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Density of Gabor Schauder bases

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ABSTRACT

A Gabor system is a fixed set of time-frequency shifts $G(g,\Lambda) = \{e^{2\pi i b\cdot x} g(x-a)\}_{(a,b)\in\Lambda}$ of a function $g \in L^2(\mathbf{R}^d)$. We prove that if $G(g,\Lambda)$ forms a Schauder basis for $L^2(\mathbf{R}^d)$ then the upper Beurling density of Λ satisfies $D^+(\Lambda) \le 1$. We also prove that if $G(g,\Lambda)$ forms a Schauder basis for $L^2(\mathbf{R}^d)$ and if g lies in a the modulation space $M^{1,1}(\mathbf{R}^d)$, which is a dense subset of $L^2(\mathbf{R}^d)$, or if $G(g,\Lambda)$ possesses at least a lower frame bound, then Λ has uniform Beurling density $D(\Lambda) = 1$. We use related techniques to show that if $g \in L^1(\mathbf{R}^d) \cap L^2(\mathbf{R}^d)$ then no collection $\{g(x-a)\}_{a\in\Gamma}$ of pure translates of g can form a Schauder basis for $L^2(\mathbf{R}^d)$. We also extend these results to the case of finitely many generating functions $g_1, \dots, g_r$.

Keywords: Beurling density, frames, Gabor systems, Schauder bases

1. INTRODUCTION

Zalik[21,22] extensively studied functions $g \in L^2(\mathbf{R})$ for which there exists a countable index set $\Gamma \subset \mathbf{R}$ such that $\{g(x-a)\}_{a\in\Gamma}$ is complete in $L^2(\mathbf{R})$, i.e., its finite linear span is dense in $L^2(\mathbf{R})$. Olson and Zalik[16] proved that no such collection of pure translates can form a Riesz basis for $L^2(\mathbf{R})$, and they further conjectured that $\{g(x-a)\}_{a\in\Gamma}$ could never form a Schauder basis for $L^2(\mathbf{R})$. Christensen, Deng, and Heil[4] proved that $\{g(x-a)\}_{a\in\Gamma}$ could never form a frame for $L^2(\mathbf{R})$. This result was obtained as a corollary of more general results on the "density" of finite unions of Gabor frames for $L^2(\mathbf{R}^d)$, i.e., frames of the form $\{e^{2\pi i b\cdot x} g_k(x-a)\}_{(a,b)\in\Lambda_k,\, k=1,\dots,r}$. Frames are possibly overcomplete systems that retain many of the properties of Riesz bases; all Riesz bases are frames but not conversely. No frame can be a Schauder basis without being a Riesz basis.

Frames, Schauder bases, and Riesz bases are defined precisely in Section 2. Intuitively, next to orthonormal bases, Riesz bases are the "nicest" bases for a Hilbert space. They provide unique expansions of all elements of the space in terms of the basis elements, and furthermore those expansions converge unconditionally, which leads to a certain robustness or stability in the convergence of the expansions. Frames likewise allow expansions of elements of the space in terms of the frame elements, also with unconditional convergence of the expansions. However, unlike Riesz bases, those expansions need not be unique. By contrast, Schauder bases do possess the property of unique expansions, but the expansions may fail to converge unconditionally. Consequently, dealing with Schauder bases that are not Riesz bases often requires a considerable amount of delicacy. In particular, almost no results are currently available about Gabor systems which are Schauder bases but not Riesz bases. In this paper, we provide an example of such a system, and we prove various necessary density conditions for Gabor Schauder bases. In particular, we show that if $\{e^{2\pi i b\cdot x} g_k(x-a)\}_{(a,b)\in\Lambda_k,\, k=1,\dots,r}$ forms a Gabor Schauder basis for $L^2(\mathbf{R}^d)$, then the "disjoint union" or amalgamation Λ of the sequences $\Lambda_1, \dots, \Lambda_r$ must have upper Beurling density $D^+(\Lambda) \le 1$. We further show that if $g_1, \dots, g_r$ lie in the modulation space $M^{1,1}(\mathbf{R}^d)$, which is a dense subspace of $L^2(\mathbf{R}^d)$, or if $\{e^{2\pi i b\cdot x} g_k(x-a)\}_{(a,b)\in\Lambda_k,\, k=1,\dots,r}$ possesses at least a lower frame bound, then Λ must actually have uniform Beurling density $D(\Lambda) = 1$. Using related techniques, we prove that if $g_1, \dots, g_r$ lie in $L^1(\mathbf{R}^d) \cap L^2(\mathbf{R}^d)$, which is a dense subset of $L^2(\mathbf{R}^d)$, then $\{g_k(x-a)\}_{a\in\Gamma_k,\, k=1,\dots,r}$ can never form a Schauder basis for $L^2(\mathbf{R}^d)$ for any sequences $\Gamma_1, \dots, \Gamma_r$.

2. DEFINITIONS AND DISCUSSION

In this section we will define our terminology and provide some background and discussion of our results.

2.1. Bases and Frames

We will briefly review the definitions and basic properties of bases and frames in this subsection, referring to the excellent books by Young[20] or Daubechies[7], the research-tutorial of Heil and Walnut[14], the introductory manuscript of Heil[13], or the encylopedic book of Singer[19] for proofs and additional information.

$\mathbf{N} = \{1, 2, 3, \ldots\}$ will denote the set of natural numbers, $\mathbf{Z}$ the integers, $\mathbf{R}$ the real numbers, and $\mathbf{C}$ the complex numbers. H will denote a Hilbert space with inner product $\langle f, g\rangle$ and norm $\|f\| = \langle f, f\rangle^{1/2}$. For example, the space $L^2(\mathbf{R}^d)$ of all complex-valued, square-integrable functions defined on $\mathbf{R}^d$ is a Hilbert space with inner product $\langle f, g\rangle = \int f(x)\,\overline{g(x)}\,dx$ and norm $\|f\|_2 = \left(\int |f(x)|^2\,dx\right)^{1/2}$.

A *Schauder basis* (or simply a *basis*) for H is a family of elements $\{f_i\}_{i\in\mathbf{N}}$ such that

$$\forall\, f \in H, \quad \exists \text{ unique } c_i(f) \in \mathbf{C} \text{ such that } f \;=\; \sum_{i=1}^{\infty} c_i(f)\, f_i. \tag{1}$$

In this case, there exist unique elements $\tilde{f}_i \in H$ such that $c_i(f) = \langle f, \tilde{f}_i\rangle$. Moreover, $\{f_i\}$ and $\{\tilde{f}_i\}$ are *biorthogonal*, i.e., $\langle f_i, \tilde{f}_j\rangle = \delta_{ij}$, and $\{\tilde{f}_i\}$ itself forms a basis for H, called the *dual basis* of $\{f_i\}$, and we have

$$\forall\, f \in H, \quad f \;=\; \sum_{i=1}^{\infty} \langle f, \tilde{f}_i\rangle\, f_i \;=\; \sum_{i=1}^{\infty} \langle f, f_i\rangle\, \tilde{f}_i, \tag{2}$$

with uniqueness of the scalars in these expansions. The *partial sum operators* associated with the basis $\{f_i\}$ are the mappings $S_N\colon H \to H$ defined by $S_N(f) = \sum_{i=1}^{N} \langle f, \tilde{f}_i\rangle\, f_i$ for $f \in H$. These operators must be uniformly bounded in norm, and the *basis constant* is $C = \sup_N \|S_N\|$, where $\|S_N\| = \sup_{\|f\|=1} \|S_N(f)\|$.

A Schauder basis is an *unconditional basis* if the series in (1) converge unconditionally for every f, i.e., if every rearrangement of the series also converges (in which case it must also converge to f). Consequently, an unconditional basis can be indexed by any countable index set, and any ordering of the set can be used in computing the expansion $f = \sum_i c_i(f)\, f_i$. By contrast, for a Schauder basis the ordering can be crucial; with a different ordering the series $\sum_i c_i(f)\, f_i$ may not converge at all, or may converge to something other than f.

A Schauder basis is *bounded* if there exist C_1, $C_2 > 0$ such that $C_1 \le \|f_i\| \le C_2$ for every i. The dual basis $\{\tilde{f}_i\}$ of a bounded Schauder basis is itself a bounded Schauder basis.

A *Riesz basis* is the image of an orthonormal basis under a continuously invertible mapping of H onto itself. In particular, every orthonormal basis is a Riesz basis, but not every Riesz basis is an orthonormal basis. A Schauder basis is a Riesz basis if and only if it is a bounded unconditional basis. The class of Schauder bases for H is strictly larger than the class of Riesz bases. The dual basis of a Riesz basis is itself a Riesz basis.

Frames are generalizations of Riesz bases, first introduced by Duffin and Schaeffer[8] in the context of nonharmonic Fourier series. A family $\{f_i\}_{i\in\mathbf{N}}$ is a *frame* for H if there exist A, $B > 0$, called *frame bounds*, such that

$$\forall\, f \in H, \quad A\,\|f\|^2 \;\le\; \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \;\le\; B\,\|f\|^2. \tag{3}$$

All Riesz bases are frames, and a frame is a Schauder basis if and only if it is a Riesz basis. The class of frames for H is strictly larger than the class of Riesz bases. If only the first inequality in (3) is satisfied then we say that $\{f_i\}$ possesses a lower frame bound, while if only the second inequality in (3) is satisfied then we say that $\{f_i\}$ possesses an upper frame bound.

