The HRT Conjecture and the Zero Divisor Conjecture for the Heisenberg Group

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Summary. This chapter reports on the current status of the HRT Conjecture (also known as the Linear Independence of Time-Frequency Shifts Conjecture), and discusses its relationship with a longstanding conjecture in algebra known as the Zero Divisor Conjecture.

Key words: Gabor systems, Heisenberg group, HRT Conjecture, indicable group, Linear Independence of Time-Frequency Shifts Conjecture, time-frequency analysis, wavelet systems, Zero Divisor Conjecture

1 Introduction

The building blocks of Gabor and wavelet systems are *translations* (or *time shifts*):

$$T_a g(x) \;=\; g(x-a), \qquad \text{where } a \in \mathbb{R};$$

modulations (or *frequency shifts*):

$$M_b g(x) \;=\; e^{2\pi i b x}\, g(x), \qquad \text{where } b \in \mathbb{R};$$

and *dilations*:

$$D_r g(x) \;=\; r^{1/2} g(rx), \qquad \text{where } r > 0.$$

Gabor systems employ the compositions

$$M_b T_a g(x) \;=\; e^{2\pi i b x} g(x-a),$$

which are called *time-frequency shifts*, while wavelet systems use the compositions

$$D_r T_a g(x) \;=\; r^{1/2} g(rx-a),$$

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which are called *time-scale shifts*.

Explicitly, a Gabor system has the form

$$\mathcal{G}(g, \Lambda) = \{M_b T_a g\}_{(a,b) \in \Lambda},$$

where Λ is an index set contained in $\mathbb{R}^2$, while a wavelet system is

$$\mathcal{W}(g, \Gamma) = \{D_r T_a g\}_{(a,r) \in \Gamma},$$

where $\Gamma \subset \mathbb{R} \times (0, \infty)$. Typically we are interested in constructing Gabor systems or wavelet systems that are bases or frames for $L^2(\mathbb{R})$. In this case Λ or Γ will be countable index sets. However, the "local" properties of Gabor and wavelet systems (by which we mean the properties of finite subsets) are often of key interest. In particular, given any set of vectors in a vector space, one of the first and most fundamental questions we can ask about this set is whether it is finitely linearly independent, i.e., whether every finite subset of the collection is linearly independent. It is known that there exist nontrivial functions $g \in L^2(\mathbb{R})$ and finite sets Γ such that the finite wavelet system $\mathcal{W}(g, \Gamma)$ is linearly dependent. It is *not known* whether there exist nontrivial functions $g \in L^2(\mathbb{R})$ and finite sets Λ such that the finite Gabor system $\mathcal{G}(g, \Lambda)$ is linearly dependent. The HRT Conjecture is that finite Gabor systems are indeed linearly independent.

In this chapter we will give a short report on the current status of the HRT Conjecture, and also comment on its relation to a longstanding conjecture in algebra known as the *Zero Divisor Conjecture*. We begin in Section 2 with some examples that illustrate that finite wavelet systems can be linearly dependent. We formulate the HRT Conjecture in Section 3, and in Section 4 we review some of the main partial results related to linear independence of time-frequency shifts that are currently known. Finally, in Section 5 we discuss the relationship between the HRT Conjecture and the Zero Divisor Conjecture.

2 Linear Dependence of Time-Scale Shifts

The fact that wavelet systems *can be linearly dependent* is the starting point for the construction of compactly supported wavelet bases through a multiresolution analysis. The first step in the construction of a compactly supported wavelet via this method is actually the construction of a compactly supported *scaling function*. A scaling function is a function $\varphi \in L^2(\mathbb{R})$ that satisfies a *refinement equation* of the form

$$\varphi(x) = \sum_{k=0}^{N} c_k \, \varphi(2x - k)$$

and additionally is such that $\{\varphi(x-k)\}_{k \in \mathbb{Z}}$ is an orthonormal system in $L^2(\mathbb{R})$. While the integer translates of a scaling function are orthonormal, and hence linearly independent, if we rewrite the refinement equation as

$$\varphi \;=\; \sum_{k=0}^{N} 2^{-1/2} c_k \, D_2 T_k \varphi,$$

then we see that the refinement equation is a statement that the collection of time-scale shifts

$$\{D_r T_a \varphi\}_{(a,r)\in\Gamma}$$

with

$$\Gamma \;=\; \{(0,1)\} \;\cup\; \{(k,2) : k = 0,\dots,N\}$$

is linearly dependent. Here is an example.

Example 1. The *Haar wavelet* is

$$\psi \;=\; \chi_{[0,\frac{1}{2}]} \;-\; \chi_{[\frac{1}{2},1]}.$$

Haar proved directly in [Haa10] that

$$\bigl\{D_{2^n} T_k \psi\bigr\}_{k,n\in\mathbb{Z}},$$

which today we call the *Haar system*, is an orthonormal basis for $L^2(\mathbb{R})$. Of course, since the Haar system is a collection of orthonormal functions, it is finitely linearly independent. However, if we wish to construct the Haar system using the modern framework of multiresolution analysis, we first begin by constructing the corresponding scaling function. For the Haar system, the scaling function is the *box function*

$$\varphi \;=\; \chi_{[0,1]}.$$

This function satisfies the refinement equation

$$\varphi(x) \;=\; \varphi(2x) \;+\; \varphi(2x-1).$$

Consequently, the set of three functions

$$\bigl\{\varphi,\, D_2\varphi,\, D_2T_1\varphi\bigr\}$$

is linearly dependent. Once we have the scaling function, the machinery of multiresolution analysis tells us that the wavelet

$$\psi(x) \;=\; \varphi(2x) \;-\; \varphi(2x-1)$$

can be used to generate an orthonormal basis for $L^2(\mathbb{R})$. $\diamondsuit$

For a detailed description of what a multiresolution analysis is and how it leads to a wavelet orthonormal basis, we refer to [Hei11, Chap. 12] or [Dau92].

