INVARIANCE OF A SHIFT-INVARIANT SPACE

AKRAM ALDROUBI, CARLOS CABRELLI, CHRISTOPHER HEIL,
KERI KORNELSON, AND URSULA MOLTER

Abstract. A shift-invariant space is a space of functions that is invariant under integer translations. Such spaces are often used as models for spaces of signals and images in mathematical and engineering applications. This paper characterizes those shift-invariant subspaces S that are also invariant under additional (non-integer) translations. For the case of finitely generated spaces, these spaces are characterized in terms of the generators of the space. As a consequence, it is shown that principal shift-invariant spaces with a compactly supported generator cannot be invariant under any non-integer translations.

1. Introduction

A *shift-invariant space* (SIS) is a space of functions that is invariant under integer translations. They have applications throughout mathematics and engineering, as such spaces are often used as models for spaces of signals and images, see [Grö01], [HW96], [Mal98].

One example of a shift-invariant space is the Paley–Wiener space of functions that are bandlimited to $[-1/2, 1/2]$:

$$PW(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}) : \operatorname{supp}(\widehat{f}\,) \subseteq [-\tfrac{1}{2}, \tfrac{1}{2}] \right\}.$$

This SIS has the property that it is not only invariant under integer translations, but it is in fact invariant under every real translation. A space with this property is said to be *translation-invariant*. A classical theorem of Fourier analysis completely characterizes the closed translation-invariant subspaces of $L^2(\mathbb{R})$ as being of the form

$$\{f \in L^2(\mathbb{R}) : \operatorname{supp}(\widehat{f}\,) \subseteq A\}$$

where $A \subseteq \mathbb{R}$ is measurable.

In many applications, it is desirable to have a shift-invariant space that possesses extra invariances [CS03], [Web00]. In this paper we characterize those shift-invariant subspaces S that are not only invariant under integer translations, but are also

Date: April 3, 2008.

Key words and phrases. Fiber space, frame, Grammian operator, Riesz basis, shift-invariant space, translation-invariant space.

The research of A. Aldroubi is supported in part by NSF Grant DMS-0504788. C. Cabrelli and U. Molter are partially supported by Grants PICT 15033, CONICET, PIP 5650, UBACyT X058 and X108. The research of C. Heil was supported in part by NSF Grant DMS-0139261. The research of K. Kornelson was supported in part by Grinnell College CSFS, the Woodrow Wilson Fellowship Foundation, and the IMMERSE program at the University of Nebraska-Lincoln.

invariant under some particular set of translations $G \subseteq \mathbb{R}$. We show that there are only two possibilities:

- either S is translation-invariant, or
- there exists an $n \in \mathbb{N}$ such that S is invariant under translations by multiples of $1/n$, but not invariant under translations by $1/m$ with $m > n$.

We give several characterizations of those shift-invariant spaces that are $\frac{1}{n}\mathbb{Z}$-invariant. A trivial way to create such a space is to fix a function $g \in L^2(\mathbb{R})$ and set

$$S = \overline{\text{span}}\big\{g(x - \tfrac{k}{n}) : k \in \mathbb{Z}\big\},$$

the closed span of the $\frac{1}{n}\mathbb{Z}$ translates of g. However, we are interested in the more subtle question of recognizing when a given SIS is $\frac{1}{n}\mathbb{Z}$-invariant. For example, in many applications one is presented with a SIS of the form

$$S = \overline{\text{span}}\big\{g(x - k) : k \in \mathbb{Z}\big\},$$

and it is not obvious whether such a space possesses any invariants other than translation by integers. We completely determine the invariances of such a space in terms of properties of g and, more generally, characterize any SIS that is $\frac{1}{n}\mathbb{Z}$-invariant.

One interesting corollary of our characterization is that the shift-invariant space generated by a compactly supported scaling function is not invariant under any translations other than $\mathbb{Z}$. Thus, the shift-invariant spaces associated with compactly supported multiresolution analyses and wavelets are already "maximally invariant."

2. Notation and Definitions

We normalize the Fourier transform of $f \in L^1(\mathbb{R})$ as

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i\omega x} dx.$$

The Fourier transform extends to a unitary operator on $L^2(\mathbb{R})$. Given $\mathcal{F} \subseteq L^2(\mathbb{R})$, we set $\widehat{\mathcal{F}} = \{\widehat{f} : f \in \mathcal{F}\}$.