Given a frame $\{f_i\}$, the *frame operator* is $Sf = \sum_i \langle f, f_i\rangle\, f_i$. The frame operator is a positive, continuously invertible mapping of H onto itself. The sequence $\{\tilde{f}_i\}$ defined by $\tilde{f}_i = S^{-1} f_i$ is also a frame for H, called the *dual frame* of $\{f_i\}$, and equation (2) is valid with unconditional convergence of those series. However, if $\{f_i\}$ is a frame which is not a Riesz basis, then given $f \in H$ there will generally be many choices of coefficients $c_i(f)$ such that $f = \sum c_i(f)\, f_i$. Thus, unlike a Schauder basis, for a frame the scalars in the expansions in (2) need not be unique. Further, the frame $\{f_i\}$ and its dual frame $\{\tilde{f}_i\}$ are biorthogonal if and only if $\{f_i\}$ is a Riesz basis.

An arbitrary sequence $\{f_i\}$ in H is *minimal* if there exists a sequence $\{\tilde{f}_i\}$ which is biorthogonal to $\{f_i\}$, i.e., $\langle f_i, f_j\rangle = \delta_{ij}$. It is *complete* if $\operatorname{span}\{f_i\}$, the set of all finite linear combinations of the f_i, is dense in H. That is, the closure $\overline{\operatorname{span}}\{f_i\}$ of the finite linear span must equal all of H. Equivalently, the only element $f \in H$ which is orthogonal to every f_i is $f = 0$. All Schauder bases are both minimal and complete, but a sequence which is minimal and complete need not be a Schauder basis (in fact, it will be a Schauder basis if and only if it has a finite basis constant). All frames are complete, but a frame is minimal if and only if it is a Riesz basis.

The definition of Schauder basis given by (1) extends without change beyond the Hilbert space setting to the case of Banach spaces X. Moreover, most of the statements made above have analogues for the Banach space setting. In particular, there exist dual elements $\tilde{f}_i$ in the dual space X^* (rather than in X) such that $f = \sum_i \langle f, \tilde{f}_i \rangle f_i$ for $f \in X$, and the basis constant $C = \sup_N \|S_N\|$ will be finite. The dual system $\{\tilde{f}_i\}$ will be a basis for X^* if X^* is a reflexive space, but in general will only be a basis for its closed linear span in X^*. As there need not be a notion of orthogonality in X, the concept of Riesz basis does not make sense, but the definition of bounded unconditional basis still applies. Extending the definition of frames to the general Banach space setting is not straightforward; see Gröchenig[10], Christensen and Heil[5], and Casazza, Han, and Larson[4].

2.2. Density

We will describe several types of density in this subsection. We make these definitions for the case of $\mathbf{R}^d$, although we will apply them to sequences in $\mathbf{R}^{2d}$ as well as $\mathbf{R}^d$. Given $x \in \mathbf{R}^d$ and $h > 0$, we let $Q_h(x)$ denote the cube in $\mathbf{R}^d$ centered at x with side lengths h, i.e., $Q_h(x) = \prod_{j=1}^d [x_j - \frac{h}{2}, x_j + \frac{h}{2})$. To distinguish between cubes in $\mathbf{R}^d$ and those in $\mathbf{R}^{2d}$, we write $\mathbf{Q}_h(x,y) = Q_h(x) \times Q_h(y)$ for a cube in $\mathbf{R}^{2d}$ centered at (x,y), where $x, y \in \mathbf{R}^d$.

Definition 2.1. Let $\Gamma = \{\gamma_i\}_{i \in I}$ be a sequence of points in $\mathbf{R}^d$, with countable or uncountable index set I. For simplicity, we often will write $\Gamma \subset \mathbf{R}^d$, but it will be clear from context that we mean that Γ is a *sequence* of points and not merely a subset of $\mathbf{R}^d$. In particular, repititions of elements in the sequence are allowed. Then the *upper* and *lower Beurling densities* of Γ are, respectively,

$$D^+(\Gamma) \;=\; \limsup_{h\to\infty} \frac{\sup_{x\in\mathbf{R}^d} \#\big(\Gamma \cap Q_h(x)\big)}{h^d} \qquad \text{and} \qquad D^-(\Gamma) \;=\; \liminf_{h\to\infty} \frac{\inf_{x\in\mathbf{R}^d} \#\big(\Gamma \cap Q_h(x)\big)}{h^d}.$$

If $D^+(\Gamma) = D^-(\Gamma)$, then we say that Γ has *uniform Beurling density* $D(\Gamma) = D^+(\Gamma) = D^-(\Gamma)$. □

The densities $D^+(\Gamma)$ and $D^-(\Gamma)$ are computed by taking the maximum or minimum number of elements of Γ in all possible cubes $Q_h(x)$ divided by the volume of $Q_h(x)$, and then letting h go to infinity. Hence they represent two possible meanings of the "average" number of elements of Γ in a cube in $\mathbf{R}^d$. Note that $0 \le D^-(\Gamma) \le D^+(\Gamma) \le \infty$. The rectangular lattice $\Gamma = a_1\mathbf{Z} \times \cdots \times a_d\mathbf{Z}$ with each $a_i > 0$ has uniform Beurling density $D(\Gamma) = 1/(a_1 \cdots a_d)$.

Given finitely many sequences $\Gamma_k = \{\gamma_{i,k}\}_{i\in I_k}$ with $k = 1, \dots, r$, we define the *disjoint union* of $\Gamma_1, \dots, \Gamma_r$ to be the sequence $\Gamma = \{\gamma_{i,k}\}_{(i,k)\in I}$ where $I = \{(i,k) : i \in I_k,\ k = 1, \dots, r\}$. This disjoint union amalgamates the individual sequences $\Gamma_1, \dots, \Gamma_r$ into a single sequence Γ, preserving all elements of the original sequences. For simplicity, we will simply write $\Gamma = \bigcup_{k=1}^r \Gamma_k$, but it will be clear from context that we mean that Γ is the disjoint union of the sequences Γ_k. We remark that, in this case, we always have $\sum_{k=1}^r D^-(\Gamma_k) \le D^-(\Gamma) \le D^+(\Gamma) \le \sum_{k=1}^r D^+(\Gamma_k)$, although some or all of these inequalities may be strict.

Sequences which have finite upper Beurling densities need not have a nonzero minimum distance between elements of the sequence (for example, consider $\Gamma = \mathbf{Z} \cup \sqrt{2}\mathbf{Z}$ in $\mathbf{R}$). However, it can be shown that a sequence with finite upper Beurling density can be written as a disjoint union of finitely many sequences which have a nonzero minimum distance between sequence elements (see Lemma 2.3). We denote the Euclidean norm on $\mathbf{R}^d$ by $|x| = (x_1^2 + \cdots + x_d^2)^{1/2}$.

Definition 2.2. Let $\Gamma = \{\gamma_i\}_{i\in I} \subset \mathbf{R}^d$.

(a) Γ is δ-*uniformly separated* if $\delta = \inf_{i\ne j} |\gamma_i - \gamma_j| > 0$.

(b) Γ is *relatively uniformly separated* if it is a finite union of uniformly separated sequences Γ_k. In this case, I can be partitioned into disjoint sets $I_1, \dots, I_r$ such that each sequence $\Gamma_k = \{\gamma_i\}_{i\in I_k}$ is δ_k-uniformly separated for some $\delta_k > 0$. □

The following lemma provides some equivalent ways to view the meaning of finite upper Beurling density[4].

Lemma 2.3. *Let $\Gamma = \{\gamma_i\}_{i\in I} \subset \mathbf{R}^d$. Then the following statements are equivalent.*

(a) $D^+(\Gamma) < \infty$.

(b) Γ *is relatively uniformly separated.*

(c) *For some (and therefore every) $h > 0$, there exists $N_h > 0$ such that every cube $Q_h(hn)$ with $n \in \mathbf{Z}^d$ contains at most N_h points of Γ.*

2.3. Gabor Systems and Historical Discussion

Given $a,\, b \in \mathbf{R}^d$, define the operations of *translation* and *modulation* of a function $g\colon \mathbf{R}^d \to \mathbf{C}$ by

$$T_a g(x) \;=\; g(x-a) \qquad \text{and} \qquad M_b g(x) \;=\; e^{2\pi i b\cdot x} g(x).$$

If $\Lambda \subset \mathbf{R}^{2d}$, then the *Gabor system* generated by g and Λ is

$$G(g,\Lambda) \;=\; \{M_b T_a g\}_{(a,b)\in\Lambda} \;=\; \{e^{2\pi i b\cdot x} g(x-a)\}_{(a,b)\in\Lambda}.$$

Gabor systems have a wide variety of applications; we refer to the book of Daubechies[7] or the research-tutorial of Heil and Walnut[14] for further information. Since $M_b T_a g = e^{2\pi i a\cdot b} T_a M_b g$, the ordering of the operators T_a, M_b in the definition of a Gabor system is irrelevant to most results.

