Invariances of Frame Sequences under Perturbations

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Abstract

This paper determines the exact relationships that hold among the major Paley–Wiener perturbation theorems for frame sequences. It is shown that major properties of a frame sequence such as excess, deficit, and rank remain invariant under Paley–Wiener perturbations, but need not be preserved by compact perturbations. For localized frames, which are frames with additional structure, it is shown that the frame measure function is also preserved by Paley–Wiener perturbations.

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1. Introduction

1.1. Overview

A frame allows robust, unconditionally convergent, basis-like but usually non-unique expansions of elements of a Hilbert space. Frame theory has seen wide application throughout mathematics and engineering, such as in wireless communications [28], Σ-Δ quantization [8], and image processing [9]. In many situations, we work with a *frame sequence* that is a frame for its closed span rather than the entire space, and it is important to know what properties of the frame sequence are stable under perturbations. There are a variety of perturbation theorems that are known to hold for frames and frame sequences. These fall into two main types, which we will call Paley–Wiener perturbations and compact perturbations of the frame synthesis operator.

We have two main goals in this paper. First, we determine the exact relationships that hold among the major Paley–Wiener perturbation theorems. Second, we study the

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invariance properties of frame sequences under perturbations. In particular, we show that major properties of a frame sequence such as excess, deficit, and rank remain invariant under Paley–Wiener perturbations, but need not be preserved by compact perturbations. For *localized frames*, which are frames with additional structure, we show that the frame measure function is preserved by Paley–Wiener perturbations.

1.2. Paley–Wiener Perturbation Theorems

The following result summarizes the main Paley–Wiener-type perturbation theorems known to hold for frame sequences. These are due to Christensen and his co-authors [12], [14], [15] (precise definitions of all terms are given in Section 2).

Theorem 1.1. Let $F = \{f_i\}_{i\in\mathbb{I}} \subset \mathcal{H}$ be a frame sequence in a separable Hilbert space $\mathcal{H}$, with synthesis operator T_F and frame bounds A_F, B_F. Let $G = \{g_i\}_{i\in\mathbb{I}} \subset \mathcal{H}$ be another sequence with synthesis operator T_G, and let $\mu \geq 0$ be fixed. Define $\mathcal{H}_F = \overline{\text{span}}(F) = \text{range}(T_F)$ and $\mathcal{H}_G = \overline{\text{span}}(G)$. If

$$\|T_F c - T_G c\| \leq \mu\, \|c\| \quad \text{for all finitely supported } c \in \ell^2(\mathbb{I}), \tag{1.1}$$

then G is a Bessel sequence. Moreover, if any one of the following conditions on the inf-angle between subspaces holds, then G is a frame sequence:

(i) $\mu < \sqrt{A_F}\, R\big(\ker(T_F), \ker(T_G)\big)$,

(ii) $\mu < \sqrt{A_F}\, R(\mathcal{H}_G, \mathcal{H}_F)$,

(iii) $\mu < \sqrt{A_F}$ and $R(\mathcal{H}_G, \mathcal{H}_F) > 0$.

In fact, there are more complicated theorems involving not only μ but also two more non-negative constants [10], [12], [15], [27]. However, the remaining parameters are rarely (if ever) used in applications of the perturbation theorems to wavelet, Gabor, shift-invariant, or exponential frames, cf. [1], [16]. Therefore, we focus in this paper on the practical versions of the perturbation theorems, with comments on the possible generalizations.

Conditions (i) and (ii) were originally expressed in terms of the 'gap' $\delta(X,Y)$ [25] instead of the inf-angle $R(X,Y)$. The above formulation follows from the fact that $\delta(X,Y) = \big(1 - R(X,Y)^2\big)^{1/2}$ [27]. That hypothesis (i) implies the lower frame condition is [12, Thm. 3.2]. The fact that (ii) also does so is [15, Thm. 3.1], and the fact that (iii) does so is [14, Thm. 3.2].

In this paper we are interested in the relationship between hypotheses (i), (ii), and (iii), and in the nature of the relationship between the original frame sequence F and the perturbed frame sequence G when the hypotheses of Theorem 1.1 are satisfied.

1.3. Invariances of Frame Sequences under Paley–Wiener-type Perturbations

We introduce some terminology in order to describe the types of frame property invariances that we will consider.

We say that the *rank* of a frame sequence F is $\text{rank}(F) = \dim(\mathcal{H}_F)$.

The *excess* of F is the cardinality of the largest subset that can be removed from F yet still leave a sequence which has the same closed span:

$$\text{excess}(F) \;=\; \sup\big\{|F'| : F' \subset F,\, \overline{\text{span}}(F \setminus F') = \overline{\text{span}}(F)\big\},$$

where $|F'|$ denotes the cardinality of the set F'. Thus, the excess is a measure of the overcompleteness or redundancy of F. It is shown in [3, Lem. 4.1(c)] that the excess is related to the synthesis operator T_F by the formula

$$\mathrm{excess}(F) = \dim\bigl(\ker(T_F)\bigr). \tag{1.2}$$

On the other hand, the *deficit* of a frame sequence F is the smallest cardinality of a sequence that must be added to F in order to obtain a sequence whose closed span is the entire space:

$$\mathrm{deficit}(F) \;=\; \inf\bigl\{|F'| : F' \subset \mathcal{H},\, \overline{\mathrm{span}}(F \cup F') = \mathcal{H}\bigr\}.$$

Thus the deficit is a measure of the undercompleteness of F. By [3, Lem. 4.1(a)], the deficit is

$$\mathrm{deficit}(F) = \dim(\mathcal{H}_F^{\perp}). \tag{1.3}$$

Our first main result characterizes the exact implications that hold among the hypothesis (i), (ii), and (iii) in the Paley–Wiener-type perturbation theorems. Moreover, we show that these types of perturbations preserve many of the fundamental properties of frames, including rank, excess, and deficit. Thus, Paley–Wiener-type perturbations preserve the "size" of a frame sequence in many ways. The proof of Theorem 1.2 will be given in Section 4.

Theorem 1.2. Let a frame sequence F and a sequence G be given as in Theorem 1.1, and suppose that (1.1) is satisfied. Then the following statements hold.

- (i) implies (iii), but not vice versa.
- (ii) implies (iii), but not vice versa.
- (i) and (ii) are independent.

Moreover, if any one of (i), (ii), or (iii) is satisfied, then G is a frame sequence, and the following statements hold.