The translation operator T_a is $T_a f(x) = f(x - a)$. Note that $(T_a f)^\wedge(\omega) = e^{-2\pi i a\omega}\widehat{f}(\omega)$.

A function f is *$b\mathbb{Z}$-periodic* if $T_{bk} f = f$ for all $k \in \mathbb{Z}$. A set $A \subseteq \mathbb{R}$ is *$b\mathbb{Z}$-periodic* if its characteristic function is $b\mathbb{Z}$-periodic.

A *shift-invariant space* (SIS) is a closed subspace S of $L^2(\mathbb{R})$ that is invariant under integer translations. We say that S is $b\mathbb{Z}$-invariant if it is invariant under translation by bk for all $k \in \mathbb{Z}$.

Given $\mathcal{F} \subseteq L^2(\mathbb{R})$, we define

$$\mathcal{T}_{\mathbb{Z}}(\mathcal{F}) = \{T_j f : f \in \mathcal{F},\, j \in \mathbb{Z}\}.$$

The SIS generated by $\mathcal{F}$ is

$$\mathfrak{S}(\mathcal{F}) = \overline{\text{span}}\big(\mathcal{T}_{\mathbb{Z}}(\mathcal{F})\big) = \overline{\text{span}}\{T_j f : f \in \mathcal{F},\, j \in \mathbb{Z}\}.$$

We call $\mathcal{F}$ a *set of generators* for $\mathfrak{S}(\mathcal{F})$. When $\mathcal{F} = \{f\}$ consists of a single function, we simply write $\mathfrak{S}(f)$.

The *length* of a SIS S is the minimum cardinality of the sets $\mathcal{F}$ such that $S = \mathfrak{S}(\mathcal{F})$. A SIS of length one is called a *principal* SIS. A SIS of finite length is a *finitely generated* SIS.

We will write $W = U \dot{\oplus} V$ to denote the *orthogonal* direct sum of closed subspaces of $L^2(\mathbb{R})$, i.e., the subspaces U, V must be closed and orthogonal, and W is their direct sum. Some other common notations for orthogonal direct sum are $U \oplus^{\perp} V$ or $U \oplus_o V$.

The Lebesgue measure of a set $E \subseteq \mathbb{R}$ is denoted by $|E|$.

The cardinality of a finite set F is denoted by $\#F$.

3. Order of Invariance

Let S be a SIS. If θ is a real number, then S is invariant under translations by θ if

$$f \in S \quad \Longrightarrow \quad T_\theta f \in S.$$

In this case S will also be invariant under translations by any integer multiple of θ.

Let G be the set of all parameters θ such that S is invariant under translations by θ. Note that $\mathbb{Z} \subseteq G$ since S is shift-invariant, and if $\theta \in G$ then we have $m + n\theta \in G$ for all integers m and n. In particular, if θ is irrational, then G is dense in $\mathbb{R}$. In this case, if a is an arbitrary real number, then we can find numbers $d_j \in G$ that converge to a. Then given any $f \in S$, we have

$$\|T_{d_j} f - T_a f\|_2^2 = \int_{-\infty}^{\infty} |f(x - d_j) - f(x-a)|^2 \, dx \to 0 \quad \text{as } j \to \infty.$$

Since S is closed, it follows that $T_a f \in S$, so S is invariant under translation by a. But a is arbitrary, so $G = \mathbb{R}$ and S is invariant under all translations, i.e., it is translation-invariant.

This reasoning applies whenever G is dense, which is certainly the case if G contains any irrational number. So, consider the case where G contains only rationals. Suppose that p/q is a rational number that belongs to G. We can assume that p/q is written in lowest terms. Then there exist m, $n \in \mathbb{Z}$ such that $mp + nq = 1$, so since G is closed under addition we have that

$$\frac{1}{q} = n + m\frac{p}{q} \in G,$$

and hence $\frac{1}{q}\mathbb{Z} \subseteq G$.