If Λ is a rectangular lattice in $\mathbf{R}^{2d}$, and if $G(g,\Lambda)$ is a basis or a frame for $L^2(\mathbf{R}^d)$, then the dual basis or dual frame is also a Gabor system, i.e., there is a function $\tilde{g}$ such that the dual basis or dual frame is $G(\tilde{g},\Lambda)$. When Λ is not a rectangular lattice, the dual basis or dual frame of a Gabor system need not itself be a Gabor system.

In this paper we are concerned with the connection between the density properties of Λ and the basis properties of $G(g,\Lambda)$. Let us briefly review some of the history and known results for this problem. For the case $d = 1$ and $\Lambda = a\mathbf{Z} \times b\mathbf{Z}$, Rieffel[18] proved (as a corollary of deep results on C^* algebras) that $G(g,\Lambda)$ must be incomplete if $D(\Lambda) = 1/(ab) < 1$. This is a kind of Nyquist density result for Gabor systems; the system must be "dense enough" in order to be complete (although density is not sufficient for completeness). The algebraic structure of the lattice $\Lambda = a\mathbf{Z} \times b\mathbf{Z}$ was crucial to this result. Daubechies[6] obtained the same result in a constructive manner in the case that ab is rational, but again the proof depended critically on the algebraic structure of the lattice $\Lambda = a\mathbf{Z} \times b\mathbf{Z}$. Ramanathan and Steger[17] proved, via an ingenious and elegant dimension-counting argument, that if Λ is an *arbitrary* subset of $\mathbf{R}^{2d}$ with finite upper Beurling density $D^+(\Lambda)$ and if the lower Beurling density of Λ satisfies $D^-(\Lambda) < 1$ then $G(g,\Lambda)$ cannot be a frame for $L^2(\mathbf{R})$. They conjectured but could not prove that $G(g,\Lambda)$ must actually be incomplete whenever $D^-(\Lambda) < 1$. In fact, Walnut and Heil[2] demonstrated that this conjecture is false by exhibiting for any $\varepsilon > 0$ a function $g \in L^2(\mathbf{R})$ and a non-lattice sequence $\Lambda \subset \mathbf{R}^{2d}$ such that $G(g,\Lambda)$ is complete yet Λ has uniform Beurling density $D(\Lambda) < \varepsilon$.

Christensen, Deng, and Heil[4] extended the results of Ramanathan and Steger to higher dimensions and to the case of multiple generating functions $g_1, \dots, g_r$, and removed some unnecessary hypotheses from the results of Ramanathan and Steger. As corollaries, they also proved that there are no frames for $L^2(\mathbf{R}^d)$ consisting purely of translates of finitely many functions. That is, no family of the form $\{g_k(x-a)\}_{a\in\Gamma_k,\, k=1,\dots,r}$ can ever form a frame for $L^2(\mathbf{R}^d)$. This generalizes the result of Olson and Zalik[16] on the nonexistence of Riesz bases of translates to the case of frames, but still leaves unsettled the question of whether there exist Schauder bases of translates.

The proofs of the frame density results mentioned above depend critically on the existence of the frame bounds, specifically in order to prove that Gabor frames satisfy a particular *Homogeneous Approximation Property*, or HAP. Results to date of any kind for Gabor Schauder bases have been elusive. Indeed, to our knowledge there are no published examples of Gabor systems which are Schauder bases but not Riesz bases. We give an example of such a system in Section 3. In Section 4 we prove that, with some restrictions on the generating functions g_k, any Gabor system which is a Schauder basis must satisfy certain density conditions. In Section 5 we show that, again with restrictions on g, no system of pure translates can form a Schauder basis for $L^2(\mathbf{R}^d)$.

2.4. Modulation Spaces

Let $\mathcal{S}(\mathbf{R}^d)$ be the Schwartz class of infinitely differentiable functions which decrease at infinity more rapidly than the reciprocal of any polynomial. For example, the Gaussian function $e^{-\frac{\pi}{2}x^2}$ is an element of $\mathcal{S}(\mathbf{R}^d)$. Let $\mathcal{S}'(\mathbf{R}^d)$ be the space of tempered distributions, i.e., the topological dual space of $\mathcal{S}(\mathbf{R}^d)$. We let $\langle\cdot,\cdot\rangle$ denote the extension of the L^2-inner product to the dual pair $\mathcal{S}'(\mathbf{R}^d)$, $\mathcal{S}(\mathbf{R}^d)$.

The *Short-Time Fourier Transform* (STFT) of a tempered distribution $f \in \mathcal{S}'(\mathbf{R}^d)$ against $\varphi \in \mathcal{S}(\mathbf{R}^d)$ is

$$V_\varphi f(a,b) \;=\; \langle f, M_b T_a \varphi\rangle.$$

The *modulation space* $M^{p,q}(\mathbf{R}^d)$ is the space of all distributions $f \in \mathcal{S}'(\mathbf{R}^d)$ for which the following norm is finite:

$$\|f\|_{M^{p,q}} \;=\; \left(\int_{\mathbf{R}^d}\left(\int_{\mathbf{R}^d}|V_\varphi f(x,y)|^p\,dx\right)^{q/p} dy\right)^{1/q}.$$

$M^{p,q}(\mathbf{R}^d)$ is a Banach space whose definition is independent of the choice of window $\varphi \in \mathcal{S}(\mathbf{R}^d)$ in the sense of equivalent norms. The modulation spaces, and their weighted versions, were introduced and extensively studied by Feichtinger and Gröchenig; for more information we refer to the book of Gröchenig[11] and to the recent application of modulation spaces to the analysis of pseudodifferential operators by Gröchenig and Heil[12].

For $p = q = 2$ we have $M^{2,2}(\mathbf{R}^d) = L^2(\mathbf{R}^d)$. In fact, if φ is fixed and has unit L^2-norm, then the mapping $g \mapsto V_\varphi g$ is an isometry of $L^2(\mathbf{R}^d)$ into $M^{2,2}(\mathbf{R}^d)$. The modulation space $M^{1,1}(\mathbf{R}^d)$, consisting of distributions f whose STFT $V_\varphi f$ is integrable on $\mathbf{R}^{2d}$, is an especially elegant space. In particular, here are some of the properties of $M^{1,1}(\mathbf{R}^d)$.

(a) $M^{1,1}(\mathbf{R}^d)$ is a dense subspace of $L^p(\mathbf{R}^d)$ for each $1 \le p < \infty$ and is a dense subspace of $C_0(\mathbf{R}^d)$.
(b) $M^{1,1}(\mathbf{R}^d)$ is a Banach algebra under both convolution and pointwise multiplication and is invariant under the Fourier transform.
(c) $M^{1,1}(\mathbf{R}^d)$ is the minimal Banach space contained in $L^1(\mathbf{R}^d)$ for which T_a and M_b are isometries.
(d) If $(1+|x|^2)^{s/2} f(x) \in L^2(\mathbf{R}^d)$ and $(1+|\omega|^2)^{s/2}\hat{f}(\omega) \in L^2(\mathbf{R}^d)$ for some $s > d$, then $f \in M^{1,1}(\mathbf{R}^d)$.

3. EXAMPLE

We will provide an example of a Gabor system which is a Schauder basis but not a Riesz basis for $L^2(\mathbf{R})$. We first require the following lemma, whose proof is analogous to the proof of Theorem 4.3.3 in Heil and Walnut[14].

Lemma 3.1. *Let $g \in L^2[-\frac{1}{2},\frac{1}{2}]$.*

(a) *$\{e^{2\pi imx}g(x)\}_{m\in\mathbf{Z}}$ is complete in $L^2[-\frac{1}{2},\frac{1}{2}]$ if and only if $g(x) \neq 0$ a.e.*

(b) *$\{e^{2\pi imx}g(x)\}_{m\in\mathbf{Z}}$ is minimal and complete in $L^2[-\frac{1}{2},\frac{1}{2}]$ if and only if $1/g \in L^2[-\frac{1}{2},\frac{1}{2}]$. In this case, $\{e^{2\pi imx}/g(x)\}_{m\in\mathbf{Z}}$ is biorthogonal to $\{e^{2\pi imx}g(x)\}_{m\in\mathbf{Z}}$.*

(c) *$\{e^{2\pi imx}g(x)\}_{m\in\mathbf{Z}}$ is a frame for $L^2[-\frac{1}{2},\frac{1}{2}]$ if and only if there exist $A, B > 0$ such that $0 < A \le |g(x)|^2 \le B < \infty$ a.e. In this case, the frame is a Riesz basis.*

So far as we are aware, there is no characterization of those functions $g \in L^2[-\frac{1}{2},\frac{1}{2}]$ such that $\{e^{2\pi imx}g(x)\}_{m\in\mathbf{Z}}$ is a Schauder basis for $L^2[-\frac{1}{2},\frac{1}{2}]$. Following is one example of a Schauder basis of this form which is not a Riesz basis.

