,\|\gamma\hat{f}(\gamma)\|_2$.

Given a Hilbert space $H$ and given (not necessarily continuous) operators $A$, $B$ mapping domains $D(A)$, $D(B) \subset H$ into $H$, respectively, define the *commutator* of $A$, $B$ to be the operator $[A,B] = AB - BA$. If $A$ is self-adjoint then the *expectation* of $A$ at $f \in D(A)$ is $E_f(A) = \langle Af, f\rangle$, and the *variance* of $A$ at $f \in D(A^2)$ is $\sigma_f^2(A) = E_f(A^2) - E_f(A)^2$. An uncertainty principle inequality can now be formulated on $H$. Its statement (Theorem 3.2) and simple computational proof are part of the folklore in the Hilbert space community.

THEOREM 3.2. *Given self-adjoint operators* $A$, $B$ *on a Hilbert space* $H$. *If* $f \in D(A^2) \cap D(B^2) \cap D(i[A,B])$ *and* $\|f\| = 1$ *then* $E_f(i[A,B])^2 \leq 4\,\sigma_f^2(A)\,\sigma_f^2(B)$.

The classical uncertainty principle (Theorem 3.1) follows as a corollary. Define the *position operator* $P$ (operating on functions $f$) by

$$Pf(t) \;=\; t\,f(t),$$

and the *momentum operator* $M$ by

$$Mf \;=\; (P\hat{f})^{\vee} \;=\; (\gamma\,\hat{f}(\gamma))^{\vee}.$$

The domains of $P$, $M$ include the Schwartz class $\mathcal{S}(\mathbf{R}) \subset L^2(\mathbf{R})$, and both $P$, $M$ are self-adjoint. If $f \in \mathcal{S}(\mathbf{R})$ then

$$\begin{aligned} (f')^{\wedge} \;&=\; 2\pi i\,P\hat{f}, \\ f' \;&=\; 2\pi i\,Mf, \\ [P,M]f \;&=\; -\tfrac{1}{2\pi i}f. \end{aligned} \tag{3.1}$$

From Theorem 3.2 we therefore have

$$E_f(-\tfrac{1}{2\pi}I)^2 \;\leq\; \sigma_f^2(P)\,\sigma_f^2(M),$$

where $I$ is the identity operator on $\mathcal{S}(\mathbf{R})$. However, $E_f(I) = \|f\|_2^2$, $\sigma_f^2(P) = \|Pf\|_2^2 - \langle Pf, f\rangle^2 \le \|Pf\|_2^2$, and $\sigma_f^2(M) = \|Mf\|_2^2 - \langle Mf, f\rangle^2 \le \|Mf\|_2^2$, from which Theorem 3.1 follows immediately for $f \in \mathcal{S}(\mathbf{R})$. A standard closure argument extends the inequality to all $f \in L^2(\mathbf{R})$.

**A. An uncertainty principle for exact Gabor frames.** If $ab = 1$ and $\{E_{mb}T_{na}g\}$ is a Gabor frame for $L^2(\mathbf{R})$ then it must be exact. Conversely, if $\{E_{mb}T_{na}g\}$ is an exact Gabor frame then necessarily $ab = 1$. By dilating $g$ if necessary we therefore assume $a = b = 1$ in this section without loss of generality, and for simplicity write $g_{mn}(x) = E_mT_ng(x) = e^{2\pi imx}\, g(x-n)$, and let $Z$ denote the Zak transform $Z = Z_a = Z_1$. Note that the dual frame to $\{g_{mn}\}$ is another exact Gabor frame $\{\tilde{g}_{mn}\}$, where $\tilde{g} = S^{-1}g$ is a uniquely determined function in $L^2(\mathbf{R})$.