- $\mathcal{H}_F$ is isomorphic to $\mathcal{H}_G$.
- $\mathcal{H}_F^{\perp}$ is isomorphic to $\mathcal{H}_G^{\perp}$.
- $\ker(T_F)$ is isomorphic to $\ker(T_G)$.
- $\mathrm{rank}(F) = \mathrm{rank}(G)$.
- $\mathrm{excess}(F) = \mathrm{excess}(G)$.
- $\mathrm{deficit}(F) = \mathrm{deficit}(G)$.

Some of the implications in Theorem 1.2 were known previously. The new contribution of Theorem 1.2 is the implication (i) implies (iii) and the invariance conclusions that follow when the weakest condition (iii) holds. The fact that $\mathcal{H}_F$ is isomorphic to $\mathcal{H}_G$ and $\mathcal{H}_F^{\perp}$ is isomorphic to $\mathcal{H}_G^{\perp}$ if (ii) is satisfied is proved in [15], and the fact that $\mathcal{H}_F$ is isomorphic to $\mathcal{H}_G$ if (iii) is satisfied is proved in [14]. The implication (ii) implies (iii) is trivial because of the fact that $R(\mathcal{H}_G, \mathcal{H}_F) \le 1$. The paper [27] also contains some of the implications of Theorem 1.2 for the restricted setting of shift-invariant frame systems. Additionally, the counterexamples constructed in [27] establish the "not vice versa" statements in Theorem 1.2 as well as the independence of (i) and (ii).

1.4. Invariance of the Frame Measure Function

Excess, deficit, and rank are measures of the overcompleteness or undercompleteness of a frame sequence, but only in a relatively crude sense. For arbitrary frames, it is extremely difficult to quantify the exact meaning of redundancy. For example, we would like to be able to say that a frame sequence has redundancy 3/2 if some subset of the frame with only 2/3 of the original elements is still frame sequence with the same closed span. This obviously has no meaning for a generic frame with infinitely many elements.

However, many practical frames have a structure which allows us to make sense of such statements. For example, a Gabor frame sequence is a frame sequence of the form

$$\mathcal{G}(g,\Lambda) \;=\; \{e^{2\pi i\beta x} g(x-\alpha)\}_{(\alpha,\beta)\in\Lambda}$$

where $g \in L^2(\mathbb{R})$ and $\Lambda \subset \mathbb{R}^2$ are fixed. A longstanding folklore for Gabor frames is that the *density of the index set* Λ *equals the redundancy of the frame* $\mathcal{G}(g,\Lambda)$. Density of the indexed set is defined in terms of *Beurling density*, which is in a sense the average number of points of Λ that lie in a unit cube (e.g., for the lattice $\Lambda = \alpha\mathbb{Z}\times\beta\mathbb{Z}$ the Beurling density is $1/(\alpha\beta)$). For a survey of the long history of results connected to Beurling density and the Nyquist Theorem for Gabor frames, including precise definitions and an extensive bibliography, we refer to [23].

Recently, the introduction of the concept of *localized frames* has allowed this folklore to be given a quantitative interpretation. Localized frames were introduced independently in [21] and [4], for quite different purposes. Applied to Gabor frames, the results of [4], [5] imply, for example, that any localized frame (including Gabor frames) must have lower density of at least 1, and if such a frame is a union of N Riesz bases then the upper and lower densities of Λ are exactly N. A deep new result from [6] is that if the density of a localized frame is $d > 1$ then there exists a subset of the frame with density $1+\varepsilon$ that is still a frame for the space. These results and others show that the density d of a localized frame, which is determined solely by the index set alone, quantifies the redundancy of the frame.

Closely related to density and redundancy issues is the question of when frame sequences are *equivalent*. A naive notion of equivalence for frame sequences is that $F = \{f_i\}_{i\in\mathbb{I}}$ and $G = \{g_i\}_{i\in\mathbb{I}}$ are equivalent if there exists a bounded bijection $U\colon \mathcal{H}\to\mathcal{H}$ such that $U(f_i) = g_i$ for each i [2]. This is the correct notion of equivalence for bases, but because of the redundancies inherent in frames, this notion of equivalent frames is too strong. For example, frames that are identical except for the ordering of their index set need not be equivalent under this definition, even though the definition and most properties of frames are independent of ordering. A new notion of frame equivalence based on the idea of a *frame measure function* was introduced in [7]. Two frames are equivalent in this definition if their frame measure functions coincide. This notion of equivalence is independent of the ordering of the index set, under multiplication of frame elements by scalars of unit modulus, and other seemingly trivial modifications that were not invariant under the earlier notion of frame equivalence.

Our second main result concerns the behavior of the frame measure function under Paley–Wiener-type perturbations. In this theorem, P_F denotes the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_F$, and $P_F|_G$ denotes the restriction of P_F to $\mathcal{H}_G$. Precise definitions of all other terminology and the proof of Theorem 1.3 are given in Sections 5 and 6.

Theorem 1.3. Let $F = \{f_n\}_{n\in\mathbb{Z}}$ be a frame sequence in $\mathcal{H}$ with frame bounds A_F, B_F, and let $G = \{g_n\}_{n\in\mathbb{Z}}$ be a sequence in $\mathcal{H}$. Suppose that equation (1.1) and hypothesis (iii) from Theorem 1.1 are satisfied, so G is a frame sequence. Let $\{\widetilde{g}_n\}_{n\in\mathbb{Z}}$ be the canonical dual frame sequence for G within $\mathcal{H}_G$. Then, in addition to the conclusions of Theorem 1.2, the following statements hold.

(a) $P_F G = \{P_F g_n\}_{n\in\mathbb{Z}}$ is a frame for $\mathcal{H}_F$, and its canonical dual frame sequence is $\{(P_G|_F)^{-1}\widetilde{g}_n\}_{n\in\mathbb{Z}}$.
(b) If (F, G) is ℓ^p-localized, then $(F, P_F G)$ is ℓ^p-localized.
(c) If $(F, P_F G)$ is ℓ^2-localized then F, $P_F G$, and G all have the same frame measure function and hence are equivalent in the sense of [7].

1.5. Compact Perturbations

The hypothesis of the second type of frame sequence perturbation theorem is that $T_F - T_G$ is a compact operator. The following is the prototypical theorem of this type; this result is [13, Thm. 4.2].

Theorem 1.4. Suppose that F is a frame for $\mathcal{H}$ and G is a sequence in $\mathcal{H}$ such that $K := T_F - T_G$ is compact. Then G is a frame sequence.

We will generalize the above theorem to a perturbation of a frame sequence F. However, before stating our result, let us note when F is a frame sequence, compactness of $T_F - T_G$ alone does not imply that G is a frame sequence.