Example 3.2. Fix $0 < \alpha < \frac{1}{2}$. Set $g(x) = |x|^\alpha$ for $x \in [-\frac{1}{2},\frac{1}{2}]$. Define $\tilde{g}(x) = |x|^{-\alpha}$, $x \in [-\frac{1}{2},\frac{1}{2}]$. The number α has been chosen so that both g and $\tilde{g}$ lie in $L^2[-\frac{1}{2},\frac{1}{2}]$. Therefore, by Lemma 3.1(b), $\{e^{2\pi imx}g(x)\}_{m\in\mathbf{Z}}$ is minimal and complete in $L^2[-\frac{1}{2},\frac{1}{2}]$, with biorthogonal sequence $\{e^{2\pi imx}\tilde{g}(x)\}_{m\in\mathbf{Z}}$. By Lemma 3.1(c), $\{e^{2\pi imx}g(x)\}_{m\in\mathbf{Z}}$ is not a Riesz basis for $L^2[-\frac{1}{2},\frac{1}{2}]$. It is a much deeper and more difficult result of Babenko[1] that $\{e^{2\pi imx}g(x)\}_{m\in\mathbf{Z}}$ forms a Schauder basis for $L^2[-\frac{1}{2},\frac{1}{2}]$ (see also the discussion on pages 351–354 of Singer[19]). □

From this example we can construct a Gabor system for $L^2(\mathbf{R})$ that is a Schauder basis but not a Riesz basis.

Example 3.3. Let $\Lambda = \mathbf{Z}\times\mathbf{Z}$. We claim that if g is as in Example 3.2, extended by zero to the real line, i.e., $g(x) = |x|^\alpha\,\chi_{[-\frac{1}{2},\frac{1}{2}]}(x)$, then $G(g,\Lambda) = \{T_nM_mg\}_{m,n\in\mathbf{Z}} = \{e^{2\pi imx}g(x-n)\}_{m,n\in\mathbf{Z}}$ is a Schauder basis for $L^2(\mathbf{R})$ that is not a Riesz basis for $L^2(\mathbf{R})$. It is again easy to see that this system is minimal and complete with biorthogonal sequence $G(\tilde{g},\Lambda) = \{T_nM_m\tilde{g}\}_{m,n\in\mathbf{Z}}$, and that it is not a Riesz basis. The fact that $\{T_nM_mg\}_{m,n\in\mathbf{Z}}$ is a Schauder basis follows immediately from the facts that $L^2(\mathbf{R})$ is a direct sum of the Hilbert spaces $L^2[n-\frac{1}{2},n+\frac{1}{2}]$ with n ranging through $\mathbf{Z}$, and that, for each fixed n, $\{T_nM_mg\}_{m\in\mathbf{Z}}$ is a Schauder basis for $L^2[n-\frac{1}{2},n+\frac{1}{2}]$. Alternatively, it is straightforward but not entirely trivial to show directly that $\{T_nM_mg\}_{m,n\in\mathbf{Z}}$ is a Schauder basis for $L^2(\mathbf{R})$. To indicate the difficulty, note that the statement in Example 3.2 that $\{M_mg\}_{m\in\mathbf{Z}}$ is a Schauder basis for $L^2[-\frac{1}{2},\frac{1}{2}]$ means that there is some specific ordering of the integers $\mathbf{Z}$ with respect to which the basis expansions converge. That is, there exists some bijection $\sigma\colon\mathbf{N}\to\mathbf{Z}$ such that

$$\forall\, f \in L^2[-\tfrac{1}{2},\tfrac{1}{2}], \quad f \;=\; \sum_{m=1}^{\infty} \big\langle f,\, M_{\sigma(m)}\tilde{g}\big\rangle\, M_{\sigma(m)}g.$$

In order to prove that $\{T_nM_mg\}_{m,n\in\mathbf{Z}}$ is a Schauder basis for $L^2(\mathbf{R})$, we must show there is likewise some bijection

$\rho\colon \mathbf{N} \to \mathbf{Z} \times \mathbf{Z}$ such that

$$\forall\, f \in L^2(\mathbf{R}), \quad f \;=\; \sum_{m=1}^{\infty} \bigl\langle f,\, T_{\rho_1(m)} M_{\rho_2(m)} \tilde{g} \bigr\rangle\, T_{\rho_1(m)} M_{\rho_2(m)} g,$$

where $\rho(m) = (\rho_1(m), \rho_2(m))$. In fact, if we let $\tau\colon \mathbf{N} \to \mathbf{Z}$ be any particular bijection, then it can be shown with some work that the following definition of ρ suffices:

$$\begin{matrix} \rho(1) \;=\; \bigl(\tau(1), \sigma(1)\bigr), & \rho(3) \;=\; \bigl(\tau(2), \sigma(1)\bigr), & \rho(5) \;=\; \bigl(\tau(2), \sigma(2)\bigr), \\ \rho(2) \;=\; \bigl(\tau(1), \sigma(2)\bigr), & \rho(4) \;=\; \bigl(\tau(1), \sigma(3)\bigr), & \rho(6) \;=\; \bigl(\tau(3), \sigma(1)\bigr), \end{matrix} \quad \text{etc.} \quad \square$$

Although the Schauder basis $G(g, \Lambda)$ of Example 3.3 is not a Riesz basis, and hence is not a frame, the generating function g satisfies $|g(x)|^2 \le 1$ for $x \in [-\frac{1}{2}, \frac{1}{2}]$ and the dual function $\tilde{g}$ satisfies $|\tilde{g}(x)|^2 \ge 1$ for $x \in [-\frac{1}{2}, \frac{1}{2}]$. An extension of Lemma 3.1(c) then shows that $\{e^{2\pi imx} g(x)\}_{m\in\mathbf{Z}}$ possesses an upper frame bound for $L^2[-\frac{1}{2}, \frac{1}{2}]$ and $\{e^{2\pi imx} \tilde{g}(x)\}_{m\in\mathbf{Z}}$ possesses a lower frame bound for $L^2[-\frac{1}{2}, \frac{1}{2}]$. The piecing together process in Example 3.3 preserves these properties, i.e., $G(g, \Lambda)$ possesses a upper frame bound and $G(\tilde{g}, \Lambda)$ possesses an lower frame bound. We will see in Corollary 4.10 that the fact that $G(\tilde{g}, \Lambda)$ possesses a lower frame bound implies that Λ must have uniform density $D(\Lambda) = 1$.

The generating function g of Example 3.3 does not lie in the modulation space $M^{1,1}(\mathbf{R})$. For the case of a rectangular lattice $\Lambda = a\mathbf{Z} \times b\mathbf{Z}$, the Balian–Low Theorem implies that if a Gabor system $G(g, \Lambda)$ is a Riesz basis then $g \notin M^{1,1}(\mathbf{R})$ (indeed, the restriction is even more severe than that). We conjecture that for arbitrary Λ, there are in fact no Schauder bases $G(g, \Lambda)$ with $g \in M^{1,1}(\mathbf{R}^d)$. For more information on the Balian–Low Theorem we refer to Benedetto, Heil, and Walnut[2].

4. DENSITY OF GABOR SCHAUDER BASES

In this section we prove our main results on the density of Gabor systems which form Schauder bases for $L^2(\mathbf{R})$.

4.1. Upper Density Estimates

Olson and Zalik[16] proved that a necessary condition for a system $\{g(x-a)\}_{a\in\Gamma}$ of pure translates to be a Schauder basis for $L^p(\mathbf{R})$ is that Γ be uniformly separated. We extend this result to Gabor systems. For this result, no Hilbert space structure is needed; instead the essential ingredients are the continuity properties of translation and modulation. For example, the spaces $L^p(\mathbf{R}^d)$ for $1 \le p < \infty$ and the space $C_0(\mathbf{R}^d)$ consisting of continuous functions vanishing at infinity satisfy the hypotheses of Lemma 4.1. The isometry hypothesis can easily be weakened further so that many more function spaces are included; we omit the details.

Lemma 4.1. *Let X be any Banach space of complex-valued functions defined on $\mathbf{R}^d$ such that:*

(a) *translation and modulation are isometries, i.e., $\forall\, a, b \in \mathbf{R}^d$, $\quad \|T_a f\| \;=\; \|f\| \;=\; \|M_b f\|$, and*

(b) *translation and modulation are strongly continuous, i.e., $\lim_{(a,b)\to 0} \|M_b T_a f - f\| \;=\; 0$.*

Let $g \in X$ and $\Lambda \subset \mathbf{R}^{2d}$ be such that $G(g, \Lambda)$ is a Schauder basis for its closed linear span in X. Then there is a $\delta > 0$ such that Λ is δ-uniformly separated.