The *Balian-Low Theorem* (BLT) is an uncertainty principle-like result for exact Gabor frames. It imposes severe restrictions on the time-frequency localization of any function $g$ which generates an exact Gabor frame:

THEOREM 3.3. *Given $g \in L^2(\mathbf{R})$. If $\{g_{mn}\}$ is an exact Gabor frame then*

$$\|t\, g(t)\|_2 \, \|\gamma\, \hat{g}(\gamma)\|_2 \;=\; +\infty.$$

In particular, note that if $g$ is the Gaussian function $g(x) = e^{-\pi x^2}$ then $\{g_{mn}\}$ is not an exact Gabor frame.

The BLT was first stated by Balian [B], and later (independently) by Low [L], for the special case of Gabor systems which are orthonormal bases. Their proofs contained a technical gap which was filled by Coifman, Semmes, and Daubechies [D1]; this group also extended the result to all exact Gabor frames. The resulting proof required the use of the Zak transform and, implicitly, both classical and distributional differentiation. Explicit justification for this proof, along with a non-distributional version, was presented in [BHW]. As all versions of this proof of the BLT are quite technical, we shall instead discuss a proof inspired by an elegant, natural, and elementary argument due to Battle [Bat], originally stated for the case of orthonormal bases only, extended formally to exact frames by Daubechies and Janssen [DJ], and justified in [BHW]. This argument depends critically on (3.1), from which the classical uncertainty principle can also be derived. Precisely, (3.1) is used in the following form:

LEMMA 3.4. *Given $f$, $g \in L^2(\mathbf{R})$. If $Pf$, $Pg$, $P\hat{f}$, $P\hat{g} \in L^2(\mathbf{R})$ then*

$$\langle Pf, Mg\rangle - \langle Mf, Pg\rangle \; = \; \tfrac{1}{2\pi i}\,\langle f, g\rangle. \tag{3.2}$$

Note that (3.2) follows immediately from (3.1) for $f$, $g \in \mathcal{S}(\mathbf{R})$, and can be extended to all $f, g \in L^2(\mathbf{R})$ satisfying the hypotheses of Lemma 3.4 by a closure argument.

Lemma 3.4 suffices to prove a weak version of the BLT:

THEOREM 3.5. *Given $g \in L^2(\mathbf{R})$. If $\{g_{mn}\}$ is an exact Gabor frame then*

$$\|t\, g(t)\|_2 \; \|\gamma\, \hat{g}(\gamma)\|_2 \; \|t\, \tilde{g}(t)\|_2 \; \|\gamma\, \hat{\tilde{g}}(\gamma)\|_2 \; = \; +\infty.$$

*Proof.* Assume $Pg$, $P\hat{g}$, $P\tilde{g}$, $P\hat{\tilde{g}} \in L^2(\mathbf{R})$. An easy computation yields the formulas

$$\begin{aligned} [E_m T_n, P] \; &= \; -n\, E_m T_n, \\ [E_m T_n, M] \; &= \; -m\, E_m T_n. \end{aligned}$$

Since $\{g_{mn}\}$ and $\{\tilde{g}_{mn}\}$ are biorthonormal, and recalling that $g = g_{00}$ and $\tilde{g} = \tilde{g}_{00}$, it follows that

$$\begin{aligned} \langle g_{mn}, P\tilde{g}\rangle \; &= \; \langle E_m T_n g, P\tilde{g}\rangle \; = \; \langle E_m T_n P g, \tilde{g}\rangle \; = \; \langle (Pg)_{mn}, \tilde{g}\rangle, \\ \langle Mg, \tilde{g}_{mn}\rangle \; &= \; \langle Mg, E_m T_n \tilde{g}\rangle \; = \; \langle g, E_m T_n M\tilde{g}\rangle \; = \; \langle g, (M\tilde{g})_{mn}\rangle. \end{aligned}$$

Since $e^{2\pi i mn} = 1$ for all $m$, $n$, we therefore have

$$\begin{aligned} \langle Pg, M\tilde{g}\rangle \; &= \; \sum \langle Pg, \tilde{g}_{mn}\rangle \, \langle g_{mn}, M\tilde{g}\rangle \\ &= \; \sum \langle (Pg)_{mn}, \tilde{g}\rangle \, \langle g, (M\tilde{g})_{mn}\rangle \\ &= \; \sum \langle g_{mn}, P\tilde{g}\rangle \, \langle Mg, \tilde{g}_{mn}\rangle \\ &= \; \langle Mg, P\tilde{g}\rangle. \end{aligned}$$