More generally, suppose that we have finitely many rationals $p_1/q_1, \dots, p_r/q_r$ in G. As above, we then have $1/q_1, \dots, 1/q_r \in G$. Let q denote the least common multiple of $q_1, \dots, q_r$. Then we can find integers α_k and n_k such that

$$q = \alpha_1 q_1 = \cdots = \alpha_r q_r \qquad \text{and} \qquad n_1\alpha_1 + \cdots + n_r\alpha_r = 1,$$

so

$$\frac{1}{q} = \frac{1}{q}\biggl(\sum_{k=1}^{r} n_k \alpha_k\biggr) = \frac{n_1}{q_1} + \cdots + \frac{n_r}{q_r} \in G.$$

In particular, S is $\frac{1}{q}\mathbb{Z}$-invariant.

Now we are reduced to two possibilities. First, there may be infinitely many rational numbers with different denominators in G. In this case, the above reasoning shows that there exist rationals of the form $1/q$ in G with q arbitrarily large. Since G is closed under addition, this implies that G is dense, and hence we actually have $G = \mathbb{R}$ and S is translation-invariant.

Second, there may be only finitely many distinct denominators among the rationals in G. In this case, the above reasoning implies that all of these rationals are multiples of $1/q$ for some integer q, with $1/q \in G$, and consequently S is $\frac{1}{q}\mathbb{Z}$-invariant.

We summarize these conclusions as follows.

Proposition 3.1. *Let S be a SIS. Then either S is translation-invariant, or there exists a maximum positive integer n such that S is $\frac{1}{n}\mathbb{Z}$-invariant.*

Proposition 3.1 suggests the following definition.

Definition 3.2. Given a shift-invariant space S, we say that S has invariance order n if n is the maximum positive integer such that S is $\frac{1}{n}\mathbb{Z}$-invariant. If this maximum does not exist, we say that S has invariance order ∞; in this case S is translation-invariant.

Remark 3.3. Note that the invariance order of any SIS is at least 1, since S is $\mathbb{Z}$-invariant. Also, if S has invariance order n, then S is *not* invariant under translation by any real number y in the range $0 < y < 1/n$. Furthermore, if $y \geq 1/n$, S can *only* be invariant under translation by y if y is a multiple of $1/d$ where d divides n. In particular, if the order of invariance of S is a prime number p, there exist no other integers $m > 1$ such that S is invariant under translations by $1/m$.

4. Characterization of $\frac{1}{n}$-Invariance

In this part we will characterize those shift-invariant spaces that are $\frac{1}{n}\mathbb{Z}$-invariant.

4.1. **Notation.** We will use the following notation throughout the remainder of this paper.

Given a fixed positive integer n, we partition the real line into n sets, each of which is $n\mathbb{Z}$-periodic, as follows. For $k = 0, \dots, n-1$, set $I_k = [k, k+1)$, and define

$$B_k = \bigcup_{j \in \mathbb{Z}} (I_k + nj).$$

Note that B_k implicitly depends on the choice of n.

Given a SIS $S \subseteq L^2(\mathbb{R})$, we associate the following subspaces:

$$U_k = \{f \in L^2(\mathbb{R}) : \widehat{f} = \widehat{g}\,\chi_{B_k} \text{ for some } g \in S\}, \qquad k = 0, \dots, n-1. \tag{1}$$

The spaces U_k are mutually orthogonal since the sets B_k are disjoint (up to sets of measure zero).

If $f \in S$ and $0 \leq k \leq n-1$, then we let f^k denote the function defined by

$$\widehat{f^k} = \widehat{f}\,\chi_{B_k}.$$

Letting P_k denote the orthogonal projection onto U_k, we have that

$$f^k = P_k f.$$

Note that integer translations commute with the projections P_k: if $j \in \mathbb{Z}$ and $k = 0, \dots, n-1$, then

$$T_j P_k = P_k T_j.$$

4.2. **Preliminary results.** We will need the following result from [dBVR94a].

Proposition 4.1 ([dBVR94a])**.** *Let $f \in L^2(\mathbb{R})$ be given. If $g \in \mathfrak{S}(f)$, then there exists a $\mathbb{Z}$-periodic function m such that $\widehat{g} = m\widehat{f}$.*

Conversely, if m is a $\mathbb{Z}$-periodic function such that $m\widehat{f} \in L^2(\mathbb{R})$, then the function g defined by $\widehat{g} = m\widehat{f}$ belongs to $\mathfrak{S}(f)$.