Proof. Schauder bases are countable, so let $\Lambda = \{(a_i, b_i)\}_{i\in\mathbf{N}} \subset \mathbf{R}^{2d}$ be the ordering of Λ with respect to which the basis expansions converge. Let $S = \overline{\operatorname{span}}\{M_{b_i} T_{a_i} g\}_{i\in\mathbf{N}}$ be the closed linear span of $G(g, \Lambda)$ in X. Since S is a Banach space under the norm inherited from X, if $G(g, \Lambda) = \{M_{b_i} T_{a_i} g\}$ is a Schauder basis for S then there exists a dual system $\{\tilde{g}_i\}$ in S^* such that each $f \in S$ has the unique expansion

$$f \;=\; \sum_{i=1}^{\infty} \langle f, \tilde{g}_i \rangle\, M_{b_i} T_{a_i} g, \qquad f \in S.$$

Suppose that Λ was not δ-uniformly separated in $\mathbf{R}^{2d}$ for any $\delta > 0$. Then

$$\inf_{m<n} |(a_m, b_m) - (a_n, b_n)| \;=\; 0. \tag{4}$$

Define the partial sum operators $S_N\colon S \to S$ by $S_N(f) = \sum_{i=1}^{N} \langle f, \tilde{g}_i \rangle\, M_{b_i} T_{a_i} g$. Since $G(g, \Lambda)$ is a Schauder basis for S, it has finite basis constant $C = \sup_N \|S_N\| < \infty$. Given $m < n \in \mathbf{N}$, consider the functions

$$f_{m,n} \;=\; M_{b_m} T_{a_m} g - M_{b_n} T_{a_n} g \;\in\; S.$$

By the biorthogonality of $G(g,\Lambda)$ and $\{\tilde{g}_i\}$, we have $S_m(f_{m,n}) = M_{b_m}T_{a_m}g$ when $n > m$, so $\|S_m(f_{m,n})\| = \|g\|$ for all $n > m$. However, translation and modulation are strongly continuous in X, so it follows from (4) that $\inf_{m<n} \|f_{m,n}\| = 0$. Therefore

$$C \;=\; \sup_N \|S_N\| \;=\; \sup_N \sup_{\|f\|=1} \|S_N(f)\| \;\geq\; \sup_m \sup_{n>m} \frac{\|S_m(f_{m,n})\|}{\|f_{n,m}\|} \;\geq\; \sup_m \sup_{n>m} \frac{\|g\|}{\|f_{n,m}\|} \;=\; \infty,$$

which is a contradiction. □

As a corollary, we find that any Gabor Schauder basis must have finite upper Beurling density, even if we allow finitely many generators.

Corollary 4.2. *Let X be as in Lemma 4.1, and assume that $g_1, \dots, g_r \in X$, and $\Lambda_1, \dots, \Lambda_r \subset \mathbf{R}^{2d}$ are such that $\bigcup_{k=1}^r G(g_k, \Lambda_k)$ is a Schauder basis for its closed linear span in X. Define $\Lambda = \bigcup_{k=1}^r \Lambda_k$. Then $D^+(\Lambda) < \infty$; in particular, Λ is relatively uniformly separated.*

Proof. Since each individual Gabor system $G(g_k, \Lambda_k)$ must be a Schauder basis for its closed linear span, it follows from Lemma 4.1 that each set Λ_k is δ_k-uniformly separated for some $\delta_k > 0$. Hence Λ is relatively uniformly separated and therefore has finite upper Beurling density by Lemma 2.3. □

4.2. The HAP and The Comparison Theorem

For Gabor frames in $L^2(\mathbf{R}^d)$, the frame expansions in (2) have been shown to have a certain kind of uniformity of convergence with respect to translation and modulations of the function f being expanded[17,4]. This property is called the Homogeneous Approximation Property (HAP), and it is a key property in deriving density results for Gabor frames. We will show that Gabor Schauder bases likewise possess a version of the HAP if the generating functions lie in the modulation space $M^{1,1}(\mathbf{R}^d)$ or if at least a lower frame bound is satisfied.

Definition 4.3 (HAP)**.** Let $g_1, \dots, g_r \in L^2(\mathbf{R}^d)$ and let $\Lambda_1, \dots, \Lambda_r \subset \mathbf{R}^{2d}$ be such that $\bigcup_{k=1}^r G(g_k, \Lambda_k)$ is a Schauder basis for $L^2(\mathbf{R}^d)$. Let $\{\tilde{g}_{k,a,b}\}_{(a,b)\in\Lambda_k,\, k=1,\dots,r}$ be the dual basis of $\bigcup_{k=1}^r G(g_k, \Lambda_k)$. Given $h > 0$ and $(p,q) \in \mathbf{R}^{2d}$, let $W(h,p,q)$ denote the following subspace of $L^2(\mathbf{R}^d)$:

$$W(h,p,q) \;=\; \operatorname{span}\bigl\{\tilde{g}_{k,a,b} : (a,b) \in \mathbf{Q}_h(p,q) \cap \Lambda_k,\; k = 1, \dots, r\bigr\}. \tag{5}$$

Note that by Corollary 4.2, Λ must be relatively uniformly separated, and hence $W(h,p,q)$ is finite-dimensional. We say that $\bigcup_{k=1}^r G(g_k, \Lambda_k)$ possesses the Homogeneous Approximation Property if for each $f \in L^2(\mathbf{R}^d)$,

$$\forall\, \varepsilon > 0, \quad \exists\, R > 0 \quad \text{such that} \quad \forall\, (x,y) \in \mathbf{R}^{2d}, \quad \operatorname{dist}\bigl(M_y T_x f,\, W(R,x,y)\bigr) < \varepsilon. \quad \square \tag{6}$$

The HAP implies the following stronger property.

Lemma 4.4 (Strong HAP)**.** *Let $g_1, \dots, g_r \in L^2(\mathbf{R}^d)$ and let $\Lambda_1, \dots, \Lambda_r \subset \mathbf{R}^{2d}$ be as in Definition 4.3. If $\bigcup_{k=1}^r G(g_k, \Lambda_k)$ posseses the HAP, then for each $f \in L^2(\mathbf{R}^d)$ and each $\varepsilon > 0$, there exists a constant $R > 0$ such that*

$$\forall\, (p,q) \in \mathbf{R}^{2d}, \quad \forall\, h > 0, \quad \forall\, (x,y) \in \mathbf{Q}_h(p,q), \quad \operatorname{dist}\bigl(M_y T_x f,\, W(h+R,p,q)\bigr) \;<\; \varepsilon.$$

Proof. Note that if $(x,y) \in \mathbf{Q}_h(p,q)$, then $W(R,x,y) \subset W(h+R,p,q)$, and therefore $\operatorname{dist}\bigl(M_y T_x f,\, W(h+R,p,q)\bigr) \leq \operatorname{dist}\bigl(M_y T_x f,\, W(R,x,y)\bigr)$. □

Now we show that Gabor Schauder bases which possess the HAP must satisfy certain density requirements in comparison to any other Gabor Schauder basis for $L^2(\mathbf{R}^d)$.

Theorem 4.5 (Comparison Theorem)**.** *Let $g_1, \dots, g_r \in L^2(\mathbf{R}^d)$ and $\Lambda_1, \dots, \Lambda_r \subset \mathbf{R}^{2d}$ be such that $\bigcup_{k=1}^r G(g_k, \Lambda_k)$ is a Schauder basis for $L^2(\mathbf{R}^d)$ which possesses the HAP, and let $\phi_1, \dots, \phi_s \in L^2(\mathbf{R}^d)$ and $\Delta_1, \dots, \Delta_s \subset \mathbf{R}^{2d}$ be such that $\bigcup_{k=1}^s G(\phi_k, \Delta_k)$ is a Schauder basis for $L^2(\mathbf{R}^d)$. Let $\Lambda = \bigcup_{k=1}^r \Lambda_k$ and let $\Delta = \bigcup_{k=1}^s \Delta_s$. Then*

$$D^-(\Delta) \;\leq\; D^-(\Lambda) \qquad \text{and} \qquad D^+(\Delta) \;\leq\; D^+(\Lambda).$$

Proof. Let $\{\tilde{\phi}_{k,a,b}\}_{(a,b)\in\Delta_k,\, k=1,\dots,s}$ be the dual basis of $\bigcup_{k=1}^s G(\phi_k, \Delta_k) = \{M_b T_a \phi_k\}_{(a,b)\in\Delta_k,\, k=1,\dots,s}$. Given $h > 0$ and $(p,q) \in \mathbf{R}^{2d}$, let $W(h,p,q)$ be the subspace of $L^2(\mathbf{R}^d)$ defined by (5), and let $V(h,p,q)$ denote the following subspace of $L^2(\mathbf{R}^d)$:

$$V(h,p,q) \;=\; \operatorname{span}\bigl\{M_b T_a \phi_k : (a,b) \in \mathbf{Q}_h(p,q) \cap \Delta_k,\; k = 1, \dots, s\bigr\}.$$

By Corollary 4.2, Δ is relatively uniformly separated, and hence $V(h,p,q)$ is finite-dimensional. Since each element of the Schauder basis $\bigcup_{k=1}^{s} G(\phi_k, \Delta_k)$ has norm equal one of the finitely many values $\|\phi_1\|_2, \ldots, \|\phi_s\|_2$, this Schauder basis is a bounded basis, and therefore its dual basis is also a bounded basis. In particular, the elements of the dual basis are uniformly bounded in norm, i.e., there exists a constant $C > 0$ such that $\|\tilde{\phi}_{k,a,b}\| \le C$ for all k, a, and b.