Therefore, $\langle g, \tilde{g}\rangle = 0$ by (3.2). However, we also have $\langle g, \tilde{g}\rangle = 1$ by biorthonormality, a contradiction. □

Note that Theorems 3.3 and 3.5 are equivalent if $\{g_{mn}\}$ is an orthonormal basis, for then $g = \tilde{g}$. We demonstrate now that Theorems 3.3 and 3.5 are equivalent in general, for which it suffices to prove:

PROPOSITION 3.6. *Given $g \in L^2(\mathbf{R})$. If $\{g_{mn}\}$ is an exact Gabor frame then*

$$Pg \in L^2(\mathbf{R}) \;\Leftrightarrow\; P\tilde{g} \in L^2(\mathbf{R}) \tag{3.3}$$

*and*

$$P\hat{g} \in L^2(\mathbf{R}) \;\Leftrightarrow\; P\hat{\tilde{g}} \in L^2(\mathbf{R}). \tag{3.4}$$

*Proof.* We give only a formal argument, due to Daubechies and Janssen [DJ] for (3.3); the proof of (3.4) is entirely symmetrical.

Given $f \in L^2(\mathbf{R})$, we formally compute

$$\begin{aligned} ZPf(t,\omega) &= t \sum f(t+k)\, e^{2\pi i k\omega} + \sum f(t+k)\, k\, e^{2\pi i k\omega} \\ &= t\, Zf(t,\omega) + \tfrac{1}{2\pi i}\, \partial_2 Zf(t,\omega). \end{aligned} \tag{3.5}$$

Note that from Lemma 2.10, $Zg_{mn} = E_{(m,n)} Zg$ and $Z\tilde{g}_{mn} = E_{(m,n)} Z\tilde{g}$, where $E_{(m,n)}(t,\omega) = e^{2\pi i m t}\, e^{2\pi i n\omega}$. It therefore follows from the biorthonormality of $\{g_{mn}\}$ and $\{\tilde{g}_{mn}\}$ that $Z\tilde{g} = 1/\overline{Zg}$. Therefore, using (3.5) we formally compute

$$\begin{aligned} \overline{ZP\tilde{g}(t,\omega)} &= \frac{t}{Zg(t,\omega)} - \tfrac{1}{2\pi i}\, \partial_2 (1/Zg)(t,\omega) \\ &= \frac{t\, Zg(t,\omega) + \tfrac{1}{2\pi i}\, \partial_2 Zg(t,\omega)}{Zg(t,\omega)^2} \\ &= \frac{ZPg(t,\omega)}{Zg(t,\omega)^2}. \end{aligned} \tag{3.6}$$

From Lemma 2.10, $|Zg|$ is an essentially constant function (i.e., is bounded below from zero and above from infinity). It therefore follows from (3.6) that $ZPg \in L^2(Q)$ if and only if $ZP\tilde{g} \in L^2(Q)$, from which (3.3) follows by the unitarity of $Z$. $\square$

The formal arguments in the proof of Proposition 3.6 above are justified in [BHW] through a mixture of classical and distributional differentiation. We therefore pose the following problem which, if answered positively, would establish Proposition 3.6 without the use of differentiation.

PROBLEM 3.7. Given $g \in L^2(\mathbf{R})$ such that $\{g_{mn}\}$ is an exact Gabor frame. Assume $f \in L^2_{\text{loc}}(\mathbf{R})$ is such that $f \cdot \overline{g_{mn}} \in L^1(\mathbf{R})$ and $\langle f, g_{mn}\rangle = 0$ for all $m, n \in \mathbf{Z}$. Does it then follow that $f = 0$?