Example 1.5. Suppose that F is a frame sequence in $\mathcal{H}$ with finite rank. Then T_F is a finite rank operator, and hence is compact. Let $\{e_n\}_{n\in\mathbb{N}}$ be the standard basis of $\ell^2(\mathbb{N})$, and set $G := \{e_n/n\}_{n\in\mathbb{N}}$. Then, G is not a frame sequence, yet $T_F - T_G$ is compact since both T_F and T_G are compact.

Instead, we must combine compactness with a statement about the inf-angle between subspaces.

Theorem 1.6. Let F be a frame sequence in $\mathcal{H}$ and let G be another sequence in $\mathcal{H}$ such that $T_F - T_G$ is compact and $R(\mathcal{H}_G, \mathcal{H}_F) > 0$. Then G is a frame sequence in $\mathcal{H}$.

We also provide examples that show that most of the invariances that hold for Paley–Wiener-type perturbations do not hold for compact perturbations.

1.6. Outline

The remainder of this article is organized in the following manner. Section 2 presents basic notations. Section 3 introduces the inf-angle and provides several lemmas needed for the proof of Theorem 1.2. The proof of Theorem 1.2 is contained in Section 4. Section 5 introduces the frame measure function and related concepts, followed by the proof of Theorem 1.3 in Section 6. Finally, Section 7 gives the proof of Theorem 1.6 for compact perturbations of the frame operator as well as additional examples.

2. Basic Notation

Throughout this paper, $\mathcal{H}$ will denote a separable complex Hilbert space. If U is a closed subspace of $\mathcal{H}$, then P_U is the orthogonal projection of $\mathcal{H}$ onto U. If V is another closed subspace of $\mathcal{H}$, then $P_U|_V$ is the restriction of P_U to V. As a mapping $P_U|_V : V \to U$, we have $(P_U|_V)^* = P_V|_U : U \to V$. We define $U \ominus V = U \cap V^\perp$, and if U, V are orthogonal then $U \oplus V$ is the orthogonal direct sum of U and V.

Given $F \subseteq \mathcal{H}$, we let

$$\mathcal{H}_F = \overline{\mathrm{span}}(F)$$

denote the closed span of F. The orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_F$ is denoted by P_F. If a sequence $G \subseteq \mathcal{H}$ is also given, then $P_F|_G$ denotes P_F restricted to $\mathcal{H}_G$.

A sequence $F = \{f_i\}_{i\in\mathbb{I}}$ is a *Bessel* sequence in $\mathcal{H}$ if the *analysis operator* $C_F(f) = \{\langle f, f_i\rangle\}_{i\in\mathbb{I}}$ maps $\mathcal{H}$ into $\ell^2(\mathbb{I})$. In this case C_F is bounded, and the square of its operator norm is the *optimal Bessel bound*. Equivalently, F is a Bessel sequence if the *synthesis* or *pre-frame operator* $T_F : \ell^2(\mathbb{I}) \to \mathcal{H}$ defined by

$$T_F c = \sum_{i\in\mathbb{I}} c(i) f_i$$

is well-defined ($T_F c$ converges for each $c \in \ell^2(\mathbb{I})$) and bounded. The operators C_F and T_F are adjoints, so the optimal Bessel bound is $\|T_F\|^2 = \|C_F\|^2$. In this terminology, we can reword equation (1.1) as saying that $\{f_i - g_i\}_{i\in\mathbb{I}}$ is a Bessel sequence with Bessel bound μ^2.

We say that $F = \{f_i\}_{i\in\mathbb{I}}$ is a *frame sequence* if T_F is well-defined, bounded, and has closed range. Equivalently, F is a frame sequence if there exist constants A_F, $B_F > 0$, called *frame bounds*, such that

$$\forall\, f \in \mathcal{H}_F, \quad A_F\, \|f\|^2 \le \sum_{i\in\mathbb{I}} |\langle f, f_i\rangle|^2 \le B_F\, \|f\|^2. \tag{2.1}$$

A frame sequence F is a *frame* for $\mathcal{H}$ if $\mathcal{H}_F = \mathcal{H}$ (equivalently, $\mathrm{range}(T_F) = \mathcal{H}$).

If $F = \{f_i\}_{i\in\mathbb{I}}$ is a frame sequence, then the *optimal frame bounds* in equation (2.1) are $A_F = \|T_F^\dagger\|^{-2}$ and $B_F = \|T_F\|^2$, where $\dagger$ denotes the pseudo-inverse (Moore-Penrose generalized inverse) of a bounded operator with closed range [22]. The *frame operator* is $S_F = T_F C_F = T_F T_F^*$. Restricted to $\mathcal{H}_F$, the frame operator is a positive invertible mapping of $\mathcal{H}_F$ onto itself. The *canonical dual frame sequence* to F in $\mathcal{H}_F$ is $\widetilde{F} = \{\widetilde{f}_i\}_{i\in\mathbb{I}}$ where $\widetilde{f}_i = (S_F|_{\mathcal{H}_F})^{-1} f_i$. The orthogonal projection P_F of $\mathcal{H}$ onto $\mathcal{H}_F$ can be expressed in the form

$$P_F f \;=\; \sum_{i\in\mathbb{I}} \langle f, \widetilde{f}_i\rangle\, f_i \;=\; \sum_{i\in\mathbb{I}} \langle f, f_i\rangle\, \widetilde{f}_i, \qquad f \in \mathcal{H},$$

where these series converge unconditionally in the norm of $\mathcal{H}$. Given scalars c_i with $\sum |c_i|^2 < \infty$, the series $\sum c_i f_i$ converges unconditionally in $\mathcal{H}$, and we have

$$\Big\|\sum_{i\in\mathbb{I}} c_i f_i\Big\|^2 \;\le\; B_F \sum_{i\in\mathbb{I}} |c_i|^2.$$

Consequently, $\|f_i\|^2 \le B_F$ for each $i \in \mathbb{I}$.

We say that $F = \{f_i\}_{i\in\mathbb{I}}$ is a *Riesz sequence* if T_F is well-defined, bounded, one-to-one, and has closed range. Equivalently, there exist constants A_F, $B_F > 0$ such that

$$\forall \text{ finite sequences } c \in \ell^2(\mathbb{I}), \quad A_F \sum_{i\in\mathbb{I}} |c_i|^2 \;\le\; \Bigl\|\sum_{i\in\mathbb{I}} c_i f_i\Bigr\|^2 \;\le\; B_F \sum_{i\in\mathbb{I}} |c_i|^2. \qquad (2.2)$$

In this case, the *optimal Riesz bounds* in equation (2.2) are again $A_F = \|T_F^\dagger\|^{-2}$ and $B_F = \|T_F\|^2$. We say that a Riesz sequence F is a *Riesz basis* for $\mathcal{H}$ if $\mathcal{H}_F = \mathcal{H}$ (equivalently, $\mathrm{range}(T_F) = \mathcal{H}$). Every Riesz sequence is a frame sequence, but the converse fails in general.