Choose now any $\varepsilon > 0$. Then, by applying Lemma 4.4 to each of the functions $f = \phi_k$, $k = 1, \ldots, s$ in turn, we see that there exists $R > 0$ such that

$$\forall\, k = 1, \ldots, s, \quad \forall\, (p,q) \in \mathbf{R}^{2d}, \quad \forall\, h > 0, \quad \forall\, (x,y) \in \mathbf{Q}_h(p,q), \quad \operatorname{dist}\big(M_y T_x \phi_k,\, W(h+R,p,q)\big) \;<\; \frac{\varepsilon}{C}. \tag{7}$$

Now let $h > 0$ and $(p,q) \in \mathbf{R}^{2d}$ be fixed. Let P_V denote the orthogonal projection of $L^2(\mathbf{R}^d)$ onto $V(h,p,q)$, and let P_W denote the orthogonal projection of $L^2(\mathbf{R}^d)$ onto $W(h+R,p,q)$. Define $T\colon V(h,p,q) \to V(h,p,q)$ by

$$T \;=\; P_V P_W.$$

Note that $\{M_b T_a \phi_k : (a,b) \in \mathbf{Q}_h(p,q) \cap \Delta_k,\ k = 1, \ldots, s\}$ is a basis for $V(h,p,q)$, and its dual basis in $V(h,p,q)$ is $\{P_V \tilde{\phi}_{k,a,b} : (a,b) \in \mathbf{Q}_h(p,q) \cap \Delta_k,\ k = 1, \ldots, s\}$. Since these dual bases are biorthogonal, the trace of T can be computed as

$$\operatorname{tr}(T) \;=\; \sum_{k=1}^{s} \sum_{(a,b) \in \mathbf{Q}_h(p,q) \cap \Delta_k} \langle T(M_b T_a \phi_k), P_V \tilde{\phi}_{k,a,b} \rangle. \tag{8}$$

However, for $(a,b) \in \mathbf{Q}_h(p,q) \cap \Delta_k$, we have

$$\langle T(M_b T_a \phi_k), P_V \tilde{\phi}_{k,a,b} \rangle \;=\; \langle P_W M_b T_a \phi_k, P_V \tilde{\phi}_{k,a,b} \rangle \;=\; \langle M_b T_a \phi_k, P_V \tilde{\phi}_{k,a,b} \rangle + \langle (P_W - I) M_b T_a \phi_k, P_V \tilde{\phi}_{k,a,b} \rangle. \tag{9}$$

Now, $P_V M_b T_a \phi_k = M_b T_a \phi_k$ since $M_b T_a \phi_k \in V(h,p,q)$. Therefore,

$$\langle M_b T_a \phi_k, P_V \tilde{\phi}_{k,a,b} \rangle \;=\; \langle P_V M_b T_a \phi_k, \tilde{\phi}_{k,a,b} \rangle \;=\; \langle M_b T_a \phi_k, \tilde{\phi}_{k,a,b} \rangle \;=\; 1, \tag{10}$$

the last equality following from biorthogonality. Further, by (7) we have

$$|\langle (P_W - I) M_b T_a \phi_k, P_V \tilde{\phi}_{k,a,b} \rangle| \;\le\; \|(P_W - I) M_b T_a \phi_k\|_2 \, \|P_V \tilde{\phi}_{k,a,b}\|_2 \;\le\; \frac{\varepsilon}{C} \cdot C \;=\; \varepsilon. \tag{11}$$

Combining (8)–(11) and using the fact that Δ is the disjoint union of $\Delta_1, \ldots, \Delta_r$, we conclude that

$$\operatorname{tr}(T) \;\ge\; \sum_{k=1}^{s} \sum_{(a,b) \in \mathbf{Q}_h(p,q) \cap \Delta_k} (1-\varepsilon) \;=\; (1-\varepsilon) \sum_{k=1}^{s} \#\big(\mathbf{Q}_h(p,q) \cap \Delta_k\big) \;=\; (1-\varepsilon)\, \#\big(\mathbf{Q}_h(p,q) \cap \Delta\big). \tag{12}$$

On the other hand, all eigenvalues of T satisfy $|\lambda| \le \|T\| \le 1$. Hence,

$$\operatorname{tr}(T) \;\le\; \operatorname{rank}(T) \;\le\; \dim\big(W(h+R,p,q)\big) \;\le\; \sum_{k=1}^{r} \#\big(\mathbf{Q}_{h+R}(p,q) \cap \Lambda_k\big) \;=\; \#\big(\mathbf{Q}_{h+R}(p,q) \cap \Lambda\big). \tag{13}$$

Therefore, by combining (12) and (13), we see that for each $h > 0$ and each $(p,q) \in \mathbf{R}^{2d}$,

$$(1-\varepsilon)\, \frac{\#\big(\mathbf{Q}_h(p,q) \cap \Delta\big)}{h^{2d}} \;\le\; \frac{\#\big(\mathbf{Q}_{h+R}(p,q) \cap \Lambda\big)}{(h+R)^{2d}}\, \frac{(h+R)^{2d}}{h^{2d}}.$$

Since R is fixed, it follows by taking the inf or sup over all $(p,q) \in \mathbf{R}^{2d}$ and then the liminf or limsup as $h \to \infty$ that $(1-\varepsilon)\, D^-(\Delta) \le D^-(\Lambda)$ and $(1-\varepsilon)\, D^+(\Delta) \le D^+(\Lambda)$. Since ε is arbitrary, the result follows. □

4.3. Lower Density Estimates

As an immediate application of the Comparison Theorem, we can show that the upper Beurling density of any Gabor Schauder basis is at most 1 by comparing it to a specific Gabor basis which is known to possess the HAP.

Corollary 4.6. *Assume that $g_1, \ldots, g_r \in L^2(\mathbf{R}^d)$ and $\Lambda_1, \ldots, \Lambda_r \subset \mathbf{R}^{2d}$ are such that $\bigcup_{k=1}^{r} G(g_k, \Lambda_k)$ is a Schauder basis for $L^2(\mathbf{R}^d)$. Let $\Lambda = \bigcup_{k=1}^{r} \Lambda_k$. Then $D^-(\Lambda) \le D^+(\Lambda) \le 1$.*

Proof. Let $\phi = \chi_{[0,1]^d}$ and $\Delta = \mathbf{Z}^d \times \mathbf{Z}^d$; then $G(\phi, \Delta)$ is an orthonormal basis for $L^2(\mathbf{R}^d)$ and $D(\Delta) = 1$. $G(g, \Delta)$ possesses the HAP by Theorem 3.4 of Christensen, Deng, Heil[4]. Therefore, Theorem 4.5 implies that $D^-(\Lambda) \le D^-(\Delta) = 1$ and $D^+(\Lambda) \le D^+(\Delta) = 1$. □

Our next goal is to show that if we add the hypothesis that $g_1, \ldots, g_r$ lie in the modulation space $M^{1,1}(\mathbf{R}^d)$, then conclusion of Corollary 4.6 can be improved to say that $D(\Lambda) = 1$. We first require the following lemma, whose proof is a simple modification of the proof of Lemma 3.3 in Christensen, Deng, and Heil[4]. Recall $V_\varphi f(a,b) = \langle f, M_b T_a \varphi \rangle$.

Lemma 4.7. *Set* $\varphi(x) = e^{-\frac{\pi}{2}x^2}$*, and let* $h > 0$ *be fixed. Then there exists a constant* $K_h > 0$ *such that*

$$\forall\, f \in L^2(\mathbf{R}^d), \quad \forall\,(p,q) \in \mathbf{R}^{2d}, \quad |V_\varphi f(p,q)| \;\le\; K_h \iint_{\mathbf{Q}_h(p,q)} |V_\varphi f(x,y)|\, dx\, dy.$$

Theorem 4.8. *Let* $g_1, \dots, g_r \in M^{1,1}(\mathbf{R}^d)$ *and* $\Lambda_1, \dots, \Lambda_r \subset \mathbf{R}^{2d}$ *be such that* $\bigcup_{k=1}^r G(g_k, \Lambda_k)$ *is a Schauder basis for* $L^2(\mathbf{R}^d)$*. Then* $\bigcup_{k=1}^r G(g_k, \Lambda_k)$ *possesses the HAP.*

Proof. By Corollary 4.2, Λ must be relatively uniformly separated. By dividing each Λ_k into subsequences which are uniformly separated if necessary, we may therefore assume without loss of generality that each Λ_k is δ_k-uniformly separated for some $\delta_k > 0$. Let $h = \min\{\delta_1/2, \dots, \delta_r/2\}$. Let $\{\tilde{g}_{k,a,b}\}_{(a,b)\in\Lambda_k,\, k=1,\dots,r}$ be the dual basis of $\bigcup_{k=1}^r G(g_k, \Lambda_k)$. Since $\bigcup_{k=1}^r G(g_k, \Lambda_k)$ is a bounded basis, the dual basis is also bounded, and therefore there is a $C > 0$ such that $\|\tilde{g}_{k,a,b}\| \le C$ for every k, a, b.

Let $\mathcal{H}$ be the subset of $L^2(\mathbf{R}^d)$ consisting of all functions f for which (6) holds. It is easy to see that $\mathcal{H}$ is closed under finite linear combinations and under L^2-limits. It therefore suffices to show that $\mathcal{H}$ contains a complete set of functions, for then we must have $\mathcal{H} = L^2(\mathbf{R}^d)$. In particular, the proof will be complete if we show that every time-frequency shift $M_t T_s \varphi$ of the Gaussian function $\varphi(x) = e^{-\frac{\pi}{2}x^2}$ belongs to $\mathcal{H}$.