Here is a modern example of a compactly supported scaling function.

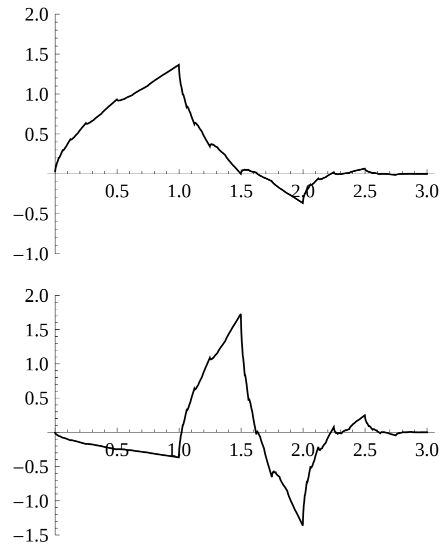

Fig. 1. The Daubechies $\mathbf{D}_4$ scaling function (top), and the corresponding wavelet $\mathbf{W}_4$ (bottom).

Example 2. The *Daubechies* $\mathbf{D}_4$ *scaling function* is the function that satisfies the refinement equation

$$\begin{aligned}\mathbf{D}_4(x) \;&=\; \tfrac{1+\sqrt{3}}{4}\,\mathbf{D}_4(2x) \;+\; \tfrac{3+\sqrt{3}}{4}\,\mathbf{D}_4(2x-1)\\ &\qquad +\; \tfrac{3-\sqrt{3}}{4}\,\mathbf{D}_4(2x-2) \;+\; \tfrac{1-\sqrt{3}}{4}\,\mathbf{D}_4(2x-3). \qquad (1)\end{aligned}$$

It can be shown that there is a unique (up to scale) compactly supported function $\mathbf{D}_4 \in L^2(\mathbb{R})$ that satisfies this refinement equation. This function, which is continuous and supported in the interval $[0,3]$ is illustrated in Figure 1. It is not obvious from the picture, but it is true that the integer translates $\{\mathbf{D}_4(x-k)\}_{k\in\mathbb{Z}}$ are orthonormal. The scaling function $\mathbf{D}_4$ generates a multiresolution analysis, and because of this it follows that the wavelet

$$\begin{aligned}\mathbf{W}_4(x) \;&=\; \tfrac{1-\sqrt{3}}{4}\,\mathbf{D}_4(2x) \;-\; \tfrac{3-\sqrt{3}}{4}\,\mathbf{D}_4(2x-1)\\ &\qquad +\; \tfrac{3+\sqrt{3}}{4}\,\mathbf{D}_4(2x-2) \;-\; \tfrac{1+\sqrt{3}}{4}\,\mathbf{D}_4(2x-3)\end{aligned}$$

can be used to generate an orthonormal basis for $L^2(\mathbb{R})$. Specifically, the wavelet system

$$\big\{D_{2^n} T_k \mathbf{W}_4\big\}_{k,n\in\mathbb{Z}}$$

is an orthonormal basis for $L^2(\mathbb{R})$. However, the point we are making here is that the refinement equation (1) implies that the finite collection of time-scale shifts

$$\big\{\mathbf{D}_4,\, D_2\mathbf{D}_4,\, D_2T_1\mathbf{D}_4,\, D_2T_2\mathbf{D}_4,\, D_2T_3\mathbf{D}_4\big\}$$

is *linearly dependent*. $\diamondsuit$

3 Gabor Systems and the HRT Conjecture

Now we turn to Gabor systems. We will need to employ the Fourier transform, which for an integrable function g we normalize as

$$\widehat{g}(\xi) \;=\; \int_{-\infty}^{\infty} g(x)\, e^{-2\pi i \xi x}\, dx.$$

The Fourier transform extends to a unitary operator that maps $L^2(\mathbb{R})$ onto itself. We have the following relations between translation, modulation, and the Fourier transform:

$$(T_a g)^\wedge \;=\; M_{-a}\widehat{g} \qquad \text{and} \qquad (M_b g)^\wedge(\xi) \;=\; T_b \widehat{g}.$$

Although translations and modulations do not commute, we have

$$M_b T_a g(x) \;=\; e^{2\pi i b x} g(x-a) \qquad \text{and} \qquad T_a M_b g(x) \;=\; e^{2\pi i b(x-a)} g(x-a),$$

and therefore

$$T_a M_b g \;=\; e^{-2\pi i a b} M_b T_a g. \tag{2}$$

We can easily show that if we only consider translations alone, then we will always obtain linearly independent collections. To see why, let $g \in L^2(\mathbb{R})$ be nontrivial (not zero a.e.) and let $a_1, \dots, a_N$ be any distinct points in $\mathbb{R}$. If

$$\sum_{k=1}^{N} c_k\, g(x-a_k) \;=\; 0 \text{ a.e.},$$

then by applying the Fourier transform we obtain

$$\sum_{k=1}^{N} c_k\, e^{-2\pi i a_k \xi}\, \widehat{g}(\xi) \;=\; 0 \text{ a.e.}$$

However, $\widehat{g}$ is not the zero function, so this implies that

$$\sum_{k=1}^{N} c_k\, e^{-2\pi i a_k \xi} \;=\; 0 \text{ a.e.}$$

Multiplying through by $e^{2\pi i a_1 \xi}$ we obtain

$$m(\xi) \;=\; c_1 \;+\; \sum_{k=2}^{N} c_k\, e^{-2\pi i (a_k - a_1)\xi} \;=\; 0 \text{ a.e.}$$

But m is a nonharmonic trigonometric polynomial, and therefore can be extended to an analytic function on the complex plane. An analytic function cannot vanish on any set that has an accumulation point without being identically zero. Hence $m(\xi) = 0$ for every ξ, which implies that $c_1 = 0$. Iterating, we obtain $c_2 = \cdots = c_N = 0$.