3. The Inf-Cosine Angle and its Symmetry

Throughout this section we let X and Y be closed subspaces of $\mathcal{H}$. If X is not trivial, the *inf-angle* between X and Y is

$$R(X,Y) = \inf_{x\in X\setminus\{0\}} \frac{\|P_Y x\|}{\|x\|},$$

where P_Y denotes the orthogonal projection onto Y. The arc-cosine of $R(X,Y)$ is usually interpreted as the largest angle between two vectors of X and Y. We define $R(\{0\},Y) = 1$ for all Y. By [30], $R(X,Y) = R(Y^\perp, X^\perp)$.

If X is a proper subspace of Y, then $R(X,Y) = 1$ and $R(Y,X) = 0$ since $Y \ominus X = Y \cap X^\perp$ is nontrivial. Hence R is not symmetric. We will show below that this is almost the only possible asymmetry of R (see Lemma 3.4).

We say that a bounded operator T is *bounded below* if there exists a positive constant c such that $\|Tf\| \ge c\,\|f\|$ for each f in the domain of T. In this case, T has closed range, hence $T^\dagger$ is well-defined and bounded. A standard fact is that T is bounded below if and only if T^* is onto.

We state next a formula for the angle involving the pseudo-inverse of $P_Y|_X$. This result is a direct generalization of [26, Lem. 3.5], and therefore we omit the proof.

Lemma 3.1. Suppose that X, Y are closed subspaces of $\mathcal{H}$, with X not trivial. Then

$$R(X,Y) = \begin{cases} 0, & \text{if } P_Y|_X \text{ is not bounded below,} \\ \|(P_Y|_X)^\dagger\|^{-1}, & \text{if } P_Y|_X \text{ is bounded below.} \end{cases}$$

In particular,

$$R(X,Y) > 0 \iff P_Y|_X \text{ is bounded below} \iff P_X|_Y \text{ is onto.}$$

The following simple observation will be an important tool in the proof of Theorem 1.2.

Lemma 3.2. If $0 = R(Y,X) < R(X,Y)$, then $Y = P_Y(X) \oplus (Y \ominus X)$ and $Y \ominus X$ is not trivial.

Proof. First, suppose that X is trivial. Then Y is not trivial since $0 = R(Y,X)$. Therefore $Y \ominus X$ is not trivial, and $P_Y(X) \oplus (Y \ominus X) = 0 \oplus Y = Y$.

Suppose, on the other hand, that X is not trivial. Lemma 3.1 implies that $P_Y|_X$ is bounded below and $(P_Y|_X)^* = P_X|_Y$ is not bounded below. Hence $P_Y|_X : X \to Y$ is bounded below and not onto. Since $P_Y|_X$ is bounded below, its range $P_Y|_X(X) = P_Y(X)$ is closed. Moreover, $Y \ominus P_Y|_X(X)$ is not trivial since $P_Y|_X$ is not onto. Hence $Y = P_Y(X) \oplus (Y \ominus P_Y|_X(X))$. Let $y \in Y \ominus P_Y|_X(X)$. In particular,

$$y \in Y \cap (\mathrm{range} P_Y|_X)^\perp = Y \cap \ker\big((P_Y|_X)^*\big) = Y \cap \ker(P_X|_Y),$$

i.e., $y \in Y$ and $P_X|_Y(y) = P_X y = 0$. This shows that $Y \ominus P_Y|_X(X) = Y \ominus X$. ☐

Let U and V be two closed subspaces of $\mathcal{H}$. If $U + V = \mathcal{H}$ and $U \cap V = \{0\}$, then we write $\mathcal{H} = U \dot{+} V$. In this case, each $f \in \mathcal{H}$ can be decomposed uniquely as $f = u + v$ for $u \in U$ and $v \in V$. Therefore, the oblique (non-orthogonal) projection $P_{U,V} f := u$ is well-defined and bounded (note the distinction between the oblique projection $P_{U,V}$ and the restriction $P_U|_V$ of the orthogonal projection P_U to the space V). The inf-angle is closely related to the existence of certain oblique projections, as the following proposition shows.

Proposition 3.3. Suppose that at least one of X or Y is nontrivial. Then the following assertions are equivalent.

(1) $0 < R(X,Y)$ and $0 < R(Y,X)$.

(2) $0 < R(X,Y) = R(Y,X)$.

(3) $P_Y|_X$ is invertible.

(4) $P_X|_Y$ is invertible.

(5) $\mathcal{H} = X \dot{+} Y^\perp$.

(6) $\mathcal{H} = Y \dot{+} X^\perp$.

Moreover, in case these hold, we have:

- $0 < R(X,Y) = R(Y,X) = \|(P_Y|_X)^{-1}\|^{-1} = \|(P_Y|_X)^{-1}\|^{-1}$,
- X and Y are isomorphic, and
- $X^\perp$ and $Y^\perp$ are isomorphic.

Proof. The equivalence of (1), (5), and (6) is folklore (see, for example, [29]). For the equivalence of (1), (2), (3), and (4) we argue as follows. By Lemma 3.1, both $R(X,Y)$ and $R(Y,X)$ are positive if and only if both $P_Y|_X$ and $P_X|_Y$ are bounded below, and since they are adjoints of each other, this holds if and only if both operators are invertible.

In case (1)–(6) hold, the formula in Lemma 3.1 implies that

$$R(X,Y) = \|(P_Y|_X)^{-1}\|^{-1} = \|\left((P_Y|_X)^*\right)^{-1}\|^{-1} = \|(P_X|_Y)^{-1}\|^{-1} = R(Y,X).$$

Further, X and Y are isomorphic by (3) or (4), and $X^\perp$ and $Y^\perp$ are isomorphic since $R(X,Y) = R(Y^\perp, X^\perp)$ [30]. ☐

As a simple consequence of the equivalence of statements (1) and (2) in Proposition 3.3, we have the following symmetry result about the inf-angle.

Corollary 3.4. Suppose that at least one of X or Y is nontrivial. Then $R(X, Y) \neq R(Y, X)$ if and only if one of these quantities is zero and the other is positive.