That Proposition 3.6 follows from a positive answer to Problem 3.7 is established as follows. Assume $\{g_{mn}\}$ is an exact Gabor frame and that $Pg \in L^2(\mathbf{R})$. Note that $P\tilde{g} \in L^2_{\text{loc}}(\mathbf{R})$ and $P\tilde{g} \cdot \overline{g_{mn}} = \tilde{g} \cdot \overline{Pg_{mn}} \in L^1(\mathbf{R})$ for all $m$, $n$. Moreover,

$$\begin{aligned} \langle P\tilde{g}, g_{mn} \rangle &= \langle \tilde{g}, Pg_{mn} \rangle \\ &= \langle Z\tilde{g}, ZPg_{mn} \rangle \\ &= \langle 1/\overline{Zg}, E_{(m,n)} ZPg \rangle \\ &= \langle \overline{ZPg/(Zg)^2}, E_{(m,n)} Zg \rangle \\ &= \langle \overline{ZPg/(Zg)^2}, Zg_{mn} \rangle. \end{aligned} \tag{3.7}$$

Note that $ZPg \in L^2(Q)$ since $Pg \in L^2(\mathbf{R})$. As $|Zg|$ is essentially constant, it follows that $ZPg/(Zg)^2 \in L^2(Q)$. Therefore $h = Z^{-1}\big(\overline{ZPg/(Zg)^2}\big) \in L^2(\mathbf{R})$. From (3.7) and the unitarity of $Z$, we therefore have

$$\langle P\tilde{g}, g_{mn} \rangle = \langle h, g_{mn} \rangle$$

for all $m$, $n$. Hence, if the answer to Problem 3.7 is positive then $P\tilde{g} = h \in L^2(\mathbf{R})$.

We note that the analogue of Problem 3.7 with the exact Gabor frame $\{g_{mn}\}$ replaced by an arbitrary (exact) frame is false. For example, if $\{\psi_{jk}\}$ is the Haar system (cf., Section 2C), then $\psi \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ and $\int \psi(x)\,dx = 0$. Therefore, taking $f \equiv 1$ we have $f \in L^2_{\text{loc}}(\mathbf{R})$ yet $\langle f, \psi_{jk} \rangle = 0$ for all $j$, $k$. Other examples relevant to Problem 3.7 are given in [BHW].

**B. An uncertainty principle for exact wavelet frames.** The previous section established that all exact Gabor frames suffer a severe time-frequency localization constraint. In this section we will consider whether exact wavelet frames suffer any similar constraints. We will use the notation $\psi_{jk}(x) = 2^{j/2}\,\psi(2^j x - k)$, cf., Section 3C.

First, note that the classical example of a wavelet orthonormal basis $\{\psi_{jk}\}$, i.e., the Haar system $\psi = \chi_{[0,1/2)} - \chi_{[1/2,1)}$, satisfies $\|t\,\psi(t)\|_2\, \|\gamma\,\hat{\psi}(\gamma)\|_2 = +\infty$. However, the "wavelet BLT" is false in general, as shown by the first modern example of a wavelet orthonormal basis. The *Meyer wavelet* is the function $\psi \in L^2(\mathbf{R})$ defined by $\hat{\psi}(\gamma) =$

$e^{i\gamma/2}\,\omega(|\gamma|)$, where

$$\omega(\gamma) \;=\; \begin{cases} 0, & \gamma \le \frac{1}{3} \text{ or } \gamma \ge \frac{4}{3}, \\ \sin \frac{\pi}{2} v(3\gamma - 1), & \frac{1}{3} \le \gamma \le \frac{2}{3}, \\ \cos \frac{\pi}{2} v(\frac{3\gamma}{2} - 1), & \frac{2}{3} \le \gamma \le \frac{4}{3}, \end{cases}$$