We will also need a version of the preceding result for spaces that are $\frac{1}{n}\mathbb{Z}$-invariant instead of shift-invariant. This follows easily by rescaling.

Corollary 4.2. *Let $f \in L^2(\mathbb{R})$ and $n \in \mathbb{N}$ be given, and set*

$$\mathfrak{S}(f, \tfrac{1}{n}\mathbb{Z}) = \overline{\operatorname{span}}\{T_{j/n} f : f \in \mathcal{F},\, j \in \mathbb{Z}\}.$$

If $g \in \mathfrak{S}(f, \frac{1}{n}\mathbb{Z})$, then there exists a $n\mathbb{Z}$-periodic function m such that $\widehat{g} = m\widehat{f}$.

Conversely, if m is an $n\mathbb{Z}$-periodic function such that $m\widehat{f} \in L^2(\mathbb{R})$, then the function g defined by $\widehat{g} = m\widehat{f}$ belongs to $\mathfrak{S}(f, \frac{1}{n}\mathbb{Z})$.

4.3. **Characterization of $\frac{1}{n}$-invariance in terms of subspaces.** The periodicity of the B_k sets yields the following lemma.

Lemma 4.3. *If S is a SIS, then for each $k = 0, \dots, n-1$, the subspace U_k is a SIS that is also $\frac{1}{n}\mathbb{Z}$-invariant.*

Proof. Fix $0 \le k \le n-1$, and choose any $f \in U_k$. There exists a $g \in S$ such that $\widehat{f} = \widehat{g}\,\chi_{B_k}$. Since S is shift-invariant and $g \in S$, we have that $e^{-2\pi i\omega}\widehat{g}(\omega)$ is in $\widehat{S}$. Hence

$$e^{-2\pi i\omega}\,\widehat{f}(\omega) = e^{-2\pi i\omega}\,\widehat{g}(\omega)\,\chi_{B_k}(\omega) \in U_k.$$

Therefore $T_1 f \in U_k$, so U_k is invariant under integer translates.

Suppose now that $f_j \in U_k$ and $f_j \to f$ in $L^2(\mathbb{R})$. Since $U_k \subseteq S$ and S is closed, f must be in S. Further,

$$\|\widehat{f_j} - \widehat{f}\,\|_2^2 = \|(\widehat{f_j} - \widehat{f}\,)\,\chi_{B_k}\|_2^2 + \|(\widehat{f_j} - \widehat{f}\,)\,\chi_{B_k^{\mathrm{C}}}\|_2^2 = \|\widehat{f_j} - \widehat{f}\,\chi_{B_k}\|_2^2 + \|\widehat{f}\,\chi_{B_k^{\mathrm{C}}}\|_2^2.$$

Since the left-hand side converges to zero, we must that have $\widehat{f}\,\chi_{B_k^{\mathrm{C}}} = 0$ a.e., and that $\widehat{f_j} \to \widehat{f}\,\chi_{B_k}$ in $L^2(\mathbb{R})$. Since we also have $\widehat{f_j} \to \widehat{f}$, we conclude that

$$\widehat{f} = \widehat{f}\,\chi_{B_k} \text{ a.e.}$$

Consequently $f \in U_k$, so U_k is closed.

Finally, to see that U_k is $\frac{1}{n}\mathbb{Z}$-invariant, define

$$h(\omega) = e^{-\frac{2\pi i\omega}{n}} \sum_{j=-k}^{n-1-k} e^{\frac{2\pi ij}{n}} \chi_{B_{k+j}}(\omega).$$

Note that $|h(\omega)| = 1$ and that h is $\mathbb{Z}$-periodic. Furthermore, if $\omega \in B_k$ and $-k \leq j \leq n-1-k$, then $\chi_{B_{k+j}}(\omega)$ can be nonzero only when $j = 0$. Hence:

$$\omega \in B_k \implies h(\omega) = e^{-\frac{2\pi i\omega}{n}}.$$

If $f \in U_k$ then, since $\operatorname{supp}(f) \subseteq B_k$, we have

$$e^{-\frac{2\pi i\omega}{n}} \widehat{f}(\omega) = h(\omega)\, \widehat{f}(\omega).$$

However, since U_k is $\mathbb{Z}$-invariant, we have by Proposition 4.1 that $h\widehat{f} \in \widehat{U_k}$. Therefore $e^{-\frac{2\pi i\omega}{n}} \widehat{f}(\omega) \in \widehat{U_k}$, which implies that $T_{1/n} f \in U_k$. □

This leads to the following characterization.