Fix $(s,t) \in \mathbf{R}^{2d}$, and consider any $(p,q) \in \mathbf{R}^{2d}$. The function $M_q T_p(M_t T_s \varphi)$ has the basis expansion

$$M_q T_p(M_t T_s \varphi) \;=\; \sum_{k=1}^{r} \sum_{(a,b)\in\Lambda_k} \langle M_q T_p(M_t T_s \varphi), M_b T_a g_k \rangle\, \tilde{g}_{k,a,b}. \tag{14}$$

More precisely, by definition of Schauder basis there is some fixed ordering of $\Lambda = \cup_{k=1}^r \Lambda_k$ with respect to which the series in (14) converge for all p, q, s, t. Given $R > 0$ we therefore have

$$\begin{aligned}
\operatorname{dist}\big(M_q T_p(M_t T_s \varphi),\, W(R,p,q)\big) \;&\le\; \Bigg\| M_q T_p(M_t T_s \varphi) \;-\; \sum_{k=1}^{r} \sum_{(a,b)\in\mathbf{Q}_R(p,q)\cap\Lambda_k} \langle M_q T_p(M_t T_s \varphi), M_b T_a g_k \rangle\, \tilde{g}_{k,a,b} \Bigg\|_2 \\
&= \Bigg\| \sum_{k=1}^{r} \sum_{(a,b)\in\Lambda_k\setminus\mathbf{Q}_R(p,q)} \langle M_q T_p(M_t T_s \varphi), M_b T_a g_k \rangle\, \tilde{g}_{k,a,b} \Bigg\|_2 \qquad (15) \\
&\le \Big(\sup_{k=1,\dots,r} \; \sup_{(a,b)\in\Lambda_k} \|\tilde{g}_{k,a,b}\|_2 \Big) \sum_{k=1}^{r} \sum_{(a,b)\in\Lambda_k\setminus\mathbf{Q}_R(p,q)} |\langle M_q T_p(M_t T_s \varphi), M_b T_a g_k \rangle| \qquad (16) \\
&\le C \sum_{k=1}^{r} \sum_{(a,b)\in\Lambda_k\setminus\mathbf{Q}_R(p,q)} |V_\varphi g_k(p+s-a, q+t-b)|. \qquad (17)
\end{aligned}$$

The equality in line (15) is valid because we have subtracted a *finite* series from the infinite series in (14). The inequality in (16) follows from Minkowski's inequality for series. Now, by Lemma 4.7, there exists a constant K_h such that

$$|V_\varphi g_k(p+s-a, q+t-b)| \;\le\; K_h \iint_{\mathbf{Q}_h(p+s-a,q+t-b)} |V_\varphi g_k(x,y)|\, dx\, dy. \tag{18}$$

Combining (17) and (18) with the fact that each Λ_k is h-separated, we conclude that

$$\begin{aligned}
\operatorname{dist}\big(M_q T_p(M_t T_s \varphi),\, W(R,p,q)\big) \;&\le\; C K_h \sum_{k=1}^{r} \sum_{(a,b)\in\Lambda_k\setminus\mathbf{Q}_R(p,q)} \iint_{\mathbf{Q}_h(p+s-a,q+t-b)} |V_\varphi g_k(x,y)|\, dx\, dy \\
&\le\; C K_h \sum_{k=1}^{r} \iint_{\mathbf{R}^{2d}\setminus\mathbf{Q}_{R-h}(s,t)} |V_\varphi g_k(x,y)|\, dx\, dy. \qquad (19)
\end{aligned}$$

Since each function g_k lies in the modulation space $M^{1,1}(\mathbf{R}^d)$, we have by definition that $V_\varphi g_k \in L^1(\mathbf{R}^{2d})$. Therefore, the last quantity in (19) can be made arbitrarily small, independently of (p,q), by taking R large enough. □

An alternative condition which implies the HAP is given in the next theorem.

Theorem 4.9. *Let $g_1, \dots, g_r \in L^2(\mathbf{R}^d)$ and $\Lambda_1, \dots, \Lambda_r \subset \mathbf{R}^{2d}$ be such that $\bigcup_{k=1}^r G(g_k, \Lambda_k)$ is a Schauder basis for $L^2(\mathbf{R}^d)$. If $\bigcup_{k=1}^r G(g_k, \Lambda_k)$ possesses a lower frame bound, then $\bigcup_{k=1}^r G(g_k, \Lambda_k)$ possesses the HAP.*

Proof. Repeat the proof of Theorem 4.8 up to equation (15), i.e.,

$$\operatorname{dist}\big(M_qT_p(M_tT_s\varphi),\, W(R,p,q)\big) \;\le\; \Bigg\| \sum_{k=1}^r \sum_{(a,b)\in\Lambda_k\setminus\mathbf{Q}_R(p,q)} \langle M_qT_p(M_tT_s\varphi), M_bT_ag_k\rangle\, \tilde{g}_{k,a,b} \Bigg\|_2. \tag{20}$$

Since $\bigcup_{k=1}^r G(g_k, \Lambda_k)$ is a basis, the series $f = \sum_{k=1}^r \sum_{(a,b)\in\Lambda_k\setminus\mathbf{Q}_R(p,q)} \langle M_qT_p(M_tT_s\varphi), M_bT_ag_k\rangle\, \tilde{g}_{k,a,b}$ is the unique representation of f in terms of this basis. Hence, applying the definition of the lower frame bound, i.e., the first inequality in (3), to this f, we obtain

$$\Bigg\| \sum_{k=1}^r \sum_{(a,b)\in\Lambda_k\setminus\mathbf{Q}_R(p,q)} \langle M_qT_p(M_tT_s\varphi), M_bT_ag_k\rangle\, \tilde{g}_{k,a,b} \Bigg\|_2^2 \;\le\; A^{-1} \sum_{k=1}^r \sum_{(a,b)\in\Lambda_k\setminus\mathbf{Q}_R(p,q)} |\langle M_qT_p(M_tT_s\varphi), M_bT_ag_k\rangle|^2. \tag{21}$$

Applying the Schwarz inequality to (18) and combining it with (20) and (21) therefore yields

$$\operatorname{dist}\big(M_qT_p(M_tT_s\varphi),\, W(R,p,q)\big)^2 \;\le\; K' \sum_{k=1}^r \iint_{\mathbf{R}^{2d}\setminus\mathbf{Q}_{R-h}(s,t)} |V_\varphi g_k(x,y)|^2\, dx\, dy \tag{22}$$

for an appropriate constant K'. Since $g_k \in L^2(\mathbf{R}^d)$ we have $V_\varphi g_k \in L^2(\mathbf{R}^{2d})$, and therefore the last quantity in (22) can be made arbitrarily small, independently of (p,q), by taking R large enough. □

Corollary 4.10. *Let $g_1, \dots, g_r \in L^2(\mathbf{R}^d)$ and $\Lambda_1, \dots, \Lambda_r \subset \mathbf{R}^{2d}$ be such that $\bigcup_{k=1}^r G(g_k, \Lambda_k)$ is a Schauder basis for $L^2(\mathbf{R}^d)$. If either*

(a) *$g_1, \dots, g_r \in M^{1,1}(\mathbf{R}^d)$, or*
(b) *$\bigcup_{k=1}^r G(g_k, \Lambda_k)$ possesses a lower frame bound,*

then $D(\Lambda) = 1$.

Proof. In either case, $\bigcup_{k=1}^r G(g_k, \Lambda_k)$ possesses the HAP. Now let $\phi = \chi_{[0,1]^d}$ and $\Delta = \mathbf{Z}^d \times \mathbf{Z}^d$. Then $G(\phi, \Delta)$ is an orthonormal basis for $L^2(\mathbf{R}^d)$ and $D(\Delta) = 1$. Therefore, Theorem 4.5 implies that $1 = D^-(\Delta) \le D^-(\Lambda)$ and $1 = D^+(\Delta) \le D^+(\Lambda)$. Combining this with the inequalities in Corollary 4.6, we see that $D^-(\Lambda) = D^+(\Lambda) = 1$. □

5. SCHAUDER BASES OF TRANSLATES

In this section we consider systems of pure translates. Given $g_1, \dots, g_r \in L^2(\mathbf{R}^d)$ and $\Gamma_1, \dots, \Gamma_r \subset \mathbf{R}^d$, define

$$T(g_k, \Gamma_k) \;=\; \{g_k(x-a)\}_{a\in\Gamma_k},$$

and suppose that $\bigcup_{k=1}^r T(g_k, \Gamma_k)$ was a Schauder basis for $L^2(\mathbf{R}^d)$. Systems of translates are simply special cases of Gabor systems where the modulation parameter is always zero, i.e., $T(g_k, \Gamma_k) = G(g_k, \Gamma_k \times \{0\})$. Setting $\Gamma = \bigcup_{k=1}^r \Gamma_k$, it therefore follows from Corollary 4.2 that $D^+(\Gamma \times \{0\}) < \infty$ (density as a subset of $\mathbf{R}^{2d}$), which implies that $D^+(\Gamma) < \infty$ (density as a subset of $\mathbf{R}^d$). Once we know that Γ has finite upper Beurling density as a subset of $\mathbf{R}^d$, it is then elementary to check that we in fact have $D^+(\Gamma \times \{0\}) = 0$. However, by Corollary 4.10, if either $g_1, \dots, g_r \in M^{1,1}(\mathbf{R}^d)$ or if $\bigcup_{k=1}^r T(g_k, \Gamma_k)$ possesses a lower frame bound, then we must have $D(\Gamma \times \{0\}) = 1$, which is a contradiction. Hence, in neither of these cases is it possible to have a Schauder basis. However, because of some simplifications that occur when the modulation parameter is absent, we can actually improve this somewhat and show that $g_1, \dots, g_r$ cannot lie in $L^1(\mathbf{R}^d) \cap L^2(\mathbf{R}^d)$, which is strictly larger than $M^{1,1}(\mathbf{R}^d)$.