We also need the following results. Given a bounded operator $T : \mathcal{H}_1 \to \mathcal{H}_2$, define

$$\gamma(T) := \inf\bigl\{\|Th_1\| : h_1 \in \ker(T)^\perp, \|h_1\| = 1\bigr\}.$$

Then T has closed range if and only if $\gamma(T) > 0$ [25]. Moreover, in this case, $\|T^\dagger\| = \gamma(T)^{-1}$ [19]. In particular, if T_F is the synthesis operator of a frame F with lower frame bound A_F, then we have

$$\sqrt{A_F} \leq \|T_F^\dagger\|^{-1} = \gamma(T_F). \tag{3.1}$$

The following result is contained in the proof of [20, Lem. 3.4] (see also the proof of [12, Thm. 2.2]).

Lemma 3.5. If $T, \widetilde{T} : \mathcal{H}_1 \to \mathcal{H}_2$ are bounded and $\ker(\widetilde{T})^\perp$ is not trivial, then

$$\inf\{\|Th_1\| : h_1 \in \ker(\widetilde{T})^\perp, \|h_1\| = 1\} \geq \gamma(T)\, R\bigl(\ker(T), \ker(\widetilde{T})\bigr).$$

4. Proof of Theorem 1.2

In this section we will prove Theorem 1.2. We will need the following result by Casazza and Christensen [11, Thm. 3.6]; see also [24, Thm. 3.1].

Proposition 4.1. Let $F := \{f_i\}_{i\in\mathbb{I}}$ be a frame for $\mathcal{H}$ with frame bounds A_F and B_F. Fix $G := \{g_i\}_{i\in\mathbb{I}} \subset \mathcal{H}$ and $\mu > 0$. If F and G satisfy equation (1.1), then

$$\dim\bigl(\ker(T_F)\bigr) < \infty \quad \iff \quad \dim\bigl(\ker(T_G)\bigr) < \infty.$$

Moreover, in this case we have $\dim\bigl(\ker(T_F)\bigr) = \dim\bigl(\ker(T_G)\bigr)$, and hence $\ker(T_F)$ and $\ker(T_G)$ are isomorphic.

Proof of Theorem 1.2. In light of the discussion following Theorem 1.2, we concentrate on proving the new portions of Theorem 1.2.

(i) $\Rightarrow$ (iii): Suppose that hypothesis (i) and equation (1.1) hold. Then Theorem 2.1 in [14] implies that $0 < R(\mathcal{H}_F, \mathcal{H}_G)$ since we have $\mu < \sqrt{A_F}$ (see also [27, Thm. 2.1]). Suppose that we had $R(\mathcal{H}_G, \mathcal{H}_F) = 0$. Since both F and G are frame sequences, we have $\mathrm{range}(T_F) = \mathcal{H}_F$ and $\mathrm{range}(T_G) = \mathcal{H}_G$. Lemma 3.2 implies that $\mathcal{H}_G \ominus \mathcal{H}_F$ is nontrivial, so there exists some $0 \neq g \in \mathrm{range}(T_G) \ominus \mathrm{range}(T_F)$. Therefore, there must exist some $c \in \ell^2(\mathbb{I})$ such that $T_G c = g \perp \mathrm{range}(T_F)$. We may assume that $\|c\| = 1$ and that $c \perp \ker(T_G)$. Then equation (1.1) and the Pythagorean Theorem imply that

$$\|T_F c\|^2 + \|T_G c\|^2 = \|T_F c - T_G c\|^2 \leq \mu^2.$$

Applying equation (3.1) and Lemma 3.5 with $T = T_F$ and $\widetilde{T} = T_G$, we see that

$$A_F\, R\bigl(\ker(T_F), \ker(T_G)\bigr)^2 \leq \gamma(T_F)^2\, R\bigl(\ker(T_F), \ker(T_G)\bigr)^2 \leq \|T_F c\|^2 \leq \mu^2,$$

which contradicts (i).

Invariance Properties. Now we establish the invariance properties that hold under Paley–Wiener-type perturbations. Since (i) and (ii) both imply (iii), we assume that (iii) holds. By [14, Thm. 2.1], we have $0 < R(\mathcal{H}_F, \mathcal{H}_G)$ since $\mu < \sqrt{A_F}$. In particular, $0 < R(\mathcal{H}_F, \mathcal{H}_G) = R(\mathcal{H}_G, \mathcal{H}_F)$ by Proposition 3.3. Proposition 3.3 also implies that $\mathcal{H}_F$ and $\mathcal{H}_G$ are isomorphic, $\mathcal{H}_F^\perp$ and $\mathcal{H}_G^\perp$ are isomorphic, and $\mathcal{H} = \mathcal{H}_G \dotplus \mathcal{H}_F^\perp$. In particular, $\mathrm{rank}(F) = \dim(\mathcal{H}_F) = \dim(\mathcal{H}_G) = \mathrm{rank}(G)$, and $\mathrm{deficit}(F) = \dim(\mathcal{H}_F^\perp) = \dim(\mathcal{H}_G^\perp) = \mathrm{deficit}(G)$.

Now,

$$\mathcal{H}_F \oplus \mathcal{H}_F^\perp = \mathcal{H} = \mathcal{H}_G \dotplus \mathcal{H}_F^\perp.$$

If $\mathcal{H}_F^\perp$ is trivial then $\ker(T_F)$ and $\ker(T_G)$ are isomorphic by Proposition 4.1. Suppose that $\mathcal{H}_F^\perp$ is not trivial, and let $H := \{h_j\}_{j\in\mathbb{J}}$ be an orthonormal basis for $\mathcal{H}_F^\perp$, with index set $\mathbb{J}$ disjoint from $\mathbb{I}$. Set $\widetilde{F} := F \cup H$ and $\widetilde{G} := G \cup H$. Then $\widetilde{F}$ is a frame for $\mathcal{H}$ with frame bounds A_F and B_F, and $\widetilde{F}$ and $\widetilde{G}$ satisfy the hypothesis of Proposition 4.1. Hence $\ker(T_{\widetilde{F}})$ and $\ker(T_{\widetilde{G}})$ are isomorphic, but since $\ker(T_F)$ is isomorphic to $\ker(T_{\widetilde{F}})$ and $\ker(T_G)$ is isomorphic to $\ker(T_{\widetilde{G}})$, we conclude that $\ker(T_F)$ and $\ker(T_G)$ are isomorphic. Consequently, the excess of F and G are equal by equation (1.2). □