and $v \in C^\infty(\mathbf{R})$ is such that $v(\gamma) = 0$ for $\gamma \le 0$, $v(\gamma) = 1$ for $\gamma \ge 1$, $0 \le v(\gamma) \le 1$ for $\gamma \in [0,1]$, and $v(\gamma) + v(1-\gamma) = 1$ for $\gamma \in [0,1]$. That $\{\psi_{jk}\}$ is an orthonormal basis for $L^2(\mathbf{R})$ is shown in [M]. As $\hat{\psi} \in C_c^\infty(\mathbf{R})$, it follows that $\psi \in \mathcal{S}(\mathbf{R})$. Thus $\psi$ has better than any polynomial localization in both time and frequency, i.e., $\|p(t)\,\psi(t)\|_2\,\|q(\gamma)\,\hat{\psi}(\gamma)\|_2 < +\infty$ for all polynomials $p$, $q$. However, $\psi$ does not possess *exponential* localization in both time and frequency, and it is therefore natural to ask whether there exist wavelet orthonormal bases, or, more generally, exact wavelet frames, which do. Battle [Ba2] answered this in the negative for the case of wavelet orthonormal bases, and we now extend this result in a weak manner to exact wavelet frames.

First, however, note that while the dual frame of any Gabor frame is itself a Gabor frame, it is not always the case that the dual frame of any wavelet frame is itself a wavelet frame [D1]. However, this is true for *exact* wavelet frames. For, if $\{\psi_{jk}\}$ is an exact wavelet frame then there exists a unique function $\tilde{\psi} \in L^2(\mathbf{R})$ such that $\langle \psi_{jk}, \tilde{\psi}\rangle = \delta_j\,\delta_k$, namely $\tilde{\psi} = S^{-1}\psi$. It follows then that $\langle \psi_{jk}, \tilde{\psi}_{j'k'}\rangle = \delta_{jj'}\,\delta_{kk'}$, whence $\{\tilde{\psi}_{jk}\}$ is the dual frame to $\{\psi_{jk}\}$. We have then:

THEOREM 3.8. *Given $\psi \in L^2(\mathbf{R})$. If $\{\psi_{jk}\}$ is an exact wavelet frame then*

$$\|e^{|t|}\,\psi(t)\|_2\;\|e^{|\gamma|}\,\hat{\psi}(\gamma)\|_2\;\|e^{|t|}\,\tilde{\psi}(t)\|_2\;\|e^{|\gamma|}\,\hat{\tilde{\psi}}(\gamma)\|_2 \;=\; +\infty.$$

The proof of Theorem 3.8 depends on the following two lemmas.

LEMMA 3.9. *Given $\psi \in L^2(\mathbf{R})$ such that $\{\psi_{jk}\}$ is an exact wavelet frame. If $\hat{\psi} \in C(\mathbf{R}) \cap L^1(\mathbf{R}) \cap L^\infty(\mathbf{R})$ and $\hat{\tilde{\psi}} \in C(\mathbf{R}) \cap L^1(\mathbf{R}) \cap L^\infty(\mathbf{R})$ then $\hat{\psi}(0) = 0 = \hat{\tilde{\psi}}(0)$.*

*Proof.* Note that $\tilde{\psi} \in C_0(\mathbf{R})$; therefore $\tilde{\psi}(x_0) \ne 0$ for some dyadic point $x_0 = 2^{-j_0}k_0$ where $j_0$, $k_0 \in \mathbf{Z}$ are not both zero. Given $j > j_0$, set $k_j = 2^{j-j_0}k_0$. Then

$$\begin{aligned} 0 \;&=\; \langle \psi_{jk_j}, \tilde{\psi}\rangle \\ &=\; \langle \hat{\psi}_{jk_j}, \hat{\tilde{\psi}}\rangle \end{aligned}$$

$$= 2^{-j/2} \int e^{-2\pi i 2^{-j}\gamma k_j}\, \hat{\psi}(2^{-j}\gamma)\, \overline{\hat{\tilde{\psi}}(\gamma)}\, d\gamma$$

$$= 2^{-j/2} \int e^{-2\pi i x_0 \gamma}\, \hat{\psi}(2^{-j}\gamma)\, \overline{\hat{\tilde{\psi}}(\gamma)}\, d\gamma. \tag{3.8}$$