Similarly, any finite set of modulations of g is linearly independent. However, as soon as we combine translations with modulations, the situation becomes much less clear. In particular, if we fix a finite Gabor system

$$\bigl\{M_{b_k} T_{a_k} g : k = 1, \dots, N\bigr\}$$

and we assume that

$$\sum_{k=1}^{N} c_k \, M_{b_k} T_{a_k} g \;=\; 0 \text{ a.e.}, \tag{3}$$

then all that we obtain by applying the Fourier transform is that

$$\sum_{k=1}^{N} c_k \, T_{b_k} M_{-a_k} \widehat{g} \;=\; 0 \text{ a.e.}$$

In view of the commutation relation in equation (2), we can rewrite this as

$$\sum_{k=1}^{N} c_k e^{2\pi i a_k b_k} \, M_{-a_k} T_{b_k} \widehat{g} \;=\; 0 \text{ a.e.},$$

which is an equation of exactly the same nature as equation (3). The Fourier transform yields no simplification here.

At first glance, this may seem to be only a minor stumbling block—surely another approach, perhaps another transform, will show that Gabor systems are finitely linearly independent. Yet this most basic question about a set of vectors in $L^2(\mathbb{R})$ remains unanswered today. The following conjecture, today known as the *Linear Independence of Time-Frequency Translates Conjecture* or the *HRT Conjecture*, first appeared in print in [HRT96].

Conjecture 1 (HRT Conjecture). If g in $L^2(\mathbb{R})$ is not the zero function and $\Lambda = \bigl\{(a_k, b_k) : k = 1, \dots, N\bigr\}$ is a set of finitely many distinct points in $\mathbb{R}^2$, then

$$\mathcal{G}(g, \Lambda) \;=\; \bigl\{M_{b_k} T_{a_k} g : k = 1, \dots, N\bigr\}$$

is linearly independent. $\diamondsuit$

Various partial results on this conjecture are known (we will discuss these in Section 4). However, even the following very restricted version of the conjecture is open as of the time of writing. Here $\mathcal{S}(\mathbb{R})$ denotes the Schwartz class of all infinitely differentiable functions which, along with all of their derivatives, have faster than polynomial decay at infinity.

Conjecture 2 (HRT Subconjecture). If $g \in \mathcal{S}(\mathbb{R}) \backslash \{0\}$, then

$$\bigl\{g(x),\; g(x-1),\; e^{2\pi i x} g(x),\; e^{2\pi i \sqrt{2} x} g(x - \sqrt{2})\bigr\} \tag{4}$$

is linearly independent. $\diamondsuit$

We observe that the set of functions that appears in equation (4) is the Gabor system $\mathcal{G}(g, \Lambda)$ where

$$\Lambda = \bigl\{(0,0),\ (1,0),\ (0,1),\ (\sqrt{2}, \sqrt{2})\bigr\}. \tag{5}$$

4 Partial Results

We will briefly list some of the main partial results that are currently available on the HRT Conjecture. These are approximately ordered chronologically, but due to vastly differing time lengths from research to publication, this particular ordering should not be interpreted as anything other than a convenience for presentation purposes. Further, no attempt has been made to state all partial results from every paper, nor to state them precisely; this list simply serves as a brief summary of the literature on Conjecture 1. A few of the results below extend to systems in $L^2(\mathbb{R}^d)$, but for the most part there are substantial obstacles in moving to higher dimensions. For a discussion of why this is so we refer to the survey paper [Hei06] (that paper also presents context, motivation, and related results that are not discussed here).

The paper [HRT96] which originally presented the HRT conjecture included the following results.

- $\mathcal{G}(g, \Lambda)$ is independent if g is compactly supported, or just supported within a half-line $[a, \infty)$ or $(-\infty, a]$.
- $\mathcal{G}(g, \Lambda)$ is independent if $g(x) = p(x)e^{-x^2}$, where p is a polynomial.
- $\mathcal{G}(g, \Lambda)$ is independent if $N \leq 3$.
- If A is a 2×2 invertible matrix with $|\det(A)| = 1$, and $z \in \mathbb{R}^2$, then $\mathcal{G}(g, \Lambda)$ is independent for all nontrivial g if and only if $\mathcal{G}(g, A(\Lambda) + z)$ is independent for all nontrivial g.
- If $\mathcal{G}(g, \Lambda)$ is independent, then there exists an $\varepsilon > 0$ such that $\mathcal{G}(g, \Lambda')$ is independent for all $\Lambda' = \{(a_k', b_k') : k = 1, \dots, N\}$ such that $|a_k - a_k'| < \varepsilon$ and $|b_k - b_k'| < \varepsilon$ for $k = 1, \dots, N$.
- If $\mathcal{G}(g, \Lambda)$ is independent, then there exists an $\varepsilon > 0$ such that $\mathcal{G}(h, \Lambda)$ is independent for all $\|h - g\|_2 < \varepsilon$.