Remark 4.2. If we assume the perturbation condition (i), we can prove that the kernels of the synthesis operators are isomorphic without using Proposition 4.1. Suppose that (i) and equation (1.1) hold. Since $R\big(\ker(T_F), \ker(T_G)\big) \le 1$, hypothesis (i) implies that $\mu < \sqrt{A_F}$. If we had $R\big(\ker(T_G), \ker(T_F)\big) = 0$, then

$$0 = R\big(\ker(T_G), \ker(T_F)\big) < R\big(\ker(T_F), \ker(T_G)\big).$$

By Lemma 3.2, there exists some $c \in \ker(T_G) \ominus \ker(T_F)$ with $\|c\| = 1$. Equations (1.1), (3.1) and the definition of $\gamma(T_F)$ then imply that

$$\sqrt{A_F} \le \gamma(T_F) \le \|T_F c\| = \|T_F c - T_G c\| \le \mu,$$

which is impossible since $\mu < \sqrt{A_F}$. Hence $R\big(\ker(T_F), \ker(T_G)\big)$ and $R\big(\ker(T_G), \ker(T_F)\big)$ must both be positive. Proposition 3.3 implies that $\ker(T_F)$ and $\ker(T_G)$ are isomorphic. Using the same argument, one can show that $\ker(T_F)$ is isomorphic to $\ker(T_G)$ if the three parameter version of (i) is satisfied.

5. Localization and the Frame Measure Function

In this section we introduce the frame measure function. The definition depends on a choice of exhaustive increasing finite subsets of the index set.

Definition 5.1. Let $I_1 \subsetneq I_2 \subsetneq \cdots$ be nested finite increasing subsets of $\mathbb{I}$ such that $\cup I_n = I$. Given a frame $F = \{f_i\}_{i\in\mathbb{I}}$ with canonical dual frame sequence $\widetilde{F} = \{\widetilde{f}_i\}_{i\in\mathbb{I}}$, the *frame measure function* of F is

$$\mu_F(p) \;=\; p\text{-}\lim \frac{1}{|I_N|} \sum_{i\in I_N} \langle f_i, \widetilde{f}_i\rangle,$$

where p is a free ultrafilter on $\mathbb{N}$.

A short review of ultrafilters and their properties can be found in [23, App. B]. The important facts for us about a free ultrafilter p are the following.

- p determines a linear functional on $\ell^\infty(\mathbb{N})$. We denote the action of this linear functional on $c = (c_N)_{N\in\mathbb{N}} \in \ell^\infty(\mathbb{N})$ by $p\text{-}\lim c_N$.
- If $c = (c_N)_{N\in\mathbb{N}} \in \ell^\infty(\mathbb{N})$ then $p\text{-}\lim c_N$ is an accumulation point of c, and hence $\liminf c_N \le p\text{-}\lim c_N \le \limsup c_N$.
- If $c = (c_N)_{N\in\mathbb{N}} \in \ell^\infty(\mathbb{N})$ and s is an accumulation point of c then there exists a free ultrafilter p such that $p\text{-}\lim c_N = s$. In particular, there exist free ultrafilters p, q such that $p\text{-}\lim c_N = \liminf c_N$ and $q\text{-}\lim c_N = \limsup c_N$.

The essential idea of the frame equivalence relation given in [7] is that two frames are equivalent if they have the same frame measure function.

In order to define "localized frames," we need to impose more structure on the index set $\mathbb{I}$. For simplicity of notation, we will take $\mathbb{I} = \mathbb{Z}$ and

$$I_N \;=\; \mathbb{Z} \cap [-N, N].$$

However, following the techniques of [4], [5], the same ideas can be adapted to more general frames, such as "irregular" Gabor frames $\mathcal{G}(g, \Lambda)$ where Λ is not a lattice (but is embedded into the additive group $\mathbb{R}^2$). Localization is formulated in terms of the off-diagonal decay of the cross-Grammian of a frame with respect to a reference sequence.

Definition 5.2. Let $F = \{f_i\}_{i\in\mathbb{Z}}$ and $G = \{g_i\}_{i\in\mathbb{Z}}$ be sequences in $\mathcal{H}$. Given $1 \le p \le \infty$, we say that (F, G) is *ℓ^p-localized* if there exists some $r \in \ell^p(\mathbb{Z})$ such that

$$|\langle f_i, g_j\rangle| \;\le\; r_{j-i}, \qquad i, j \in \mathbb{Z}.$$

6. Perturbation of the Frame Measure Function

We will prove Theorem 1.3 in this section.

Proof of Theorem 1.3. (a) By Theorem 1.2, $\mathcal{H}_F$ is isomorphic to $\mathcal{H}_G$ and the isomorphism is given by $P_F|_G \colon \mathcal{H}_G \to \mathcal{H}_F$. Letting $S_G \colon \mathcal{H}_G \to \mathcal{H}_G$ denote the frame operator for G, the canonical dual frame sequence for G is $\widetilde{G} = S_G^{-1} G$. The frame operator for $P_F G$ is $S_{P_F G} = (P_F|_G) S_G (P_F|_G)^* = (P_F|_G) S_G (P_G|_F)$, so the canonical dual frame sequence for $P_F G$ is

$$S_{P_F G}^{-1} P_F G \;=\; (P_G|_F)^{-1} S_G^{-1} G \;=\; (P_G|_F)^{-1} \widetilde{G}.$$

(b) This follows from the fact that $\langle f_i, P_F g_j\rangle = \langle P_F f_i, g_j\rangle = \langle f_i, g_j\rangle$.

(c) Let A_F, B_F be frame bounds for F, and let $A_{P_F G}$, $B_{P_F G}$ be frame bounds for the frame sequence $P_F G$. Let $\widetilde{F} = \{\widetilde{f}_i\}_{i\in\mathbb{I}}$ be the canonical dual frame sequence for F. For simplicity of notation, in this proof we will write P_G for $P_G|_F$ and P_F for $P_F|_G$. Thus, from part (a) we have that the canonical dual frame sequence for $P_F G$ is $P_G^{-1} \widetilde{G}$.