Now,

$$e^{-2\pi i x_0 \gamma}\, \hat{\psi}(2^{-j}\gamma)\, \overline{\hat{\tilde{\psi}}(\gamma)} \;\to\; e^{-2\pi i x_0 \gamma}\, \hat{\psi}(0)\, \overline{\hat{\tilde{\psi}}(\gamma)}$$

pointwise as $j \to \infty$. Also,

$$|e^{-2\pi i x_0 \gamma}\, \hat{\psi}(2^{-j}\gamma)\, \overline{\hat{\tilde{\psi}}(\gamma)}| \;\le\; \|\hat{\psi}\|_\infty\, |\hat{\tilde{\psi}}(\gamma)|$$

and $\hat{\tilde{\psi}} \in L^1(\mathbf{R})$, so by (3.8) and the Lebesgue Dominated Convergence Theorem,

$$\begin{aligned} 0 &= \lim_{j\to\infty} \int e^{-2\pi i x_0 \gamma}\, \hat{\psi}(2^{-j}\gamma)\, \overline{\hat{\tilde{\psi}}(\gamma)}\, d\gamma \\ &= \hat{\psi}(0)\, \overline{\int e^{2\pi i x_0 \gamma}\, \hat{\tilde{\psi}}(\gamma)\, d\gamma} \\ &= \hat{\psi}(0)\, \overline{\tilde{\psi}(x_0)}. \end{aligned}$$

As $\tilde{\psi}(x_0) \neq 0$ we conclude that $\hat{\psi}(0) = 0$. By symmetry, $\hat{\tilde{\psi}}(0) = 0$ as well. □

The following result extends Lemma 3.9 to general moments.

LEMMA 3.10. *Given $\psi \in L^2(\mathbf{R})$ such that $\{\psi_{jk}\}$ is an exact wavelet frame. If $\hat{\psi}$, $\hat{\tilde{\psi}} \in C^{N+1}(\mathbf{R}) \cap L^\infty(\mathbf{R})$ and*

$$(1+|\gamma|)^{N+1}\, \hat{\psi}(\gamma),\;\; (1+|\gamma|)^{N+1}\, \hat{\tilde{\psi}}(\gamma) \;\in\; L^1(\mathbf{R})$$

*then*

$$\int x^m\, \psi(x)\, dx \;=\; \int x^m\, \tilde{\psi}(x)\, dx \;=\; 0$$

*for $m = 0, \ldots, N$.*

*Proof.* That $\int \psi(x)\, dx = \hat{\psi}(0) = 0$ follows from Lemma 3.9. Assume now that

$$i^m\, (D^m \hat{\psi})(0) \;=\; \int x^m\, \psi(x)\, dx \;=\; 0$$

for $m = 0, \ldots, k-1$ for some $k \le N$, and assume $D^k \hat{\psi}(0) \neq 0$. Then

$$\hat{\psi}(\gamma) \;=\; \hat{\psi}(0) \;+\; \gamma\, D\hat{\psi}(0) \;+\; \cdots \;+\; \frac{\gamma^k\, (D^k \hat{\psi})(0)}{k!} \;+\; R_k(\gamma) \;=\; \frac{\gamma^k\, (D^k \hat{\psi})(0)}{k!} \;+\; R_k(\gamma),$$

where $R_k$ is the Taylor remainder

$$R_k(\gamma) \;=\; \frac{\gamma^{k+1}\,(D^{k+1}\hat{\psi})(\xi)}{(k+1)!}$$

for some $\xi$ between 0 and $\gamma$. As $D^k\hat{\psi}$ is continuous, there exists some dyadic point $x_0 = 2^{-j_0}k_0$ such that $D^k\hat{\psi}(x_0) \neq 0$. Given $j > j_0$, set $k_j = 2^{j-j_0}k_0$. Then, as in (3.8),

$$\begin{aligned} 0 \;&=\; \int e^{-2\pi i x_0\gamma}\,\hat{\psi}(2^{-j}\gamma)\,\overline{\hat{\tilde{\psi}}(\gamma)}\,d\gamma \\ &=\; \int e^{-2\pi i x_0\gamma}\,\frac{\gamma^k\,(D^k\hat{\psi})(0)}{2^{jk}\,k!}\,\overline{\hat{\tilde{\psi}}(\gamma)}\,d\gamma \;+\; \int e^{-2\pi i x_0\gamma}\,R_k(2^{-j}\gamma)\,\overline{\hat{\tilde{\psi}}(\gamma)}\,d\gamma \\ &=\; I_1(j) \;+\; I_2(j). \end{aligned}$$