Since the set of compactly supported functions is dense in $L^2(\mathbb{R})$, if Λ is a fixed finite set then by applying the final perturbation result listed above it follows that there exists an open, dense subset $\mathcal{U}$ of $L^2(\mathbb{R})$ such that $\mathcal{G}(g, \Lambda)$ is linearly independent for all $g \in \mathcal{U}$.

Linnell proved in [Lin99] that

- $\mathcal{G}(g, \Lambda)$ is independent if Λ is a finite subset of a translate of a full-rank lattice in $\mathbb{R}^2$, i.e., if $\Lambda \subset A(\mathbb{Z}^2) + z$ where A is an invertible 2×2 matrix (with any nonzero determinant) and $z \in \mathbb{R}^2$.

In particular, any set of three points in $\mathbb{R}^2$ is contained in a translate of a full-rank lattice, so this gives another proof that $\mathcal{G}(g,\Lambda)$ is independent if $N \le 3$. However, four points in $\mathbb{R}^2$ need not be contained in a translate of a full-rank lattice. In particular, the set of points Λ given in equation (5) is not contained in any translate of any full-rank lattice.

In [CL01], Christensen and Lindner obtain

- estimates of the frame bounds of a finite Gabor system $\mathcal{G}(g,\Lambda)$.

In principle, sufficiently strong estimates of the frame bounds of finite Gabor systems could be combined with the perturbation theorems in [HRT96] to yield a proof of the HRT Conjecture. Unfortunately, the results of [CL01] do not seem to be conducive for advancing this type of approach. A short introduction to frame theory can be found in [Hei13].

Rzeszotnik, in an unpublished work, proved that

- $\mathcal{G}(g,\Lambda)$ is independent if

$$\Lambda \;=\; \bigl\{(0,0),\ (1,0),\ (0,1),\ (\sqrt{2},0)\bigr\}.$$

This set Λ, like the one given in equation (5), is not contained in any translate of a full-rank lattice in $\mathbb{R}^2$. On the other hand, the points in this set Λ lie on two parallel lines, while the four points in equation (5) do not.

An equivalent formulation of the HRT Conjecture is that if $c_1,\dots,c_N$ are not all zero, then the kernel of the operator

$$T \;=\; \sum_{k=1}^{N} c_k M_{b_k} T_{a_k}$$

is $\{0\}$, or, in other words, 0 is not an eigenvalue of T. This suggests that it would be interesting to investigate the spectrum of such a linear combination of time-frequency shift operators. Now, if T had a nonzero eigenvalue λ, then for some nonzero function g we would have

$$Tg \;=\; \sum_{k=1}^{N} c_k M_{b_k} T_{a_k} g \;=\; \lambda g.$$

Setting $a_0 = b_0 = 0$ and $c_0 = -\lambda$, it follows that

$$\sum_{k=0}^{N} c_k M_{b_k} T_{a_k} g \;=\; 0,$$

and hence 0 is an eigenvalue of the operator $S = \sum_{k=0}^{N} c_k M_{b_k} T_{a_k}$, which is simply another finite linear combination of time-frequency shift operators. Consequently we can restate the HRT Conjecture in terms of eigenvalues as follows.

Conjecture 3 (Spectral Version of the HRT Conjecture). If

$$\Lambda \;=\; \bigl\{(a_k, b_k) : k = 1, \dots, N\bigr\}$$

is a set of finitely many distinct points in $\mathbb{R}^2$ and $c_1, \dots, c_N$ are not all zero, then the point spectrum of

$$T \;=\; \sum_{k=1}^{N} c_k M_{b_k} T_{a_k}$$

is empty (i.e., T has no eigenvalues). ♢

Balan proved in [Bal08] that

- the operator T cannot have an isolated eigenvalue of finite multiplicity.

Consequently, *if* the point spectrum of T is not empty, then it can only contain eigenvalues of infinite multiplicity, or eigenvalues that also belong to the continuous spectrum of T. Further results on the spectral properties of T have been obtained by Balan and Krishtal in [BK10]. In particular, a consequence of the results of that paper is that if Λ is the set of four points specified in equation (5), then the corresponding operator T has no isolated eigenvalues.

Linnell's proof that $\mathcal{G}(g, \Lambda)$ is linearly independent if Λ is a finite subset of a shift of a full-rank lattice was obtained through the machinery of von Neumann algebras. In [BS10],

- a different proof of Linnell's result was derived through time-frequency methods.

Interestingly, Demeter and Gautam obtained in [DG12]

- yet another proof of Linnell's result, this time based on the spectral theory of random Schrödinger operators.

Demeter in [Dem10], and Demeter and Zaharescu in [DZ12], focused on the case where Λ contains four distinct points. These two papers prove that

- $\mathcal{G}(g, \Lambda)$ is independent if $\#\Lambda = 4$ and Λ is a subset of two parallel lines in $\mathbb{R}^2$.

In particular, this recovers the result by Rzeszotnik that was stated earlier.