To prove that F and $P_F G$ have the same measure function, it suffices to show that

$$\lim_{N\to\infty} \left(\frac{1}{|I_N|} \sum_{i\in I_N} \langle f_i, \widetilde{f}_i\rangle \;-\; \frac{1}{|I_N|} \sum_{j\in I_N} \langle P_F g_j, P_G^{-1} \widetilde{g}_j\rangle \right) \;=\; 0.$$

By definition, there exists some $r \in l^2(I)$ such that $|\langle f_i, g_j\rangle| = |\langle f_i, P_F g_j\rangle| \le r_{j-i}$. Fix $\varepsilon > 0$, and choose N_ε so that

$$\sum_{k \in I - I_{N_\varepsilon}} r_k^2 \; < \; \varepsilon.$$

Then

$$\begin{aligned}
&\sum_{i \in I_N} \langle f_i, \widetilde{f}_i\rangle \; - \; \sum_{j \in I_N} \langle P_F g_j, P_G^{-1} \widetilde{g}_j\rangle \\
&= \; \sum_{i \in I_N} \langle \widetilde{f}_i, f_i\rangle \; - \; \sum_{j \in I_N} \langle P_F g_j, P_G^{-1} \widetilde{g}_j\rangle \\
&= \; \sum_{i \in I_N} \sum_{j \in I} \langle \widetilde{f}_i, P_G^{-1} \widetilde{g}_j\rangle \, \langle P_F g_j, f_i\rangle \; - \; \sum_{j \in I_N} \sum_{i \in I} \langle \widetilde{f}_i, P_G^{-1} \widetilde{g}_j\rangle \, \langle P_F g_j, f_i\rangle \\
&= \; \sum_{i \in I_N} \sum_{j \in I - I_N} \langle \widetilde{f}_i, P_G^{-1} \widetilde{g}_j\rangle \, \langle P_F g_j, f_i\rangle \; - \; \sum_{j \in I_N} \sum_{i \in I - I_N} \langle \widetilde{f}_i, P_G^{-1} \widetilde{g}_j\rangle \, \langle P_F g_j, f_i\rangle \\
&= \; \sum_{i \in I_N} \sum_{j \in I - I_{N+N_\varepsilon}} \langle \widetilde{f}_i, P_G^{-1} \widetilde{g}_j\rangle \, \langle P_F g_j, f_i\rangle \\
&\qquad + \sum_{i \in I_N} \sum_{j \in I_{N+N_\varepsilon} - I_N} \langle \widetilde{f}_i, P_G^{-1} \widetilde{g}_j\rangle \, \langle P_F g_j, f_i\rangle \\
&\qquad - \sum_{j \in I_N} \sum_{i \in I - I_{N+N_\varepsilon}} \langle \widetilde{f}_i, P_G^{-1} \widetilde{g}_j\rangle \, \langle P_F g_j, f_i\rangle \\
&\qquad - \sum_{j \in I_N} \sum_{i \in I_{N+N_\varepsilon} - I_N} \langle \widetilde{f}_i, P_G^{-1} \widetilde{g}_j\rangle \, \langle P_F g_j, f_i\rangle \\
&= \; T_1 + T_2 - T_3 - T_4.
\end{aligned}$$

Since the frame bounds for $\widetilde{F}$ are B_F^{-1}, A_B^{-1} and those for $P_G^{-1}\widetilde{G}$ are $B_{P_F G}^{-1}$, $A_{P_F G}^{-1}$, we have

$$\sum_j |\langle \widetilde{f}_i, P_G^{-1} \widetilde{g}_j\rangle|^2 \; \le \; A_{P_F G}^{-1} \, \|\widetilde{f}_i\|^2 \; \le \; \bigl(A_F \, A_{P_F G}\bigr)^{-1}.$$

Therefore,

$$\begin{aligned}
|T_1| \; &\le \; \sum_{i \in I_N} \sum_{j \in I - I_{N+N_\varepsilon}} |\langle \widetilde{f}_i, P_G^{-1} \widetilde{g}_j\rangle| \, |\langle P_F g_j, f_i\rangle| \\
&\le \; \sum_{i \in I_N} \Biggl(\sum_{j \in I - I_{N+N_\varepsilon}} |\langle \widetilde{f}_i, P_G^{-1} \widetilde{g}_j\rangle|^2 \Biggr)^{1/2} \Biggl(\sum_{j \in I - I_{N+N_\varepsilon}} |\langle P_F g_j, f_i\rangle|^2 \Biggr)^{1/2} \\
&\le \; \bigl(A_F \, A_{P_F G}\bigr)^{-1/2} \sum_{i \in I_N} \Biggl(\sum_{j \in I - I_{N+N_\varepsilon}} |\langle P_F g_j, f_i\rangle|^2 \Biggr)^{1/2}
\end{aligned}$$

$$\leq \big(A_F\,A_{P_FG}\big)^{-1/2} \sum_{i\in I_N} \Bigg(\sum_{j\in I - I_{N+N_\varepsilon}} r_{j-i}^2 \Bigg)^{1/2}$$

$$\leq \left(\frac{\varepsilon}{A_F\,A_{P_FG}}\right)^{1/2} |I_N|,$$

and it similarly follows that

$$|T_3| \;\leq\; \left(\frac{\varepsilon}{A_F\,A_{P_FG}}\right)^{1/2} |I_N|.$$

Also,

$$\begin{aligned}
|T_2| \;&\leq\; \sum_{i\in I_N} \sum_{j\in I_{N+N_\varepsilon} - I_N} |\langle \widetilde{f}_i, P_G^{-1}\widetilde{g}_j\rangle|\,|\langle P_F g_j, f_i\rangle| \\
&\leq\; \big(A_F\,A_{P_FG}\big)^{-1/2} \sum_{j\in I_{N+N_\varepsilon} - I_N} \Bigg(\sum_{i\in I_N} |\langle P_F g_j, f_i\rangle|^2 \Bigg)^{1/2} \\
&\leq\; \left(\frac{B_F\,B_{P_FG}}{A_F\,A_{P_FG}}\right)^{1/2} |I_{N+N_\varepsilon} - I_N|, \\
&=\; \left(\frac{B_F\,B_{P_FG}}{A_F\,A_{P_FG}}\right)^{1/2} 2N_\varepsilon,
\end{aligned}$$

and similarly

$$|T_4| \;\leq\; \left(\frac{B_F\,B_{P_FG}}{A_F\,A_{P_FG}}\right)^{1/2} 2N_\varepsilon.$$

Combining the previous estimates, we see that

$$\begin{aligned}
&\limsup_{N\to\infty} \left| \frac{1}{|I_N|} \sum_{i\in I_N} \langle f_i, \widetilde{f}_i\rangle - \frac{1}{|I_N|} \sum_{j\in I_N} \langle P_F g_j, P_G^{-1}\widetilde{g}_j\rangle \right| \\
&\qquad\leq\; \limsup_{N\to\infty} \frac{|T_1| + |T_2| + |T_3| + |T_4|}{|I_N|} \\
&\qquad\leq\; \limsup_{N\to\infty} \left(2\left(\frac{\varepsilon}{A_F\,A_{P_FG}}\right)^{1/2} \;+\; 2\left(\frac{B_F\,B_{P_FG}}{A_F\,A_{P_FG}}\right)^{1/2} \frac{2N_\varepsilon}{2N} \right) \\
&\qquad=\; 2\left(\frac{\varepsilon}{A_F\,A_{P_FG}}\right)^{1/2}.
\end{aligned}$$

Since ε is arbitrary, our claim that F and P_FG have the same measure function follows. Finally, since

$$\begin{aligned}
\big\langle P_F g_i,\, P_G^{-1}\widetilde{g}_j \big\rangle \;&=\; \big\langle (P_F|_G) g_i,\, (P_G|_F)^{-1}\widetilde{g}_j \big\rangle \\
&=\; \big\langle g_i,\, (P_F|_G)^*(P_G|_F)^{-1}\widetilde{g}_j \big\rangle \\
&=\; \big\langle g_i,\, (P_G|_F)(P_G|_F)^{-1}\widetilde{g}_j \big\rangle \;=\; \langle g_i, \widetilde{g}_j\rangle,
\end{aligned}$$

we conclude that G and P_FG have the same frame measure functions. $\square$

7. Compact Perturbations

We will give the proof of Theorem 1.6 in this section.