Theorem 4.4. *If $S \subseteq L^2(\mathbb{R})$ is a SIS, then the following are equivalent.*

(a) *S is $\frac{1}{n}\mathbb{Z}$-invariant.*

(b) *$U_k \subseteq S$ for $k = 0, \dots, n-1$.*

(c) *If $f \in S$, then $f^k = P_k f \in S$ for each $k = 0, \dots, n-1$.*

Moreover, in case these hold we have that S is the orthogonal direct sum

$$S = U_0 \dot{\oplus} \dots \dot{\oplus} U_{n-1},$$

with each U_k being a (possibly trivial) $\frac{1}{n}\mathbb{Z}$-invariant SIS.

Proof. (a) ⇒ (b). Assume that S is $\frac{1}{n}\mathbb{Z}$-invariant and fix $0 \leq k \leq n-1$ and $f \in U_k$. By definition of U_k, we have that $\widehat{f} = \widehat{g}\,\chi_{B_k}$ for some $g \in S$. Since χ_{B_k} is $n\mathbb{Z}$-periodic and bounded, Corollary 4.2 implies that $f \in \mathfrak{S}(g, \frac{1}{n}\mathbb{Z}) \subseteq S$.

(b) ⇒ (a). Suppose that $U_k \subseteq S$ for each $k = 0, \dots, n-1$. Note that Lemma 4.3 implies that U_k is $\frac{1}{n}\mathbb{Z}$-invariant, and we also have that the U_k are mutually orthogonal since the sets B_k are disjoint.

Suppose that $f \in S$. Then $f = f^0 + \cdots + f^{n-1}$ where $\widehat{f^k} = \widehat{f}\,\chi_{B_k}$. This implies that $f \in U_0 \dot{\oplus} \dots \dot{\oplus} U_{n-1}$, and consequently S is the orthogonal direct sum

$$S = U_0 \dot{\oplus} \dots \dot{\oplus} U_{n-1}.$$

As each U_k is $\frac{1}{n}\mathbb{Z}$-invariant, it follows that S is $\frac{1}{n}\mathbb{Z}$-invariant as well.

(b) ⇔ (c). This is a restatement of the definition of U_k. □

Corollary 4.5. *Let S be a SIS. If there exists a $k \in \{0, \dots, n-1\}$ such that* $\operatorname{supp}(\widehat{f}\,) \subseteq B_k$ *for all $f \in S$, then S is $\frac{1}{n}\mathbb{Z}$-invariant.*

4.4. **Characterization of $\frac{1}{n}$-invariance in terms of generators.** We will show now that the conditions for $\frac{1}{n}\mathbb{Z}$-invariance can be formulated in terms of properties of a set of generators of the SIS.

Theorem 4.6. *Let $\mathcal{F}$ be a set of generators for a SIS S, i.e., $S = \mathfrak{S}(\mathcal{F})$. Then the following statements are equivalent.*

(a) *S is $\frac{1}{n}\mathbb{Z}$-invariant.*

(b) *$P_k\mathcal{F} = \{f^k : f \in \mathcal{F}\} \subseteq S$ for $k = 0, \dots, n-1$.*

Proof. (a) $\Rightarrow$ (b). This is a consequence of Theorem 4.4.

(b) $\Rightarrow$ (a). Suppose that statement (b) holds. Then, by hypothesis, $V_k = \mathfrak{S}(P_k\mathcal{F}) \subseteq S$, and, by Corollary 4.5, V_k is $\frac{1}{n}\mathbb{Z}$-invariant. Furthermore, $V_j \perp V_k$ when $j \neq k$. If $f \in \mathcal{F}$, then $f = f^0 + \cdots + f^{n-1} \in V_1 \dot{\oplus} \cdots \dot{\oplus} V_{n-1}$. Consequently, $S = \mathfrak{S}(\mathcal{F}) = V_1 \dot{\oplus} \cdots \dot{\oplus} V_{n-1}$. As each V_k is $\frac{1}{n}\mathbb{Z}$-invariant, it follows that S is as well. □

It is known that it is always possible to choose a (possibly infinite) set of generators of a SIS in such a way that the integer translates of the generators actually forms a frame for the SIS. (See Theorem 4.14). This is particularly important in applications, and we examine this situation next.