One of the results obtained in [HRT96] is that $\mathcal{G}(g, \Lambda)$ is linearly independent if g has the particular form $g(x) = p(x) e^{-x^2}$ where p is a polynomial. No other results related to decay conditions seem to have been obtained until quite recently. In [BS13] it is proved that

- $\mathcal{G}(g, \Lambda)$ is linearly independent if

$$\lim_{x \to \infty} |g(x)|\, e^{cx^2} \;=\; 0 \;\; \text{for all } c > 0,$$

and

- $\mathcal{G}(g, \Lambda)$ is linearly independent if

$$\lim_{x\to\infty} |g(x)|\, e^{cx \log x} \;=\; 0 \quad \text{for all } c > 0.$$

However, the case where $|g(x)|\, e^{cx} \to 0$ for all $c > 0$ remains open. Note that while functions in the Schwartz class $\mathcal{S}(\mathbb{R})$ have extremely rapid decay, they need not decay at an exponential rate.

Benedetto and Bourouihiya also obtained partial results related to decay. Without stating their results precisely, they proved in their paper [BB14] that

- $\mathcal{G}(g, \Lambda)$ is linearly independent if g is ultimately positive and $b_1, \dots, b_N$ are independent over $\mathbb{Q}$,

and

- $\mathcal{G}(g, \Lambda)$ is linearly independent if $\#\Lambda = 4$, g is ultimately positive, and $g(x)$ and $g(-x)$ are ultimately decreasing.

The most recent paper related to the HRT Conjecture of which we are aware is [Grö14] by Gröchenig. He proves that

- the lower Riesz bound of a finite section of a Gabor frame that is not a Riesz basis converges to zero, and in many cases this convergence is super-fast.

A Gabor frame that is not a Riesz basis is "globally redundant" in some sense. Even so, if the HRT Conjecture is true then every finite subset of such a frame must be linearly independent. Gröchenig's result implies that, from a numerical point of view, such finite subsets rapidly become "nearly dependent" as their size increases. To quote Gröchenig, this "illustrates the spectacular difference between a conjectured mathematical truth and a computationally observable truth."

Finally, although they do not obtain results directly about Conjecture 1, we mention that the papers of Kutyniok [Kut02] and Rosenblatt [Ros08] consider some generalizations of the conjecture.

5 The Zero Divisor Conjecture

In this section we discuss the relation (or lack thereof) between the HRT Conjecture and the Zero Divisor Conjecture.

First we review some terminology. Let G be a group (with the group operation written multiplicatively). The *complex group algebra* of G is the set of all formal finite linear combinations of elements of G. We write this as

$$\mathbb{C}G \;=\; \biggl\{ \sum_{g\in G} c_g g : c_g \in \mathbb{C} \text{ with only finitely many } c_g \neq 0 \biggr\}.$$

The natural operation of addition in $\mathbb{C}G$ is defined by

$$\sum_{g\in G} c_g g \;+\; \sum_{g\in G} d_g g \;=\; \sum_{g\in G} (c_g + d_g) g,$$

and we define multiplication in $\mathbb{C}G$ by

$$\Biggl(\sum_{g\in G} c_g g\Biggr)\Biggl(\sum_{h\in G} d_h g\Biggr) \;=\; \sum_{g\in G}\sum_{h\in G} c_g d_h gh \;=\; \sum_{g\in G}\Biggl(\sum_{h\in G} c_{gh^{-1}} d_h\Biggr) g.$$

All of the above sums are well-defined since only finitely many terms in any sum are nonzero. More generally, the field $\mathbb{C}$ can be replaced by other fields, but we will restrict our attention here to the complex field.

Suppose that g is an element of G that has finite order $n > 1$. If we let e denote the identity element of G and set

$$\alpha \;=\; g - e \qquad \text{and} \qquad \beta \;=\; g^{n-1} + \cdots + g + e,$$

then

$$\begin{aligned} \alpha\beta &= (g-e)(g^{n-1} + \cdots + g + e) \\ &= (g^n + \cdots + g^2 + g) \;-\; (g^{n-1} + \cdots + g + e) \\ &= g^n - e \\ &= 0. \end{aligned}$$

Thus, if G has any nontrivial elements of finite order then $\mathbb{C}G$ has zero divisors. What happens if there are no nontrivial elements of G that have finite order? In general the answer is unknown, and this is the context of the following *Zero Divisor Conjecture*.

Conjecture 4 (Zero Divisor Conjecture). Let G be a torsion-free group (i.e., G contains no elements of finite order other than the identity). If α, $\beta \in \mathbb{C}G$, then

$$\alpha \neq 0 \text{ and } \beta \neq 0 \implies \alpha\beta \neq 0. \qquad \diamondsuit$$

This conjecture is sometimes attributed to Kaplansky (for example, this is the attribution stated by Wikipedia in the article “Group Ring”). Variants of the conjecture seem to have appeared in the literature over time. Higman proved one version of the Zero Divisor Conjecture for “locally indicable” groups in 1940 [Hig40]. Two surveys of the conjecture published in 1977 are the paper [Sni77] by Snider and Chapter 13 of Passman’s text [Pas77]. According to Snider, Higman’s result was “essentially all that was known until 1974, when Formanek proved the conjecture for supersolvable groups” (see [For73]). Since then various results have been obtained, but the conjecture remains open in the generality stated. A survey by Linnell of *analytic versions of the zero divisor conjecture* can be found in [Lin98].