Now,

$$|R_k(2^{-j}\gamma) \;\le\; \frac{|\gamma|^{k+1}\,\|D^{k+1}\hat{\psi}\|_\infty}{2^{j(k+1)}\,(k+1)!} \;=\; \frac{C_1\,|\gamma|^{k+1}}{2^{j(k+1)}} \;\le\; \frac{C_1\,(1+|\gamma|)^{N+1}}{2^{j(k+1)}}.$$

Thus

$$I_2(j) \;\le\; \frac{C_1}{2^{j(k+1)}}\int (1+|\gamma|)^{N+1}\,|\hat{\tilde{\psi}}(\gamma)|\,d\gamma \;=\; \frac{C_2}{2^{j(k+1)}},$$

so $I_2(j) \in \mathcal{O}(2^{-j(k+1)})$. Also,

$$\begin{aligned} I_1(j) \;&=\; \frac{D^k\hat{\psi}(0)}{2^{jk}\,k!}\int e^{-2\pi i x_0\gamma}\,\gamma^k\,\overline{\hat{\tilde{\psi}}(\gamma)}\,d\gamma \\ &=\; \frac{C_3}{2^{jk}}\,\overline{\int \gamma^k\,\hat{\tilde{\psi}}(\gamma)\,e^{2\pi i x_0\gamma}\,d\gamma} \\ &=\; \frac{C_3\,i^k}{2^{jk}}\,\overline{\big((i\gamma)^k\,\hat{\tilde{\psi}}(\gamma)\big)^\vee(x_0)} \\ &=\; \frac{C_3\,i^k}{2^{jk}}\,\overline{D^k\tilde{\psi}(x_0)}. \end{aligned}$$

Thus $I_1(j) \neq 0$ for all $j$ and $I_1(j) \in \mathcal{O}(2^{-jk}) \setminus \mathcal{O}(2^{-j(k+1)})$. As $I_1(j) = -I_2(j)$, this is a contradiction. □

The proof of Theorem 3.8 now follows immediately.

*Proof of Theorem 3.8.* Assume $e^{|t|}\,\psi(t)$, $e^{|\gamma|}\,\hat{\psi}(\gamma)$, $e^{|t|}\,\tilde{\psi}(t)$, $e^{|\gamma|}\,\hat{\tilde{\psi}}(\gamma) \in L^2(\mathbf{R})$. It then follows that both $\psi$ and $\tilde{\psi} \in \mathcal{S}(\mathbf{R})$. Therefore, from Lemma 3.10, all moments of $\psi$ and $\tilde{\psi}$ vanish, which implies $\psi = \tilde{\psi} = 0$, a contradiction. □

Theorem 3.8 is closer in analogy to the weak BLT, Theorem 3.5, than to the BLT, Theorem 3.3. In the Gabor case, the weak BLT and the BLT have been shown to be equivalent; we leave as an open problem whether Theorem 3.8 can be improved to a strong form which states that if $\{\psi_{jk}\}$ is an exact wavelet frame then neither $\psi$ nor $\tilde{\psi}$ can have exponential localization both in time and frequency.

## References

[AT] L. Auslander and R. Tolimieri, *Abelian Harmonic Analysis, Theta Functions and Function Algebras on a Nilmanifold*, Lecture Notes in Mathematics, No. 436, Springer–Verlag, New York, 1975.

[B] R. Balian, *Un principe d'incertitude fort en théorie du signal ou en mécanique quantique*, C. R. Acad. Sci. Paris **292** (1981), 1357–1362.

[Ba1] G. Battle, *Heisenberg proof of the Balian-Low theorem*, Lett. Math. Phys. **15** (1988), 175–177.