Theorem 5.1. *Let $g_1, \dots, g_r \in L^2(\mathbf{R}^d)$ and let $\Gamma_1, \dots, \Gamma_r \subset \mathbf{R}^d$ be fixed. Then $\bigcup_{k=1}^r T(g_k, \Gamma_k)$ cannot form a Schauder basis for $L^2(\mathbf{R}^d)$ if either*

(a) *$\bigcup_{k=1}^r T(g_k, \Gamma_k)$ possesses a lower frame bound, or*
(b) *$g_1, \dots, g_r \in L^1(\mathbf{R}^d) \cap L^2(\mathbf{R}^d)$.*

Proof. Part (a) has already been proved, so we will concentrate on part (b). We will show that $\bigcup_{k=1}^r T(g_k, \Gamma_k)$ satisfies a pure-translation analogue of the HAP and that a Comparison Theorem follows from this. For simplicity, we will present only a limited version of the HAP and Comparision Theorem that is sufficient for our present needs.

Since $\Gamma = \bigcup_{k=1}^\infty \Gamma_k$ is relatively uniformly separated, we may, by splitting the Γ_k into subsequences if necessary, assume that each Γ_k is δ_k-uniformly separated for some $\delta_k > 0$. Let $\delta = \min\{\delta_1/2, \dots, \delta_r/2\}$, and denote the dual basis of $\bigcup_{k=1}^r T(g_k,\Gamma_k)$ by $\{\tilde g_{k,a} : a \in \Gamma_k,\ k = 1,\dots,r\}$. Since $\bigcup_{k=1}^r T(g_k,\Gamma_k)$ is a bounded basis, there is a constant $C > 0$ such that $\|\tilde g_{k,a}\| \le C$ for every k and a. For each $p \in \mathbf{R}^d$ and $R > 0$, define

$$W(R,p) \;=\; \operatorname{span}\bigl\{\tilde g_{k,a} : a \in Q_R(p)\cap\Gamma_k,\ k = 1,\dots,r\bigr\}.$$

Let $\chi = \delta^{-d/2}\,\chi_{Q_\delta(0)}$, the characteristic function of $Q_\delta(0) = [-\frac{\delta}{2},\frac{\delta}{2}]^d$ normalized to unit L^2-norm. Then we claim that the following HAP is satisfied:

$$\forall\,\varepsilon > 0, \quad \exists\,R > 0 \quad \text{such that} \quad \forall\,(p,q) \in \mathbf{R}^{2d}, \quad \operatorname{dist}\bigl(M_qT_p\chi,\ W(R,p)\bigr) \;<\; \varepsilon. \tag{23}$$

To see this, consider $(p,q) \in \mathbf{R}^{2d}$. Using the basis expansion $M_qT_p\chi = \sum_{k=1}^r \sum_{a\in\Gamma_k} \langle M_qT_p\chi, T_ag_k\rangle\,\tilde g_{k,a}$, we have

$$\begin{aligned}
\operatorname{dist}\bigl(M_qT_p\chi,\ W(R,p)\bigr) \;&\le\; \Bigl\|M_qT_p\chi \;-\; \sum_{k=1}^r \sum_{a\in Q_R(p)\cap\Gamma_k} \langle M_qT_p\chi, T_ag_k\rangle\,\tilde g_{k,a}\Bigr\|_2 \\
&=\; \Bigl\|\sum_{k=1}^r \sum_{a\in\Gamma_k\setminus Q_R(p)} \langle M_qT_p\chi, T_ag_k\rangle\,\tilde g_{k,a}\Bigr\|_2 \\
&\le\; \Bigl(\sup_{k=1,\dots,r}\ \sup_{a\in\Gamma_k} \|\tilde g_{k,a}\|_2\Bigr) \sum_{k=1}^r \sum_{a\in\Gamma_k\setminus Q_R(p)} |\langle M_qT_p\chi, T_ag_k\rangle| \\
&\le\; C\delta^{-d/2} \sum_{k=1}^r \sum_{a\in\Gamma_k\setminus Q_R(p)} \int_{Q_\delta(0)} |g(x-(a-p)|\,dx \\
&\le\; C\delta^{-d/2} \int_{\mathbf{R}^d\setminus Q_{R-\delta}(0)} |g(x)|\,dx.
\end{aligned} \tag{24}$$

Since $g \in L^1(\mathbf{R}^d)$, the last quantity in (24) can be made arbitrarily small, independently of (p,q), by taking R large enough. Hence (23) follows.

Now set $\Delta = \delta\mathbf{Z}^d \times \frac{1}{\delta}\mathbf{Z}^d$. Then $G(\chi,\Delta)$ is an orthonormal basis for $L^2(\mathbf{R}^d)$, and $D(\Delta) = 1$. Fix $\varepsilon > 0$. Then by (23), there is an $R > 0$ such that $\operatorname{dist}\bigl(M_yT_x\chi,\ W(R,x)\bigr) < \varepsilon$ for every $(x,y) \in \mathbf{R}^{2d}$. Fix $h > 0$ and $(p,q) \in \mathbf{R}^{2d}$. Then for $(a,b) \in \mathbf{Q}_h(p,q)$ we have $W(R,a) \subset W(h+R,p)$, and therefore

$$\operatorname{dist}\bigl(M_bT_a\chi,\ W(h+R,p)\bigr) \;\le\; \operatorname{dist}\bigl(M_bT_a\chi,\ W(R,a)\bigr) \;<\; \varepsilon. \tag{25}$$

Define

$$V(h,p,q) \;=\; \operatorname{span}\{M_bT_a\chi : (a,b) \in \mathbf{Q}_h(p,q)\cap\Delta\}.$$

Define $T : V(h,p,q) \to V(h,p,q)$ by $T = P_VP_W$, where P_V is the orthogonal projection of $L^2(\mathbf{R}^d)$ onto $V(h,p,q)$ and P_W is the orthogonal projection onto $W(h+R,p)$. Then, since $\{M_bT_a\chi : (a,b) \in \mathbf{Q}_h(p,q)\cap\Delta\}$ is an orthonormal basis for $V(h,p,q)$,

$$\operatorname{tr}(T) \;=\; \sum_{(a,b)\in\mathbf{Q}_h(p,q)\cap\Delta} \langle T(M_bT_a\chi), M_bT_a\chi\rangle. \tag{26}$$

For $(a,b) \in \mathbf{Q}_h(p,q)\cap\Delta$, we have

$$\langle T(M_bT_a\chi), M_bT_a\chi\rangle \;=\; \langle P_WM_bT_a\chi, M_bT_a\chi\rangle \;=\; 1 \;+\; \langle (P_W - I)M_bT_a\chi, M_bT_a\chi\rangle. \tag{27}$$

However, by (25) we have

$$|\langle (P_W - I)M_bT_a\chi, M_bT_a\chi\rangle| \;\le\; \|(P_W - I)M_bT_a\chi\|_2\,\|M_bT_a\chi\|_2 \;\le\; \varepsilon\cdot 1 \;=\; \varepsilon. \tag{28}$$

Combining (26)–(28), we obtain

$$\operatorname{tr}(T) \;\ge\; (1-\varepsilon)\,\#\bigl(\mathbf{Q}_h(p,q)\cap\Delta\bigr).$$

However, we also have

$$\operatorname{tr}(T) \;\le\; \operatorname{rank}(T) \;\le\; \dim\bigl(W(h+R,p)\bigr) \;\le\; \#\bigl(Q_{h+R}(p)\cap\Gamma\bigr).$$

Therefore,

$$(1-\varepsilon)\,\frac{\#\big(\mathbf{Q}_h(p,q)\cap\Delta\big)}{h^{2d}} \;\le\; \frac{1}{h^d}\,\frac{\#\big(\mathbf{Q}_{h+R}(p)\cap\Gamma\big)}{(h+R)^d}\,\frac{(h+R)^d}{h^d}.$$

Keeping in mind that Γ is a sequence in $\mathbf{R}^d$ while Δ is a sequence in $\mathbf{R}^{2d}$, we take the inf over all $(p,q)\in\mathbf{R}^{2d}$ and then the liminf as $h\to\infty$, and conclude that

$$(1-\varepsilon)\,D(\Delta) \;\le\; 0\cdot D^-(\Gamma).$$

However, $D(\Delta)=1$, so this is a contradiction. □

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