There is a natural group associated with time-frequency analysis, the *Heisenberg group* $\mathbf{H}$, and therefore we can consider the Zero Divisor Conjecture for the special case that $G = \mathbf{H}$. There are several versions of the Heisenberg group and many isomorphic definitions of each of these. For our purposes, it is simplest to consider the *reduced Heisenberg group* defined by

$$\mathbf{H} \;=\; \bigl\{zM_bT_a : z \in \mathbb{T}, a, b \in \mathbb{R}\bigr\},$$

where $\mathbb{T}$ is the unit circle in the complex plane, i.e.,

$$\mathbb{T} \;=\; \{z \in \mathbb{C} : |z| = 1\}.$$

In other words, with this definition $\mathbf{H}$ is the set of all unit modulus scalar multiples of time-frequency operators. The group operation is simply composition of operators, which by equation (2) follows the rule

$$(zM_bT_a)\,(wM_dT_c) \;=\; zwe^{-2\pi iad}M_{b+d}T_{a+c}.$$

The Heisenberg group is noncommutative (but even so, as a locally compact group it turns out that left and right Haar measure on $\mathbf{H}$ coincide, and therefore $\mathbf{H}$ is *unimodular*). If we fix any element zM_bT_a of $\mathbf{H}$, then the nth power of this element has the form

$$(zM_bT_a)^n \;=\; w_nM_{nb}T_{na},$$

where w_n is a scalar that has unit modulus. Therefore no element of $\mathbf{H}$ other than the identity $I = M_0T_0$ has finite order. Consequently, $\mathbf{H}$ is torsion-free.

Since $\mathbf{H}$ consists of scalar multiples of time-frequency shift operators, its group algebra $\mathbb{C}\mathbf{H}$ is the vector space of all finite linear combinations of time-frequency shift operators:

$$\mathbb{C}\mathbf{H} \;=\; \left\{\sum_{k=1}^{N} c_kM_{b_k}T_{a_k} : N > 0, c_k \in \mathbb{C}, a_k \in \mathbb{R}, b_k \in \mathbb{R}\right\}.$$

The question we wish to answer is whether $\mathbb{C}\mathbf{H}$ has any zero divisors. So, suppose that α and β are nonzero elements of $\mathbb{C}\mathbf{H}$ such that $\alpha\beta = 0$. Since α and β belong to $\mathbb{C}\mathbf{H}$, we can write

$$\alpha \;=\; \sum_{j=1}^{M} z_jM_{b_j}T_{a_j} \qquad\text{and}\qquad \beta \;=\; \sum_{k=1}^{N} w_kM_{d_k}T_{c_k},$$

with $(a_j, b_j) \neq (a_{j'}, b_{j'})$ whenever $j \neq j'$, and similarly for (c_k, d_k). Since α and β are each nonzero, we can assume without loss of generality that z_j and w_k are nonzero complex scalars for every j and k. For simplicity of notation, let t_{jk} be the nonzero scalar

$$t_{jk} \;=\; z_jw_ke^{-2\pi ia_jd_k}.$$

Then we have

$$\alpha\beta \;=\; \sum_{j=1}^{M}\sum_{k=1}^{N} t_{jk} M_{b_j+d_k} T_{a_j+c_k} \;=\; 0. \tag{6}$$

Many of the values of $a_j + c_k$ or $b_j + d_k$ in equation (6) may coincide. For convenience, and without loss of generality, we can assume that the a_j and c_k are ordered (possibly with duplicates). That is, we can assume that

$$a_1 \;\le\; a_2 \;\le\; \cdots \;\le\; a_M \qquad \text{and} \qquad c_1 \;\le\; c_2 \;\le\; \cdots \;\le\; c_N.$$

Let

$$I \;=\; \big\{(j,k) : a_j + c_k = a_M + c_N\big\}.$$

Since $\alpha\beta = 0$, for every $f \in L^2(\mathbb{R})$ we have

$$\sum_{(j,k)\notin I} t_{jk} M_{b_j+d_k} T_{a_j+c_k} f(x) \;+\; \sum_{(j,k)\in I} t_{jk} M_{b_j+d_k} T_{a_M+c_N} f(x) \;=\; 0 \;\text{ a.e.} \tag{7}$$

Now, if $(j,k) \notin I$, then $a_j + c_k < a_M + c_N$. Since there are only finitely many choices, if we set

$$r \;=\; \max\{a_j + c_k : (j,k) \notin I\} \qquad \text{and} \qquad s \;=\; a_M + c_N,$$

then we have $r < s$.

Let $f \in L^2(\mathbb{R})$ be any function that is nonzero everywhere on $(-\infty, 0)$ and zero on $[0, \infty)$. For example, we could take

$$f(x) \;=\; e^x \,\chi_{(-\infty,0)}(x).$$

If $r < x < s$, then for each $(j,k) \notin I$ we have $x - a_j - c_k \ge 0$ and therefore $f(x - a_j - c_k) = 0$. Thus for such x the first summation in equation (7) is zero. Consequently, only the second summation in equation (7) remains for such x. Simplifying, it follows that for a.e. x in the interval (r, s) we have

$$\begin{aligned} 0 \;&=\; \sum_{(j,k)\in I} t_{jk} M_{b_j+d_k} T_{a_M+c_N} f(x) \\ &=\; \sum_{(j,k)\in I} t_{jk} e^{2\pi i(b_j+d_k)x} f(x-s) \\ &=\; p(x)\, f(x-s), \end{aligned}$$

where

$$p(x) \;=\; \sum_{(j,k)\in I} t_{jk} e^{2\pi i(b_j+d_k)x}. \tag{8}$$

All of the sums above are finite and $f(x-s) \neq 0$ for $r < x < s$, so this implies that

$$p \;=\; 0 \text{ a.e. on } (r, s).$$

But p is a nonharmonic trigonometric polynomial and therefore cannot vanish on any set of positive measure (indeed, p can have at most countably many zeros). Therefore, we must have $p(x) = 0$ for every $x \in \mathbb{R}$.

We are tempted to conclude from this that every t_{jk} is zero, but we cannot do this because some of the values $b_j + d_k$ may coincide. To deal with this, note that if $(j,k) \in I$ then $a_j = a_M$ and $c_k = c_N$. Therefore, if we choose two distinct points (j,k) and (j',k') in I, then $a_j = a_M = a_{j'}$. Since (a_j, b_j) must be distinct from $(a_{j'}, b_{j'})$, this implies that $b_j \neq b_{j'}$. Consequently there is a *unique* j_0 such that

$$b_{j_0} \;=\; \max\{b_j : (j,k) \in I\}.$$

Similarly, there is a unique k_0 such that

$$d_{k_0} \;=\; \max\{d_k : (j,k) \in I\}.$$

Therefore equation (8) can be rewritten as

$$p(x) \;=\; t_{j_0k_0} e^{2\pi i(b_{j_0}+d_{k_0})x} \;\;+ \sum_{(j,k)\in I,\, b_j+d_k<b_{j_0}+d_{k_0}} t_{jk} e^{2\pi i(b_j+d_k)x}.$$

As p is identically zero, this implies that $t_{j_0k_0} = 0$. However, this contradicts the fact that every t_{jk} is nonzero.

In summary, we have shown that the Zero Divisor Conjecture is true when we take G to be the Heisenberg group $\mathbf{H}$. We state this formally as a theorem.

Theorem 1. *If α and β are nonzero elements of $\mathbb{C}\mathbf{H}$, then $\alpha\beta \neq 0$.* $\diamondsuit$

The proof of Theorem 1 uses an ordering, or indexing, of the elements in the Heisenberg group. First we ordered the translations and examined the largest pair, then we ordered the modulations and again examined the largest pair. It will not be surprising, then, to see that there are general arguments which imply that the Heisenberg group satisfies the Zero Divisor Conjecture. In fact, we will see below that Higman's original work already implies that the Heisenberg group satisfies the Zero Divisor Conjecture.

A group is said to be *locally indicable* if each of its non-identity finitely generated subgroups maps homomorphically onto $\mathbb{Z}$. A locally indicable group has no non-trivial elements of finite order. Indeed, if $g \in G$ is an element of finite order, then the subgroup $\langle g \rangle$ generated by g is finite, so there cannot exist a homomorphism from $\langle g \rangle$ onto $\mathbb{Z}$.

Given a homomorphism $\gamma : H \to \mathbb{Z}$, we define the *degree* of $h \in H$ (relative to γ) to be $\gamma(h)$. The degree of an element depends on the particular homomorphism chosen, but in what follows we will not explicitly indicate this dependence. We say an element of the group ring $\mathbb{C}H$ is *homogeneous of degree a* (relative to γ) if it can be put in the form $\sum_{j=1}^{k} c_j e_j$, where for each $1 \leq j \leq k$ we have $\gamma(e_j) = a$. Any element of the group ring $\mathbb{C}H$ can be written as

$$P_1 + \cdots + P_p, \tag{9}$$

where each P_i is homogeneous of order a_i and $a_1 < a_2 < \cdots < a_p$.

Here is Higman's result for locally indicable groups, proved in [Hig40].

Theorem 2 (Higman). *If G is a locally indicable group, and if α and β are nonzero elements of $\mathbb{C}G$, then $\alpha\beta \neq 0$.*

Proof. Write $\alpha = m_1 g_1 + \cdots + m_K g_K$ and $\beta = n_1 h_1 + \cdots + n_L h_L$, where $m_1, \ldots, m_K$ and $n_1, \ldots, n_K$ are nonzero complex numbers and $g_1, \ldots, g_K$ and $h_1, \ldots, h_K$ are in G.

We proceed by induction on $K + L$. If $K + L = 2$, then $\alpha\beta = m_1 n_1 g_1 h_1$, which is not zero since $m_1 n_1 \neq 0$.

Let $n > 2$ and assume that $\alpha\beta \neq 0$ whenever the sum of the number of terms of α and β is less than n; that is when $K + L < n$. We will show that $\alpha\beta \neq 0$ whenever $K + L = n$.

Note that if $\alpha\beta = 0$, then $g_1^{-1}\alpha\beta h_1^{-1} = 0$ as well, so we may assume without loss of generality that g_1 and h_1 are the group identity element. Let H be the subgroup of G generated by

$$\{g_1, \ldots, g_K, h_1, \ldots, h_L\},$$

and let ϕ be a homomorphism from H onto $\mathbb{Z}$. Write

$$\alpha = \sum_{i=1}^{r} P_i \qquad \text{and} \qquad \beta = \sum_{j=1}^{s} Q_j$$

as in equation (9), where the degrees of $P_1, \ldots, P_r$ are $a_1 < \cdots < a_r$ and the degrees of $Q_1, \ldots, Q_s$ are $b_1 < \cdots < b_s$. Since we have assumed that g_1 and h_1 are the identity, we have that $\phi(g_1) = \phi(h_1) = 0$. Moreover, since ϕ is onto, it cannot map every element of H to zero, and therefore at least one of r or s must exceed one. The product $\alpha\beta$ has the form

$$\alpha\beta = P_1 Q_1 + \cdots + P_r Q_s,$$

where $P_1 Q_1$ is homogeneous of degree $a_1 + b_1$, $P_r Q_s$ is homogeneous of degree $a_r + b_s$, and the terms not listed have degrees strictly between $a_1 + b_1$ and $a_r + b_s$. Since $\alpha\beta \neq P_1 Q_1$, it follows that the sum of the number of terms in P_1 and Q_1 is less than n. Therefore, by the induction hypothesis, $P_1 Q_1 \neq 0$ and hence $\alpha\beta \neq 0$. $\square$

An easy first example of a group which admits a homomorphism onto $\mathbb{Z}$ is $\mathbb{Z}^N$; one choice of homomorphism is $\phi(m_1, \ldots, m_N) = m_1$. Since every finitely generated subgroup of an Abelian group is isomorphic to $\mathbb{Z}^N \bigoplus_{i=1}^{K} \mathbb{Z}_{a_i}$, it follows that every torsion-free Abelian group is locally indicable. In particular, $(\mathbb{R}, +)$, the real line under addition, is locally indicable.

We provide a direct proof that the Heisenberg group is locally indicable. For this argument, it will be most convenient to represent the Heisenberg group as

$$\mathbf{H} = \{(a,b,c) : a,b,c \in \mathbb{R}\},$$

with product

$$(a,b,c) \cdot (x,y,z) = (a+x, b+y, c+z+ay).$$

We gather some basic facts about $\mathbf{H}$ in the following lemma.

Lemma 1. *Let* $\mathbf{H}$ *denote the Heisenberg group.*

(a) $(x,y,z)^{-1} = (-x,-y,-z+xy)$.

(b) *The commutator* $\mathbf{H}' = \{aba^{-1}b^{-1} : a,b \in \mathbf{H}\}$ *is* $\{(0,0,z) : z \in \mathbb{R}\}$, *which is isomorphic to* $(\mathbb{R},+)$.

(c) $\mathbf{H}'$ *is the center of* $\mathbf{H}$ *and is a normal subgroup of* $\mathbf{H}$.

(d) $\mathbf{H}/\mathbf{H}'$ *is isomorphic to* $(\mathbb{R}^2,+)$. $\diamondsuit$

Proposition 1. *The Heisenberg group* $\mathbf{H}$ *is locally indicable.*

Proof. Let G be any finitely generated subgroup of $\mathbf{H}$. We must show that there exists a homomorphism that maps G onto $\mathbb{Z}$.

Case I: $G \subset \mathbf{H}'$. Since $\mathbf{H}'$ is isomorphic to $(\mathbb{R},+)$, this case follows from the local indicability of $(\mathbb{R},+)$.

Case II: $G \not\subset \mathbf{H}'$. Note that $G\mathbf{H}'/\mathbf{H}'$ is a normal subgroup of $\mathbf{H}/\mathbf{H}'$, and that $\mathbf{H}/\mathbf{H}'$ is isomorphic to $\mathbb{R}^2$. By the Second Isomorphism Theorem, there exists an isomorphism

$$\eta : G/(G \cap \mathbf{H}') \to G\mathbf{H}'/\mathbf{H}'.$$

Since G is finitely generated, so is $G/(G \cap \mathbf{H}')$, and therefore $G\mathbf{H}'/\mathbf{H}'$ is finitely generated as well. Therefore, since $\mathbb{R}^2$ is locally indicable, there is a homomorphism ϕ from $G\mathbf{H}'/\mathbf{H}'$ onto $\mathbb{Z}$. Consequently, if we let

$$\psi : G \to G/(G \cap \mathbf{H}')$$

be the natural onto homomorphism, then the composition $\phi \circ \eta \circ \psi$ is a surjective homomorphism of G onto $\mathbb{Z}$. $\square$

Corollary 1. *If* α *and* β *are non-zero elements of* $\mathbb{C}\mathbf{H}$, *then* $\alpha\beta \neq 0$. $\diamondsuit$

A group is said to be *indicable throughout* if *every* subgroup admits a homomorphism onto $\mathbb{Z}$. Since $\mathbf{H}$ contains $\mathbb{R}$ as a subgroup, and there is no homomorphism from $\mathbb{R}$ onto $\mathbb{Z}$, the Heisenberg group is not indicable throughout.

We remark that arguments similar to the ones above can be used to show that the affine group also has no non-trivial zero divisors. Even so, as we described in Section 2, time-scale shifts of functions in $L^2(\mathbb{R})$ are not necessarily linearly independent.

We have shown that no product of two nontrivial finite linear combinations of time-frequency shifts operators can be the zero operator. We close with a related, but simpler, observation.

Lemma 2. *The set of time-frequency shift operators*

$$\{M_b T_a : a, b \in \mathbb{R}\}$$

is a finitely linearly independent set in $\mathbb{C}\mathbf{H}$. *That is, if*

$$\Lambda \;=\; \{(a_k, b_k) : k = 1, \dots, N\}$$

is a set of finitely many distinct points in $\mathbb{R}^2$ *and*

$$\sum_{k=1}^{N} c_k M_{b_k} T_{a_k} \;=\; 0,$$

then $c_1 = \cdots = c_N = 0$.

Proof. If $\sum_{k=1}^{N} c_k M_{b_k} T_{a_k}$ is the zero operator, then

$$\sum_{k=1}^{N} c_k M_{b_k} T_{a_k} f \;=\; 0 \quad \text{a.e.} \tag{10}$$

for *every* function $f \in L^2(\mathbb{R})$. Yet we know that there are at least *some* functions that have linearly independent time-frequency translates. For example, by the partial results reviewed earlier this is true for every compactly supported function, and for the Gaussian function. Taking f to be one of these functions, equation (10) implies that $c_1 = \cdots = c_N = 0$. □

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