First, we recall some facts on the *sup-angle* between subspaces, cf. [18], [29], [30]. Given closed subspaces $X \neq \{0\}$ and Y in $\mathcal{H}$, define

$$S(X,Y) \;=\; \sup_{x\in X\setminus\{0\}} \frac{\|P_Y x\|}{\|x\|} = \|P_Y|_X\|.$$

The arc-cosine of $S(X,Y)$ is the smallest angle between vectors in X and Y. We define $S(\{0\},Y)=0$. By [30],

$$R(X,Y) \;=\; \bigl(1 - S(X,Y^{\perp})^2\bigr)^{1/2}.$$

Moreover, $X+Y$ is closed and $X\cap Y=\{0\}$ if and only if $S(X,Y)<1$ [18], [29].

Proof of Theorem 1.6. Assume that F is a frame sequence in $\mathcal{H}$, and G is another sequence such that $T_F - T_G$ is compact. If $\mathcal{H}_F^{\perp}$ is trivial, then G is a frame sequence by Theorem 1.4. Hence we suppose that $\mathcal{H}_F^{\perp}$ is not trivial. Let $H := \{h_j\}_{j\in\mathbb{J}}$ be an orthonormal basis for $\mathcal{H}_F^{\perp}$, with an index set $\mathbb{J}$ that is disjoint from $\mathbb{I}$. Set $\widetilde{F} := F\cup H$ and $\widetilde{G} := G\cup H$. Then, $\widetilde{F}$ is a frame for $\mathcal{H}$ and $T_{\widetilde{F}} - T_{\widetilde{G}}$ is compact, so $\widetilde{G}$ is a frame sequence by Theorem 1.4.

By hypothesis, $R(\mathcal{H}_G,\mathcal{H}_F)>0$, so

$$S(\mathcal{H}_G,\mathcal{H}_F^{\perp}) \;=\; \bigl(1 - R(\mathcal{H}_G,\mathcal{H}_F)^2\bigr)^{1/2} \;<\; 1.$$

Consequently $\mathcal{H}_G + \mathcal{H}_F^{\perp}$ is closed and $\mathcal{H}_G \cap \mathcal{H}_F^{\perp} = \{0\}$, and therefore $\widetilde{G} = G\cup H$ is a frame for $\mathcal{H}_G \dotplus \mathcal{H}_F^{\perp}$. In particular, $T_{\widetilde{G}} : \ell^2(\mathbb{I}\cup\mathbb{J}) \to \mathcal{H}_G \dotplus \mathcal{H}_F^{\perp}$ is onto. It is enough to show that $T_G : \ell^2(\mathbb{I}) \to \mathcal{H}_G$ is onto.

Fix any $g\in\mathcal{H}_G$. Then since $T_{\widetilde{G}}$ is onto, there exists some $c\in\ell^2(\mathbb{I}\cup\mathbb{J})$ such that $T_{\widetilde{G}}c = g$. Write $c=(a,b)$ where $a\in\ell^2(\mathbb{I})$ and $b\in\ell^2(\mathbb{J})$. Then

$$g \;=\; T_{\widetilde{G}}c = T_G a + T_H b,$$

so

$$g - T_G a \;=\; T_H b \;\in\; \mathcal{H}_G \cap \mathcal{H}_F^{\perp} \;=\; \{0\}.$$

Hence $T_G : \ell^2(\mathbb{I}) \to \mathcal{H}_G$ is onto, so G is a frame sequence. □

Note that if F is a frame for $\mathcal{H}$, then $\mathcal{H}_F = \mathcal{H}$, so $R(\mathcal{H}_G,\mathcal{H}_F) = 1 > 0$. Thus Theorem 1.4 is a special case of Theorem 1.6, even though Theorem 1.4 is used in the proof of Theorem 1.6.

We now show that a perturbation of the type in Theorem 1.6 does not preserve frame properties, and is independent of the conditions in Theorem 1.1.

The fact that a compact perturbation preserves neither excess nor deficit is apparent from [11, Cor. 3.4]. For example, let $\mathcal{H} := \ell^2(\mathbb{N})$, let $B := \{e_n\}_{n\in\mathbb{N}}$ be the standard basis for $\ell^2(\mathbb{N})$, set $F := B$, and let $G := \{0, e_2, e_3, \cdots\}$.

The angle condition in Theorem 1.6 implies that $\dim(\mathcal{H}_G) \le \dim(\mathcal{H}_F)$ since $P_F|_G : \mathcal{H}_G \to \mathcal{H}_F$ is injective. Hence the perturbation in Theorem 1.6 can only decrease the rank, and in fact the rank can be strictly decreased (consider $F := \{e_1, e_2\}$ and $G := \{e_1, 0\}$). The more subtle question is: Do there exist F and G satisfying Theorem 1.6 such that $\dim(\mathcal{H}_F) = \infty$ and $\dim(\mathcal{H}_G) < \infty$? We will show that this is impossible.

Suppose that F is a frame sequence such that $\dim(\operatorname{range} T_F) = \infty$, $K := T_F - T_G$ is compact, and $\dim(\operatorname{range} T_G) < \infty$. Then $T_F = K + T_G$ is compact since K and T_G are compact. Moreover, $\operatorname{range}(T_F)$ is closed since F is a frame sequence. Therefore $\operatorname{range}(T_F)$ is finite-dimensional since T_F is compact [17, p. 177]. This shows that if F is a frame sequence of infinite rank, then a compact perturbation preserves the rank.

Note that equation (1.1) is satisfied if the norm of $T_F - T_G$ is small enough, and $T_F - T_G$ can be compact even if its norm is large. Hence it is easy to see that equation (1.1) is independent of the conditions in Theorem 1.6.

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