[Ba2] ———, *Phase space localization theorem for ondelettes*, J. Math. Phys. **30** (1989), 2195–2196.

[BHW] J. Benedetto, C. Heil, and D. Walnut, *Remarks on the proof of the Balian–Low theorem*, preprint.

[C] A. Cohen, *Ondelettes, analysis multirésolutions et filters mirroirs en quadrature*, Annales de l'institute Henri Poincaré **7** (1990), 439–459.

[D1] I. Daubechies, *The wavelet transform, time-frequency localization and signal analysis*, IEEE Trans. Inform. Th. **39** (1990), 961–1005.

[D2] ———, *Orthonormal bases of compactly suported wavelets*, Comm. Pure Appl. Math. **41** (1988), 909–996.

[DGM] I. Daubechies, A. Grossmann, and Y. Meyer, *Painless nonorthogonal expansions*, J. Math. Phys. **27** (1986), 1271–1283.

[DJ] I. Daubechies and A.J.E.M. Janssen, *Two theorems on lattice expansions*, IEEE Trans. Inform. Th., to appear.

[DS] R.J. Duffin and A.C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc. **72** (1952), 341–366.

[G] D. Gabor, *Theory of communications*, J. Inst. Elec. Eng. (London) **93** (1946), 429–457.

[HW] C. Heil and D. Walnut, *Continuous and discrete wavelet transforms*, SIAM Review **31** (1989), 628–666.

[J] A.J.E.M. Janssen, *Bargmann transform, Zak transform, and coherent states*, J. Math. Phys. **23** (1982), 720–731.

[L] F. Low, *Complete sets of wave packets*, "A Passion for Physics—Essays in Honor of Geoffrey Chew," C. DeTar, et al., ed., World Scientific, Singapore (1985), 17–22.

[Ma] S. Mallat, *Multiresolution approximations and wavelet orthonormal bases of* $L^2(\mathbf{R})$, Trans. Amer. Math. Soc. **315** (1989), 69–87.

[M] Y. Meyer, *Principe d'incertitude, bases hilbertiennes et algèbres d'opérateurs*, Séminaire Bourbaki **662** (1985-86).

The MITRE Corporation
McLean, Virginia 22102

MSC 1991: Primary 42A99
Secondary 42A65, 46B15

# ERRATA

**Note:** This errata listing is not included in the published version of this paper.

The following paragraph, which appears on page 21 of the published version of this paper, is incorrect:

> First, however, note that while the dual frame of any Gabor frame is itself a Gabor frame, it is not always the case that the dual frame of any wavelet frame is itself a wavelet frame [D1]. However, this is true for *exact* wavelet frames. For, if $\{\psi_{jk}\}$ is an exact wavelet frame then there exists a unique function $\tilde{\psi} \in L^2(\mathbf{R})$ such that $\langle \psi_{jk}, \tilde{\psi} \rangle = \delta_j \, \delta_k$, namely $\tilde{\psi} = S^{-1}\psi$. It follows then that $\langle \psi_{jk}, \tilde{\psi}_{j'k'} \rangle = \delta_{jj'} \, \delta_{kk'}$, whence $\{\tilde{\psi}_{jk}\}$ is the dual frame to $\{\psi_{jk}\}$. We have then:

In fact, the dual frame of an arbitrary exact wavelet frame need not be a wavelet frame (see [D1] for a counterexample). The paragraph above should be replaced by the following corrected paragraph:

> First, however, note that while the dual frame of any Gabor frame is itself a Gabor frame, it is not always the case that the dual of a wavelet frame is also a wavelet frame [D1]. This may fail even if the frame is exact. However, a wavelet frame is of practical interest only when its dual is also a wavelet frame. Therefore, we restrict our attention to frames of this type, i.e., whenever we consider a wavelet frame $\{\psi_{jk}\}$ in this section, it is understood that its dual frame has the form $\{\tilde{\psi}_{jk}\}$ for some $\hat{\psi} \in L^2(\mathbf{R})$. We have then: