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MODULATION SPACES AND PSEUDODIFFERENTIAL OPERATORS

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We use methods from time-frequency analysis to study boundedness and trace-class properties of pseudodifferential operators. As natural symbol classes, we use the modulation spaces on $\mathbf{R}^{2d}$, which quantify the notion of the time-frequency content of a function or distribution. We show that if a symbol σ lies in the modulation space $M_{\infty,1}(\mathbf{R}^{2d})$, then the corresponding pseudodifferential operator is bounded on $L^2(\mathbf{R}^d)$ and, more generally, on the modulation spaces $M_{p,p}(\mathbf{R}^d)$ for $1 \le p \le \infty$. If σ lies in the modulation space $M_{2,2}^s(\mathbf{R}^{2d}) = L_s^2(\mathbf{R}^{2d}) \cap H^s(\mathbf{R}^{2d})$, i.e., the intersection of a weighted L^2-space and a Sobolev space, then the corresponding operator lies in a specified Schatten class. These results hold for both the Weyl and the Kohn-Nirenberg correspondences. Using recent embedding theorems of Lipschitz and Fourier spaces into modulation spaces, we show that these results improve on the classical Calderòn–Vaillancourt boundedness theorem and on Daubechies' trace-class results.

1. Introduction.

The Weyl and Kohn-Nirenberg correspondences are formalisms which bijectively associate to any continuous linear operator $L: \mathcal{S}(\mathbf{R}^d) \to \mathcal{S}'(\mathbf{R}^d)$ a distributional symbol $\sigma \in \mathcal{S}'(\mathbf{R}^{2d})$. These correspondences play important roles in many fields, including partial differential equations, quantum mechanics, and signal processing. Because of their invariance properties under translations and dilations, it is customary to treat pseudodifferential operators with Littlewood–Paley theory [CM78], [Ste93]. This approach culminates in Meyer's treatment of pseudodifferential operators with wavelet bases [Mey90]. Alternatively, pseudodifferential operators can be represented as superpositions of time-frequency shifts [Fol89],[GLS68], [How80]. Thus methods from time-frequency analysis can be employed to study pseudodifferential operators. The common feature of both approaches lies in the decomposition of the operator or its symbol into simpler parts that are easier to analyze.

In this paper, we use a decomposition of a pseudodifferential operator that is induced from a decomposition of the associated symbol into a superposition of time-frequency shifts $e^{2\pi i\beta\cdot t}\phi(t-\alpha)$ of a fixed window ϕ. Such a decomposition corresponds to a uniform partition of the time-frequency plane (also called phase space), and is called a coherent state decomposition in physics. The idea of using time-frequency or coherent state expansions for the analysis of operators is well-known and has been used by Daubechies and Grossmann [Dau80], [Dau83], [DG80], Howe [How80], and others.

The technical innovation in this paper is the introduction of new symbol classes in the form of function spaces which quantify the time-frequency content of symbols. These so-called *modulation spaces* are defined by decay properties of the *short time Fourier transform* of the symbol σ with respect to a window function $\phi \in \mathcal{S}(\mathbf{R}^{2d})$,

$$S_\phi\sigma(\alpha,\beta) \;=\; \int_{\mathbf{R}^{2d}} e^{-2\pi i\beta\cdot t}\,\overline{\phi(t-\alpha)}\,\sigma(t)\,dt.$$

Decay of $S_\phi\sigma$ amounts to simultaneous decay and smoothness properties of σ. The time-frequency behavior of σ will control the properties of the corresponding pseudodifferential operator.

Our first result is a simple condition for the boundedness of pseudodifferential operators in the style of Calderón–Vaillancourt. The function space $M_{\infty,1}(\mathbf{R}^{2d})$ will be the symbol class for this result; it consists of all distributions $\sigma \in \mathcal{S}'(\mathbf{R}^{2d})$ such that

$$\|\sigma\|_{M_{\infty,1}} \;=\; \int_{\mathbf{R}^{2d}} \sup_{\alpha} |S_\phi\sigma(\alpha,\beta)|\,d\beta < \infty.$$

We will prove the following boundedness result in Section 3; L_σ denotes the Weyl transform and K_σ the Kohn-Nirenberg transform of the symbol σ.

THEOREM 1.1. *If $\sigma \in M_{\infty,1}(\mathbf{R}^{2d})$, then L_σ and K_σ are bounded mappings of $M_{p,p}(\mathbf{R}^d)$ into itself for each $1 \le p \le \infty$. Further, there exists a constant C_p such that*

$$\|L_\sigma\|_{M_{p,p}\to M_{p,p}},\; \|K_\sigma\|_{M_{p,p}\to M_{p,p}} \;\le\; C_p\,\|\sigma\|_{M_{\infty,1}}.$$

In particular, both L_σ and K_σ are bounded on $L^2(\mathbf{R}^d) = M_{2,2}(\mathbf{R}^d)$ when $\sigma \in M_{\infty,1}(\mathbf{R}^{2d})$. In contrast to the traditional symbol classes, membership in $M_{\infty,1}$ does not imply any smoothness of σ, and Theorem 1.1 applies to certain non-smooth symbols. On the other hand, we shall show that the Lipschitz classes $\Lambda^s(\mathbf{R}^{2d})$ for $s > 2d$, and hence $C^{2d+1}(\mathbf{R}^{2d})$ in particular, are embedded in $M_{\infty,1}(\mathbf{R}^{2d})$. Thus Theorem 1.1 is an improvement of the classical Calderón–Vaillancourt theorem [CV72].

Our second result concerns the asymptotic behavior of the singular values of compact pseudodifferential operators. We give a simple and transparent proof of a result for the Weyl correspondence presented in [HRT97]. Moreover, we show that this result is also valid for the Kohn-Nirenberg correspondence. Let $L^2_s(\mathbf{R}^{2d})$ denote the L^2 space weighted by the factor $(1+|\xi|+|x|)^s$ on the time side, and let $H^s(\mathbf{R}^{2d})$ denote the Sobolev space with the same weight on the frequency side. Additionally, let $\mathcal{I}_p$ denote the Schatten ideal of compact operators whose singular values lie in ℓ^p, and let $\mathcal{I}_{p,q}$ be the analogous Schatten quasi-ideal corresponding to the Lorentz space $\ell^{p,q}$. We will prove the following result in Section 4.

Theorem 1.2. *If $\sigma \in L^2_s(\mathbf{R}^{2d}) \cap H^s(\mathbf{R}^{2d})$ with $s \geq 0$, then the singular values $\{s_k(L_\sigma)\}$ satisfy*

$$s_k(L_\sigma) \;=\; \mathcal{O}(k^{-\frac{s}{2d}-\frac{1}{2}}).$$

Thus $L_\sigma \in \mathcal{I}_{\frac{2d}{d+s},\infty} \subset \mathcal{I}_p$ for each $p > 2d/(d+s)$. In particular, L_σ is trace-class if $s > d$. The same statements hold for the Kohn-Nirenberg transform K_σ.

The trace-class statement in Theorem 1.2 is a significant improvement of analogous trace-class results in [Dau80] or [Grö96], which require $s > 2d$. Distinct results in a similar spirit for certain integral operators were obtained by Birman and Solomyak in [BS77].

The hypotheses of Theorem 1.2 can also be formulated as the membership of σ in a modulation space. Although the proof of Theorem 1.2 is simple, it relies on deep results on Gabor expansions of functions and distributions. The recent progress in understanding Gabor expansions has renewed interest in the time-frequency approach to pseudodifferential operators and led to a number of new investigations in this direction. Discrete Gabor frame expansions of the symbol were the key tool in [HRT97]. Tachizawa [Tac94] used traditional symbol classes and studied the mapping properties of pseudodifferential operators between modulation spaces. Rochberg and Tachizawa used Gabor frame expansions in [RT97].

To summarize, our new contribution is the use of modulation spaces as symbol classes for pseudodifferential operators. With such symbol classes, easy and natural conditions for the boundedness and Schatten-class properties of pseudodifferential operators can be given. Compared to the standard treatment of pseudodifferential operators, it is remarkable that these results hold for both the Weyl correspondence and the Kohn-Nirenberg correspondence.

Our paper is organized as follows. We present background and preliminary results on modulation spaces and pseudodifferential operators in Section 2. Section 3 contains the proof of Theorem 1.1, along with discussion and related results, and Section 4 contains the proof of Theorem 1.2 and related results.

2. Preliminaries.

2.1. General notation.

We define $t^2 = t \cdot t$ for $t \in \mathbf{R}^d$, and set $[x, y] = \frac{1}{2}(x_2 \cdot y_1 - x_1 \cdot y_2)$ for $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbf{R}^{2d}$.

$\mathcal{S}(\mathbf{R}^d)$ is the Schwartz class, and $\mathcal{S}'(\mathbf{R}^d)$ is its topological dual, the space of tempered distributions. We use $\langle f, g\rangle$ to denote the extension to $\mathcal{S}(\mathbf{R}^d) \times \mathcal{S}'(\mathbf{R}^d)$ of the inner product $\langle f, g\rangle = \int f(t)\, \overline{g(t)}\, dt$ on $L^2(\mathbf{R}^d)$. The Fourier transform is $\hat{f}(\omega) = \mathcal{F}f(\omega) = \int f(t)\, e^{-2\pi i t\cdot\omega}\, dt$, and the inverse Fourier transform is $\check{f}(\omega) = \mathcal{F}^{-1}f(\omega) = \hat{f}(-\omega)$. Translation and modulation of $f \in \mathcal{S}'(\mathbf{R}^d)$ are defined, respectively, by

$$T_x f(t) \;=\; f(t-x) \qquad \text{and} \qquad M_y f(t) \;=\; e^{2\pi i y\cdot t}\, f(t).$$

We have the formulas $(T_x f)^\wedge = M_{-x}\hat{f}$, $(M_y f)^\wedge = T_y \hat{f}$, and $M_y T_x = e^{2\pi i x\cdot y}\, T_x M_y$. We define $\tilde{g}(t) = \overline{g(-t)}$.

The singular values $\{s_k(L)\}_{k=1}^\infty$ of a compact operator $L \in \mathcal{B}(L^2(\mathbf{R}^d))$ are defined in terms of the eigenvalues of the positive operator L^*L, i.e., $s_k(L) = \lambda_k(L^*L)^{1/2}$. Because $L^2(\mathbf{R}^d)$ is a Hilbert space, the singular values of L coincide with its approximation numbers, and therefore

$$s_k(L) \;=\; \inf\{\|L - T\|_{L^2\to L^2} : \operatorname{rank}(T) < k\}. \tag{1}$$

The Schatten class $\mathcal{I}_p$ is the space of all compact operators whose singular values lie in ℓ^p. In particular, $\mathcal{I}_2$ is the space of Hilbert–Schmidt operators, and $\mathcal{I}_1$ is the space of trace-class operators. The Schatten quasi-ideal $\mathcal{I}_{p,q}$ consists of compact operators whose singular values lie in the Lorentz space $\ell^{p,q}$ defined by

$$\|\{c_k\}\|_{\ell^{p,q}} \;=\; \left(\sum_{k=1}^{\infty} |k^{\frac{1}{p}-\frac{1}{q}}\, c_k|^q\right)^{1/q}.$$

2.2. Time-frequency distributions and the Weyl Correspondence.

We adopt most of the notation and conventions of Folland's book [Fol89].

The *Short-Time Fourier Transform* (STFT) of a distribution $f \in \mathcal{S}'(\mathbf{R}^d)$ with respect to a window $g \in \mathcal{S}(\mathbf{R}^d)$ is

$$S_g f(x,y) \;=\; \langle f, M_y T_x g\rangle \;=\; \int_{\mathbf{R}^d} e^{-2\pi i y\cdot t}\, \overline{g(t-x)}\, f(t)\, dt.$$

The following equivalent forms of the STFT are useful.

$$\begin{aligned} S_g f(x,y) \;&=\; \langle f, M_y T_x g\rangle \;=\; (f\cdot T_x\bar{g})^\wedge(y) \;=\; e^{-2\pi i x\cdot y}\,(f*(M_y g)^\sim)(x) \;=\; \langle \hat{f}, T_y M_{-x}\hat{g}\rangle \\ &=\; (\hat{f}*(M_{-x}\hat{g})^\sim)(y) \;=\; e^{-2\pi i x\cdot y}\,(\hat{f}\cdot T_y\bar{\hat{g}})^\vee(x) \;=\; e^{2\pi i x\cdot y}\, S_{\hat{g}}\hat{f}(y,-x). \end{aligned}$$

The Schrödinger representation ρ_0 of the Heisenberg group $\mathbf{H}^d = \mathbf{R}^d \times \mathbf{R}^d \times \mathbf{T}$ on $L^2(\mathbf{R}^d)$ is $\rho_0(x,y,z)f(t) = z\, e^{\pi i x\cdot y}\, e^{2\pi i y\cdot t}\, f(t+x)$. In many considerations the toral variable z is unimportant. Therefore, for $\alpha = (x,y) \in \mathbf{R}^{2d}$ we define

$$\rho_0(\alpha)f(t) \;=\; \rho_0(x,y)f(t) \;=\; \rho_0(x,y,1)f(t) \;=\; e^{\pi i x\cdot y}\, e^{2\pi i y\cdot t}\, f(t+x) \;=\; e^{\pi i x\cdot y}\, M_y T_{-x} f(t).$$

The *ambiguity function* of f and g is

$$A(f,g)(x,y) \;=\; \langle \rho_0(x,y)f, g\rangle \;=\; e^{-\pi i x\cdot y}\, S_g f(x,-y).$$

The *Wigner distribution* of f and g is the $\mathbf{R}^{2d}$-Fourier transform of the ambiguity function, $W(f,g) = (A(f,g))^\wedge$. The ambiguity function and the Wigner distribution map $\mathcal{S}(\mathbf{R}^d) \times \mathcal{S}(\mathbf{R}^d)$ into $\mathcal{S}(\mathbf{R}^{2d})$ and extend to maps from $\mathcal{S}'(\mathbf{R}^d)\times\mathcal{S}'(\mathbf{R}^d)$ into $\mathcal{S}'(\mathbf{R}^{2d})$. Further, they are sesqui-unitary on $L^2(\mathbf{R}^d)$, i.e.,

$$\big\langle A(f_1,g_1),\, A(f_2,g_2)\big\rangle \;=\; \langle f_1, f_2\rangle\, \langle g_2, g_1\rangle \;=\; \big\langle W(f_1,g_1),\, W(f_2,g_2)\big\rangle. \tag{2}$$

The second equality in (2) is often referred to as *Moyal's identity.*

Define the linear transformation $M\colon \mathbf{R}^{4d} \to \mathbf{R}^{4d}$ by

$$M \;=\; \begin{bmatrix} 0 & -1/2 & 0 & -1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}, \tag{3}$$

where each block is a multiple of the $d \times d$ identity block. Then, by [Fol89, Prop. 1.94],

$$W(\rho_0(\alpha)f,\, \rho_0(\beta)g) \;=\; \rho(M(\alpha,\beta))W(f,g),$$

where ρ_0 represents the Schrödinger representation on $L^2(\mathbf{R}^d)$ while ρ represents the Schrödinger representation on $L^2(\mathbf{R}^{2d})$.

The Weyl correspondence is the $1-1$ correspondence between symbols $\sigma \in \mathcal{S}'(\mathbf{R}^{2d})$ and pseudodifferential operators $L_\sigma = \sigma(D,X)\colon \mathcal{S}(\mathbf{R}^d) \to \mathcal{S}'(\mathbf{R}^d)$ defined by

$$\langle L_\sigma f, g\rangle \;=\; \langle \hat{\sigma}, A(g,f)\rangle \;=\; \langle \sigma, W(g,f)\rangle, \qquad f,g \in \mathcal{S}(\mathbf{R}^d).$$

L_σ is the *Weyl transform* of σ. One particularly useful characterization is that the mapping $\sigma \mapsto L_\sigma$ is an isometry of $L^2(\mathbf{R}^{2d})$ onto the class $\mathcal{I}_2$ of Hilbert–Schmidt operators [Poo66].

By contrast, the Kohn-Nirenberg correspondence assigns to a symbol σ the operator K_σ defined by

$$\langle K_\sigma f, g\rangle \;=\; \langle \hat{\sigma}, e^{\pi i x\cdot y} A(g,f)\rangle, \qquad f,g \in \mathcal{S}(\mathbf{R}^d).$$

Thus the operators L_σ in the Weyl correspondence and K_τ in the Kohn-Nirenberg correspondence are equal if and only if their symbols are related by

$$\hat{\sigma}(x,y) = e^{-\pi i x\cdot y}\, \hat{\tau}(x,y). \tag{4}$$

Therefore statements that are preserved under a multiplication by a chirp $e^{\pi i x\cdot y}$ will be valid for one correspondence if and only if they are valid for the other.

2.3. Modulation spaces.

The modulation spaces measure the joint time-frequency distribution of $f \in \mathcal{S}'(\mathbf{R}^d)$. For background and information on their basic properties we refer to [FG89a], [FG89b], [FG92], [Grö91].

A *weight function* on $\mathbf{R}^d$ is a function $w\colon \mathbf{R}^d \to (0,\infty)$ which satisfies $w(x+y) \le C\,(1+|x|)^s\, w(y)$ for some $s \ge 0$. Given $1 \le p,\, q \le \infty$, given a weight w on $\mathbf{R}^{2d}$, and given a window function $g \in \mathcal{S}(\mathbf{R}^d)$, we define $M^w_{p,q}(\mathbf{R}^d)$ to be the space of all distributions $f \in \mathcal{S}'(\mathbf{R}^d)$ for which the following norm is finite:

$$\|f\|_{M^w_{p,q}} \;=\; \left(\int_{\mathbf{R}^d} \left(\int_{\mathbf{R}^d} |S_g f(x,y)|^p\, w(x,y)^p\, dx\right)^{q/p} dy\right)^{1/q},$$

with obvious modifications if p or $q = \infty$. $M^w_{p,q}(\mathbf{R}^d)$ is a Banach space whose definition is independent of the choice of window g. That is, different windows yield equivalent norms. If $w \equiv 1$ then we write $M_{p,q}(\mathbf{R}^d)$. If $w(x,y) = (1+|x|+|y|)^s$ then we write $M^s_{p,q}(\mathbf{R}^d)$.

If $w(\pm x, \pm y) = w(y,x)$, then $M^w_{p,p}(\mathbf{R}^d)$ is invariant under the Fourier transform since $|S_g f(x,y)| = |S_{\hat{g}} \hat{f}(y,-x)|$, cf. [FG92, Thm. 29].

Among the modulation spaces the following well-known function spaces occur:

(a) $M_{2,2}(\mathbf{R}^d) = L^2(\mathbf{R}^d)$.

(b) Weighted L^2-spaces, $w(x,y) = w_s(x) = (1+|x|)^s$:

$$M^w_{2,2}(\mathbf{R}^d) = L^2_s(\mathbf{R}^d) \;=\; \bigl\{f : f(x)\,(1+|x|)^s \in L^2(\mathbf{R}^d)\bigr\}.$$

(c) Sobolev spaces, $w(x,y) = w_s(y) = (1+|y|)^s$:

$$M^w_{2,2}(\mathbf{R}^d) \;=\; H^s(\mathbf{R}^d) \;=\; \bigl\{f : \hat{f}(y)\,(1+|y|)^s \in L^2(\mathbf{R}^d)\bigr\}.$$

(d) $w(x,y) = (1+|x|+|y|)^s$:

$$\begin{aligned} M^s_{2,2}(\mathbf{R}^d) \;=\; M^w_{2,2}(\mathbf{R}^d) \;&=\; L^2_s(\mathbf{R}^d) \cap H^s(\mathbf{R}^d) \\ &=\; \bigl\{f : (|f(x)| + |\hat{f}(x)|)\,(1+|x|)^s \in L^2(\mathbf{R}^d)\bigr\}. \end{aligned}$$

In some contexts, the equivalent norm $\|(1-x^2-\Delta)^s f\|_{L^2}$ for this space is employed, e.g., [Dau80].

(e) (Feichtinger's algebra) $M_{1,1}(\mathbf{R}^d) = S_0(\mathbf{R}^d)$.

The following lemma belongs to a group of statements on the invariance of the modulation spaces under the metaplectic representation, cf. [FG92, Thm. 29]. We need the following version.

LEMMA 2.1. *Let A be a symmetric, real-valued $d \times d$-matrix, and let convolution T and multiplication U with the chirp $e^{-\pi i x\cdot Ax}$ be defined by*

$$(Tf)^\wedge(\omega) \;=\; e^{-\pi i \omega\cdot A\omega}\hat{f}(\omega) \;=\; U\hat{f}(\omega).$$

Then T leaves $M^s_{p,q}(\mathbf{R}^d)$ invariant for each $s \ge 0$ and $1 \le p,\, q \le \infty$.

PROOF: As in [Fol89, p. 179], we calculate that $U^{-1}T_yM_{-x}U\hat{g} \;=\; e^{\pi i y\cdot Ay}\,T_yM_{-x+Ay}\,\hat{g}$. Then for $f \in \mathcal{S}'(\mathbf{R}^d)$, the short time Fourier transform of Tf is

$$\begin{aligned} S_g(Tf)(x,y) \;&=\; \langle Tf, M_yT_xg\rangle \\ &=\; \langle U\hat{f}, T_yM_{-x}\hat{g}\rangle \\ &=\; \langle \hat{f}, U^{-1}T_yM_{-x}UU^{-1}\hat{g}\rangle \end{aligned}$$

$$= e^{-\pi i y\cdot Ay}\,\langle \hat f, T_y M_{-x+Ay} U^{-1}\hat g\rangle$$
$$= e^{-\pi i y\cdot Ay}\,\langle f, M_y T_{x-Ay} T^{-1} g\rangle$$
$$= e^{-\pi i y\cdot Ay}\,S_{T^{-1}g}(f)(x-Ay,y).$$

Since the Schwartz class is invariant under multiplication with bounded C^∞ functions and under the Fourier transform, $T^{-1}g \in \mathcal{S}(\mathbf{R}^d)$ is simply another admissible window. Consequently,

$$\begin{aligned}
\|Tf\|^q_{M^s_{p,q}} &= \int_{\mathbf{R}^d}\left(\int_{\mathbf{R}^d} |S_g(Tf)(x,y)|^p\,(1+|x|+|y|)^{ps}\,dx\right)^{q/p} dy\\
&= \int_{\mathbf{R}^d}\left(\int_{\mathbf{R}^d} |S_{T^{-1}g}(f)(x-Ay,y)|^p\,(1+|x|+|y|)^{ps}\,dx\right)^{q/p} dy\\
&= \int_{\mathbf{R}^d}\left(\int_{\mathbf{R}^d} |S_{T^{-1}g}(f)(x,y)|^p\,(1+|x+Ay|+|y|)^{ps}\,dx\right)^{q/p} dy\\
&\le C\,\|f\|^q_{M^s_{p,q}},
\end{aligned}$$

Here the constant results from the change of the window, which corresponds to using an equivalent norm, and from the obvious inequality $1+|x+Ay|+|y| \le C'(1+|x|+|y|)$. □

3. Boundedness of Pseudodifferential Operators.

In this section we shall prove Theorem 1.1, which states that symbols in $M_{\infty,1}(\mathbf{R}^{2d})$ yield Weyl and Kohn-Nirenberg transforms that are bounded on each $M_{p,p}(\mathbf{R}^d)$, including $M_{2,2}(\mathbf{R}^d) = L^2(\mathbf{R}^d)$. Our approach is to realize a symbol σ as a superposition of time-frequency shifts. Specifically, we use the following standard inversion formula. This formula is proved in an abstract context in [FG89a, Thm. 4.1, Cor. 4.5], but it could also be easily proved for the special case of the short time Fourier transform by using arguments of the type found in [HW89, Sect. 3].

Theorem 3.1. *If $\phi \in \mathcal{S}(\mathbf{R}^{2d})$ and $\|\phi\|_{L^2} = 1$, then*

$$\sigma = \iint_{\mathbf{R}^{4d}} S_\phi\sigma(\alpha,\beta)\,M_\beta T_\alpha \phi\,d\alpha\,d\beta. \tag{5}$$

If $\sigma \in M^w_{p,q}(\mathbf{R}^{2d})$ with $1 \le p, q < \infty$, then this integral converges in the norm of this space. If $p=\infty$ or $q=\infty$ or if $\sigma \in \mathcal{S}'(\mathbf{R}^{2d})$, then this integral converges weakly.

As a consequence of this result, we have

$$L_\sigma \;=\; \iint_{\mathbf{R}^{4d}} S_\phi\sigma(\alpha,\beta)\, L_{M_\beta T_\alpha \phi}\, d\alpha\, d\beta. \qquad (6)$$

Thus arbitrary operators L_σ can be studied in terms of the "elementary" pseudodifferential operators $L_{M_\beta T_\alpha \phi}$ and the STFT of σ.

We choose to fix as a convenient window on $\mathbf{R}^{2d}$ the function

$$\phi(x,y) \;=\; 2^d\, e^{-2\pi(x^2+y^2)} \;=\; W(\varphi,\varphi)(x,y), \qquad (x,y)\in\mathbf{R}^{2d},$$

where φ is the Gaussian function

$$\varphi(x) \;=\; 2^{d/4}\, e^{-\pi x^2}, \qquad x\in\mathbf{R}^d.$$

The following lemma shows that for this window, the elementary operators $L_{M_\beta T_\alpha \phi}$ are rank-one projections. We use a linear transformation $N\colon \mathbf{R}^{4d}\to\mathbf{R}^{4d}$ closely related to the linear transformation M defined in (3). Precisely,

$$N \;=\; \mathrm{diag}(-1,-1,1,1)\, M \;=\; \begin{bmatrix} 0 & 1/2 & 0 & 1/2 \\ -1/2 & 0 & -1/2 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}.$$

Note that $(\alpha,\beta)=N(\xi,\eta)$ if and only if $(-\alpha,\beta)=M(\xi,\eta)$.

LEMMA 3.2. *Using the definition $(\alpha,\beta)=N(\xi,\eta)$, the following statements hold.*

(a) $M_\beta T_\alpha\phi \;=\; e^{-2\pi i[\xi,\eta]}\, W(\rho_0(\xi)\varphi,\, \rho_0(\eta)\varphi)$.

(b) $L_{M_\beta T_\alpha \phi} f \;=\; e^{-2\pi i[\xi,\eta]}\, \langle f, \rho_0(\eta)\varphi\rangle\, \rho_0(\xi)\varphi$.

(c) $L_\sigma f \;=\; \displaystyle\iint_{\mathbf{R}^{4d}} S_\phi\sigma(N(\xi,\eta))\, e^{-2\pi i[\xi,\eta]}\, \langle f, \rho_0(\eta)\varphi\rangle\, \rho_0(\xi)\varphi\, d\xi\, d\eta$.

PROOF: (a) Since $\alpha\cdot\beta = -2[\xi,\eta]$, we have

$$\begin{aligned} M_\beta T_\alpha \phi \;&=\; e^{\pi i \alpha\cdot\beta}\, \rho(-\alpha,\beta)\phi \;=\; e^{\pi i\alpha\cdot\beta}\, \rho(-\alpha,\beta) W(\varphi,\varphi) \\ &=\; e^{\pi i\alpha\cdot\beta}\, \rho(M(\xi,\eta)) W(\varphi,\varphi) \\ &=\; e^{\pi i\alpha\cdot\beta}\, W(\rho_0(\xi)\varphi,\, \rho_0(\eta)\varphi) \\ &=\; e^{-2\pi i[\xi,\eta]}\, W(\rho_0(\xi)\varphi,\, \rho_0(\eta)\varphi). \end{aligned}$$

(b) Given $f \in \mathcal{S}(\mathbf{R}^d)$ and $g \in \mathcal{S}'(\mathbf{R}^d)$ we have

$$\begin{aligned}\langle L_{M_\beta T_\alpha \phi} f, g\rangle &= \big\langle M_\beta T_\alpha \phi,\, W(g,f)\big\rangle \\ &= e^{-2\pi i[\xi,\eta]}\, \big\langle W(\rho_0(\xi)\varphi,\, \rho_0(\eta)\varphi),\, W(g,f)\big\rangle \\ &= e^{-2\pi i[\xi,\eta]}\, \langle f, \rho_0(\eta)\varphi\rangle\, \langle \rho_0(\xi)\varphi, g\rangle,\end{aligned}$$

the last inequality following from Moyal's formula (2).

(c) Since $\det(N) = 1$, the change of variables $(\alpha, \beta) = N(\xi, \eta)$ applied to equation (6) yields the desired result. □

Lemma 3.3. *If $\sigma \in M_{\infty,1}(\mathbf{R}^{2d})$ then*

$$\begin{aligned}\sup_\eta \int_{\mathbf{R}^{2d}} |S_\phi\sigma(N(\xi,\eta))|\, d\xi &\le \|\sigma\|_{M_{\infty,1}} < \infty, \\ \sup_\xi \int_{\mathbf{R}^{2d}} |S_\phi\sigma(N(\xi,\eta))|\, d\eta &\le \|\sigma\|_{M_{\infty,1}} < \infty.\end{aligned}$$

As a consequence, the mapping T defined by

$$TG(\xi) = \int_{R^{2d}} S_\phi\sigma(N(\xi,\eta))\, G(\eta)\, d\eta$$

is a bounded mapping of $L^p(\mathbf{R}^{2d})$ into itself for each $1 \le p \le \infty$, and

$$\|T\|_{L^p \to L^p} \le \|\sigma\|_{M_{\infty,1}}.$$

Proof: Note that $S_\phi\sigma(N(\xi,\eta)) = S_\phi\sigma(\theta, \xi - \eta)$, where $\theta = \big(\frac{\xi_2+\eta_2}{2}, -\frac{\xi_1+\eta_1}{2}\big)$. Hence,

$$\begin{aligned}\int_{\mathbf{R}^{2d}} |S_\phi\sigma(N(\xi,\eta))|\, d\xi &\le \int_{\mathbf{R}^{2d}} \sup_\theta |S_\phi\sigma(\theta, \xi-\eta))|\, d\xi \\ &= \int_{\mathbf{R}^{2d}} \sup_\theta |S_\phi\sigma(\theta, \xi))|\, d\xi \\ &= \|\sigma\|_{M_{\infty,1}}.\end{aligned}$$

The other inequality is similar. The fact that T is bounded is then an immediate consequence of these inequalities and Schur's lemma, see, e.g., [Mey90]. □

Introduce now the weighted mixed-norm space $L^w_{p,q}(\mathbf{R}^{2d})$, with

$$\|F\|_{L^w_{p,q}} \;=\; \left(\int_{\mathbf{R}^d}\left(\int_{\mathbf{R}^d} |F(x,y)|^p\, w(x,y)^p\, dx\right)^{q/p} dy\right)^{1/q}.$$

The following estimate is proved in [FG89a, Cor. 4.5].

PROPOSITION 3.4. *Let w be a weight function on $\mathbf{R}^{2d}$, and fix $\varphi \in \mathcal{S}(\mathbf{R}^d)$. Then there exists a constant $C = C(\varphi)$ such that if $F \in L^w_{p,q}(\mathbf{R}^{2d})$, then*

$$f \;=\; \iint_{\mathbf{R}^{2d}} F(x,y)\,\rho_0(x,y)\varphi\, dx\, dy \;\in\; M^w_{p,q}(\mathbf{R}^d),$$

where the integral converges in $M^w_{p,q}(\mathbf{R}^d)$ for $p, q < \infty$, and $\|f\|_{M^w_{p,q}} \le C\,\|F\|_{L^w_{p,q}}$.

These results combine to give the proof of Theorem 1.1.

PROOF OF THEOREM 1.1: Assume that $\sigma \in M_{\infty,1}(\mathbf{R}^{2d})$. Recall from Lemma 3.2 that

$$L_\sigma f \;=\; \int_{\mathbf{R}^{2d}}\left(\int_{\mathbf{R}^{2d}} S_\phi\sigma(N(\xi,\eta))\, e^{-2\pi i[\xi,\eta]}\,\langle f, \rho_0(\eta)\varphi\rangle\, d\eta\right)\rho_0(\xi)\varphi\, d\xi.$$

If $f \in M_{p,p}(\mathbf{R}^d)$, then $G(\eta) = e^{-2\pi i[\xi,\eta]}\,\langle f, \rho_0(\eta)\varphi\rangle$ is an element of $L^p(\mathbf{R}^{2d})$ by definition. It therefore follows from Lemma 3.3 that $\int_{\mathbf{R}^{2d}} S_\phi\sigma(N(\xi,\eta))\, e^{-2\pi i[\xi,\eta]}\,\langle f, \rho_0(\eta)\varphi\rangle\, d\eta$ is an element of $L^p(\mathbf{R}^{2d}) = L_{p,p}(\mathbf{R}^{2d})$. Proposition 3.4 therefore implies that $L_\sigma f \in M_{p,p}(\mathbf{R}^d)$, and that the mapping is bounded.

To transfer this result to the Kohn-Nirenberg correspondence, recall from (4) that $K_\sigma = L_{T\sigma}$, where T is the convolution on $\mathbf{R}^{2d}$ with the chirp corresponding to the $2d \times 2d$ matrix $A = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$ with $d \times d$ blocks consisting of multiples of the identity matrix. By Lemma 2.1 we have $T\sigma \in M_{\infty,1}(\mathbf{R}^{2d})$, so the boundedness of K_σ follows. □

The following boundedness result then follows by interpolation.

COROLLARY 3.5. *If $\sigma \in M_{p,p'}(\mathbf{R}^{2d})$ with $2 \le p < \infty$, then $L_\sigma,\, K_\sigma \in \mathcal{I}_p(\mathbf{R}^d)$.*

PROOF: By [Poo66], the Weyl transform $\sigma \mapsto L_\sigma$ is a bounded mapping of $L^2(\mathbf{R}^{2d}) = M_{2,2}(\mathbf{R}^{2d})$ onto $\mathcal{I}_2(\mathbf{R}^d)$. By Theorem 1.1, the Weyl transform is a bounded mapping of $M_{\infty,1}(\mathbf{R}^{2d})$ into $\mathcal{B}(L^2(\mathbf{R}^d))$. By [Fei81],

$$(M_{2,2}(\mathbf{R}^{2d}),\, M_{\infty,1}(\mathbf{R}^{2d}))_{\theta,p} \;=\; M_{p,p'}(\mathbf{R}^{2d}).$$

Moreover, by the proof of and remark following Theorem 2.c.6 in [Kön86],

$$(\mathcal{I}_2(\mathbf{R}^d),\, \mathcal{B}(L^2(\mathbf{R}^d)))_{\theta,p} \;=\; \mathcal{I}_p(\mathbf{R}^d).$$

It therefore follows that the Weyl correspondence is a bounded mapping of $M_{p,p'}(\mathbf{R}^{2d})$ into $\mathcal{I}_p(\mathbf{R}^d)$ for $2 < p < \infty$. A similar argument applies to the Kohn-Nirenberg correspondence. □

The classical version of the Calderón–Vaillancourt theorem states that L_σ and K_σ are bounded on $L^2(\mathbf{R}^{2d})$ if $\sigma \in C^{2d+1}(\mathbf{R}^{2d})$ [CV72]. We end this section by observing that several particular function classes are properly contained in $M_{\infty,1}(\mathbf{R}^{2d})$, including $C^{2d+1}(\mathbf{R}^{2d})$ in particular. We let $\Lambda^s(\mathbf{R}^{2d})$ denote the Lipschitz or Hölder–Zygmund classes. $\Lambda^s(\mathbf{T}^{2d})$ denotes the subset of $\Lambda^s(\mathbf{R}^{2d})$ containing functions that are 1-periodic in each component. $A(\mathbf{T}^{2d})$ denotes the space of 1-periodic functions which possess absolutely convergent Fourier series.

Proposition 3.6.

(a) $\Lambda^s(\mathbf{T}^{2d}) \subsetneq A(\mathbf{T}^{2d}) \subsetneq M_{\infty,1}(\mathbf{R}^{2d})$ *for* $s > d$.

(b) $\Lambda^s(\mathbf{R}^{2d}) \subsetneq M_{\infty,1}(\mathbf{R}^{2d})$ *for* $s > 2d$.

Proof: (a) The fact that $\Lambda^s(\mathbf{T}^{2d}) \subset A(\mathbf{R}^{2d})$ for $s > d$ is the $2d$-dimensional version of the Bernstein–Zygmund Lemma [Zyg68]. This inclusion is proper since, for example, if the integers n_k increase fast enough then $\sigma(\gamma) = \sum_{k=1}^{\infty} k^{-2}\, e^{2\pi i n_k \gamma_1}$ has an absolutely convergent Fourier series but cannot lie in $\Lambda^s(\mathbf{T}^{2d})$ for any $s > 0$.

Now suppose that $\sigma \in A(\mathbf{T}^{2d})$, and write $\sigma(\gamma) = \sum_{k \in \mathbf{Z}^{2d}} a_k\, e^{2\pi i k\cdot\gamma}$ with $\sum |a_k| < \infty$. Since $S_\phi(e^{2\pi i k\cdot\gamma})(\alpha,\beta) = e^{-2\pi i \alpha\cdot(\beta-k)}\, \hat{\phi}(\beta-k)$, we have

$$\begin{aligned}\|\sigma\|_{M_{\infty,1}} &= \int_{\mathbf{R}^{2d}} \sup_{\alpha\in\mathbf{R}^{2d}} |S_\phi\sigma(\alpha,\beta)|\, d\beta \\ &\le \sum_{k\in\mathbf{Z}^{2d}} |a_k| \int_{\mathbf{R}^{2d}} |\hat{\phi}(\beta-k)|\, d\beta \\ &= \|\sigma\|_{A(\mathbf{T}^{2d})}\, \|\phi\|_{A(\mathbf{R}^d)}.\end{aligned}$$

(b) Choose $\sigma \in \Lambda^s(\mathbf{R}^{2d})$ with $s > 2d$, and fix $2d < t < s$. Then, by [HRT97, Prop. 6.3], there exists a constant C such that

$$\sup_{\alpha\in\mathbf{R}^{2d}} |S_\phi\sigma(\alpha,\beta)| \;=\; \sup_{\alpha\in\mathbf{R}^{2d}} |(\sigma\cdot T_\alpha\phi)^\wedge(\beta)| \;\le\; C\,(1+|\beta|)^{-t}.$$

Since $t > 2d$, we therefore have $\|\sigma\|_{M_{\infty,1}} \le C \int (1+|\beta|)^{-t}\, d\beta < \infty$. This inclusion is proper since the example constructed in part (a) lies in $M_{\infty,1}(\mathbf{R}^{2d})$ but not in $\Lambda^s(\mathbf{R}^{2d})$ for any $s > 0$. □

The preceeding result suggests that embedding theorems of classical function spaces into modulation spaces are of an importance similar to Sobolev embeddings in the field of partial differential equations. In particular, such embeddings allow sufficient conditions on the operator symbol to be stated in forms appropriate to a particular application.

4. Singular Values of Compact Pseudodifferential Operators.

We shall prove Theorem 1.2 in this section. First, however, we note that the continuous-type expansions used in (5) lead immediately to the following observation that operators whose symbols lie in the Feichtinger algebra are trace-class. This result was first proved in [Grö96], although it was known to Feichtinger.

Proposition 4.1. *If $\sigma \in S_0(\mathbf{R}^{2d})$, then L_σ, $K_\sigma \in \mathcal{I}_1$.*

Proof: We shall show that the integral in (6) converges absolutely in trace-class norm when $\sigma \in S_0(\mathbf{R}^{2d})$. Certainly the elementary operators $L_{M_\beta T_\alpha \phi} f = e^{-2\pi i[\xi,\eta]}\, \langle f, \rho_0(\eta)\varphi\rangle\, \rho_0(\xi)\varphi$ are all trace-class since they are rank-one projections. Indeed, their trace-class norms satisfy

$$\|L_{M_\beta T_\alpha \phi}\|_{\mathcal{I}_1} \;=\; \|\rho_0(\eta)\varphi\|_{L^2}\, \|\rho_0(\xi)\varphi\|_{L^2} \;=\; \|\varphi\|_{L^2}^2 \;=\; 1.$$

Therefore, applying the $\mathcal{I}_1$ norm to (6) yields

$$\|L_\sigma\|_{\mathcal{I}_1} \;\le\; \iint_{\mathbf{R}^{4d}} |S_\phi\sigma(\alpha,\beta)|\, \|L_{M_\beta T_\alpha \phi}\|_{\mathcal{I}_1}\, d\alpha\, d\beta \;=\; \iint_{\mathbf{R}^{4d}} |S_\phi\sigma(\alpha,\beta)|\, d\alpha\, d\beta \;=\; \|\sigma\|_{S_0}.$$

The same argument as in the proof of Theorem 1.1 allows us to transfer this result to the Kohn-Nirenberg correspondence. □

To interpret this result we quote the following special case of embeddings of $L_a^p \cap \mathcal{F}L_b^q$ into S_0 proved in [Grö96].

Proposition 4.2.

(a) *If $s > 2d$ then $L_s^2(\mathbf{R}^{2d}) \cap H^s(\mathbf{R}^{2d}) \subset S_0(\mathbf{R}^{2d})$.*

(b) *If $s \le 2d$ then $L_s^2(\mathbf{R}^{2d}) \cap H^s(\mathbf{R}^{2d}) \not\subset S_0(\mathbf{R}^{2d})$.*

Thus Proposition 4.1 is sufficient to recover and improve the trace-class result of Daubechies [Dau80], which states that $L_\sigma \in \mathcal{I}_1$ if $\sigma \in L^2_s(\mathbf{R}^{2d}) \cap H^s(\mathbf{R}^{2d})$ with $s > 2d$. Note, however, that the trace-class conclusion of Theorem 1.2 requires only the far weaker hypothesis that $s > d$.

To prove Theorem 1.2, we shall use a discrete series expansion of the symbol, rather than the continuous expansion in (5). The motivation for this approach is that singular values can be estimated via finite-rank approximations, and series expansions lead naturally to finite-rank approximants. The following refinement of (1) allows us to estimate the singular values from Hilbert–Schmidt, rather than operator norm, estimates.

Lemma 4.3. *If $L \in \mathcal{I}_2$, then*

$$s_{2K}(L)^2 \;\le\; \frac{1}{K}\sum_{k>K} s_k(L)^2 \;\le\; \frac{1}{K}\,\inf\{\|L-T\|^2_{\mathcal{I}_2} : \operatorname{rank}(T) \le K\}.$$

Proof: The first inequality follows from the fact that the singular values are decreasing. The second follows from the standard inequality $s_{k+\ell+1}(L_1+L_2) \le s_{k+1}(L_1) + s_{\ell+1}(L_2)$ [DS88, p. 1089]. □

The series expansions that we use are based on the theory of Gabor frames. For basic exposition we refer to [Dau90], [HW89]. Specifically, we shall use the following results. These statements are quite deep; as a consequence, Theorem 1.2 is more substantial than may be apparent.

(A) If $\phi = 2^d e^{-2\pi(x^2+y^2)} = W(\varphi,\varphi)$ and $a,\, b > 0$ are chosen so that $ab < 1$, then there exist constants $A,\, B > 0$ such that

$$A\,\|f\|^2_{L^2} \;\le\; \sum_{m,n\in\mathbf{Z}^{2d}} |S_\phi f(na,mb)|^2 \;\le\; B\,\|f\|^2_{L^2}$$

for all $f \in L^2(\mathbf{R}^{2d})$. This was proved by Seip and Wallstén [Sei92], [SW92] for $L^2(\mathbf{R})$ and follows immediately for higher dimensions by a tensor product argument. The collection $\{M_{mb}T_{na}\phi\}_{m,n\in\mathbf{Z}^{2d}}$ is called a Gabor frame for $L^2(\mathbf{R}^{2d})$.

(B) There exists $\gamma \in \mathcal{S}(\mathbf{R}^{2d})$, the so-called dual window, such that

$$\begin{aligned} f \;&=\; \sum_{m,n\in\mathbf{Z}^{2d}} S_\gamma f(na,mb)\, M_{mb}T_{na}\phi \\ &=\; \sum_{m,n\in\mathbf{Z}^{2d}} S_\phi f(na,mb)\, M_{mb}T_{na}\gamma \end{aligned}$$

for all $f \in L^2(\mathbf{R}^{2d})$, with unconditional convergence of the series in L^2-norm [Jan95].

(C) The frame expansion in (B) also holds for all $f \in M^w_{p,q}(\mathbf{R}^{2d})$, with unconditional convergence in the norm of $M^w_{p,q}$ if $1 \le p,\ q < \infty$, and weak convergence if p or $q = \infty$. Furthermore, the following is an equivalent norm for $M^w_{p,q}(\mathbf{R}^{2d})$ for each $1 \le p,\ q \le \infty$:

$$\|f\|_{M^w_{p,q}} \sim \left(\sum_{m \in \mathbf{Z}^{2d}} \left(\sum_{n \in \mathbf{Z}^{2d}} |S_\gamma f(na, mb)|^p \, w(na, mb)^p \right)^{q/p} \right)^{1/q}. \tag{7}$$

See [Grö91], [FG97].

We shall use the following further property, which holds for arbitrary windows in the Wiener algebra:

$$\forall \{c_{mn}\} \in \ell^2, \qquad \left\| \sum_{m,n} c_{mn} \, M_{ma} T_{na} \phi \right\|^2_{L^2} \le B \sum_{m,n} |c_{mn}|^2. \tag{8}$$

In the following we choose $a = b < 1$ and the constants A, B accordingly.

Proof of Theorem 1.2: Fix $s \ge 0$ and choose $\sigma \in M^s_{2,2}(\mathbf{R}^{2d}) = L^2_s(\mathbf{R}^{2d}) \cap H^s(\mathbf{R}^{2d})$. Then $\sigma \in L^2(\mathbf{R}^{2d})$, so the frame expansion

$$\sigma = \sum_{m,n} S_\gamma \sigma(na, ma) \, M_{ma} T_{na} \phi$$

converges in L^2 and in $M^s_{2,2}$. For each $K > 0$, define

$$\sigma_K = \sum_{|m|,|n| \le K} S_\gamma \sigma(na, ma) \, M_{ma} T_{na} \phi,$$

and note that

$$L_{\sigma_K} = \sum_{|m|,|n| \le K} S_\gamma \sigma(na, ma) \, L_{M_{ma} T_{na} \phi}.$$

By Lemma 3.2, the operator $L_{M_{ma}T_{na}\phi} f = e^{-2\pi i [\xi, \eta]} \langle f, \rho_0(\eta)\varphi \rangle \, \rho_0(\xi)\varphi$ is a rank-one operator, where $(\xi, \eta) = N^{-1}(na, ma)$.

If we write $N = \begin{bmatrix} \mathcal{J} & \mathcal{J} \\ I & -I \end{bmatrix}$, where I is the $2d \times 2d$ identity matrix and $\mathcal{J} = \begin{bmatrix} 0 & 1/2 \\ -1/2 & 0 \end{bmatrix}$, then $N(\xi, \eta) = (na, ma)$ implies $\xi = -2a\mathcal{J}n + \frac{1}{2}am \in \frac{a}{2}\mathbf{Z}^{2d}$. Consequently, we obtain the following estimate for the rank of L_{σ_K}:

$$\begin{aligned} \operatorname{rank}(L_{\sigma_K}) &\le \#\{\xi \in \mathbf{R}^{2d} : (\xi, \eta) = N^{-1}(na, ma),\ |m|, |n| \le K\} \\ &\le \#\{\xi \in \tfrac{a}{2}\mathbf{Z}^{2d} : |\xi| \le \tfrac{3}{2}aK\} \\ &\le (6K+1)^{2d}. \end{aligned}$$

Let R be an integer such that $\operatorname{rank}(L_{\sigma_K}) \le RK^{2d}$ for all K, and set

$$I_K = \bigl\{(m,n) \in \mathbf{Z}^{4d} : |m|, |n| \le K\bigr\}.$$

Then,

$$\begin{aligned}
&\sum_{k > RK^{2d}} s_k(L_\sigma)^2 \\
&\le \sum_{k > \operatorname{rank}(L_{\sigma_K})} s_k(L_\sigma)^2 \\
&\le \|L_\sigma - L_{\sigma_K}\|_{\mathcal{I}_2}^2 && \text{by Lemma 4.3} \\
&= \|\sigma - \sigma_K\|_{L^2}^2 && \text{by [Poo66]} \\
&= \Bigl\| \sum_{(m,n)\notin I_K} S_\gamma\sigma(na,ma)\, M_{ma}T_{na}\phi \Bigr\|_{L^2}^2 \\
&\le B \sum_{(m,n)\notin I_K} |S_\gamma\sigma(na,ma)|^2 && \text{by (8)} \\
&\le B\Bigl(\sup_{(m,n)\notin I_K} (1+|ma|+|na|)^{-2s}\Bigr)\Bigl(\sum_{(m,n)\notin I_K} |S_\gamma\sigma(na,ma)|^2\,(1+|ma|+|na|)^{2s}\Bigr) \\
&\le B\,K^{-2s}\,C\,\|\sigma\|_{M^s_{2,2}}^2, && \text{by (7)}
\end{aligned}$$

where the constant C is determined from the constants of proportionality in the norm equivalence in (7). Applying the first inequality in Lemma 4.3, we therefore have

$$s_{2RK^{2d}}(L_\sigma)^2 \;\le\; \frac{BC}{2R}\,\|\sigma\|_{M^s_{2,2}}^2\,K^{-2s-2d}.$$

Upon reindexing, we obtain

$$s_k(L_\sigma) \;\le\; D\,k^{-\frac{s}{2d}-\frac{1}{2}},$$

with D a constant independent of k.

Finally, since $K_\sigma = L_{T\sigma}$ and $M^s_{2,2}$ is invariant under T, the result also holds for the singular values of K_σ. □

Corollary 4.4. *If $\sigma \in L^2_s(\mathbf{R}^{2d}) \cap H^s(\mathbf{R}^{2d})$ with $s \ge 0$, then L_σ, $K_\sigma \in \mathcal{I}_{\frac{2d}{d+s},2}$.*

Proof: This proof follows as in [HRT97, Thm. 5.5] by interpolating between Theorem 1.2 and the fact that $\sigma \mapsto L_\sigma$ and $\sigma \mapsto K_\sigma$ are isometries of $L^2(\mathbf{R}^{2d})$ onto $\mathcal{I}_2$. □

We close by noting that the quantity $\sum_{(m,n)\notin I_K} |S_\gamma\sigma(na,ma)|^2$ that occurs in the proof of Theorem 1.2 is essentially the quantity that is estimated in the phase-space localization theorem of Daubechies. The proof of Theorem 1.2 could also be completed by adapting the proof of [Dau90, Thm. 3.1] to give the following.

Proposition 4.5. *Let $B_r = \{(x,y) \in \mathbf{R}^{4d} : |x|, |y| \le r\}$. If $\sigma \in L^2(\mathbf{R}^{2d})$, then there exist $C > 0$ and $\varepsilon_K \to 0$ such that for each K,*

$$\sum_{(m,n)\notin I_K} |S_\gamma\sigma(na,ma)|^2 \;\le\; C\|\sigma \cdot \chi_{B^{\mathrm{C}}_{K/2}}\|^2_{L^2} \;+\; C\|\hat{\sigma} \cdot \chi_{B^{\mathrm{C}}_{K/2}}\|^2_{L^2} \;+\; \varepsilon_K \|\sigma\|^2_{L^2}.$$

The value of ε_K in Proposition 4.5 is determined by the decay rate of the window γ.

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