Recall that a countable collection of vectors $\{v_\alpha : \alpha \in \Lambda\}$ forms a *frame* for a Hilbert space H if there exist constants A, B (called *frame bounds*) such that

$$\forall\, w \in H, \quad A\,\|w\|^2 \,\leq\, \sum_{\alpha\in\Lambda} |\langle w, v_\alpha\rangle|^2 \,\leq\, B\|w\|^2. \tag{2}$$

If we can take $A = B = 1$, then the frame is called a *Parseval frame.*

The next result shows that if the integer translates of the generators of a SIS form a frame, then the set of integer translations of the "cutoffs" of the generators remains a frame.

Theorem 4.7. *Assume that S is a SIS that is $\frac{1}{n}\mathbb{Z}$-invariant, and that $\mathcal{F} \subseteq S$ is such that $\mathcal{T}_{\mathbb{Z}}(\mathcal{F})$ is a frame for S with frame bounds A, B. Then*

$$\mathcal{T}_{\mathbb{Z}}(P_k\mathcal{F}) \,=\, \big\{T_j f^k : f \in \mathcal{F},\, j \in \mathbb{Z}\big\}$$

is a frame for $U_k = \mathfrak{S}(P_k\mathcal{F})$ with frame bounds A, B. Further,

$$\mathcal{T}_{\mathbb{Z}}\bigg(\bigcup_{k=0}^{n-1} P_k\mathcal{F}\bigg) \,=\, \big\{T_j f^k : f \in \mathcal{F},\, j \in \mathbb{Z},\, k = 0, \dots, n-1\big\}$$

is a frame for S with frame bounds A, B.

Proof. By hypothesis,

$$\forall\, g \in S, \quad A\,\|g\|_2^2 \leq \sum_{j\in\mathbb{Z}} \sum_{f\in\mathcal{F}} |\langle g, T_j f\rangle|^2 \,\leq\, B\,\|g\|_2^2. \tag{3}$$

Suppose that $g \in U_k$. Then since P_k commutes with integer translations, we have

$$\begin{aligned}\sum_{j\in\mathbb{Z}}\sum_{f\in\mathcal{F}} |\langle g, T_j P_k f\rangle|^2 &= \sum_{j\in\mathbb{Z}}\sum_{f\in\mathcal{F}} |\langle g, P_k T_j f\rangle|^2 \\ &= \sum_{j\in\mathbb{Z}}\sum_{f\in\mathcal{F}} |\langle P_k g, T_j f\rangle|^2 \\ &= \sum_{j\in\mathbb{Z}}\sum_{f\in\mathcal{F}} |\langle g, T_j f\rangle|^2.\end{aligned}$$

Combining this with (3), we see that $\mathcal{T}_{\mathbb{Z}}(P_k\mathcal{F})$ is a frame for U_k with frame bounds A, B.

Suppose now that $g \in S$. Then since S is the orthogonal direct sum of the U_k, we have that

$$\sum_{j\in\mathbb{Z}}\sum_{f\in\mathcal{F}}\sum_{k=0}^{n-1} |\langle g, T_j P_k f\rangle|^2 = \sum_{k=0}^{n-1}\sum_{j\in\mathbb{Z}}\sum_{f\in\mathcal{F}} |\langle P_k g, T_j f\rangle|^2 \leq B\sum_{k=0}^{n-1} \|P_k g\|_2^2 = B\,\|g\|_2^2.$$

The estimate from below is similar, so we see that $\mathcal{T}_{\mathbb{Z}}\big(\bigcup_{k=0}^{n-1} P_k\mathcal{F}\big)$ is a frame for S, with frame bounds A, B. □

4.5. **Characterization of $\frac{1}{n}$-invariance in terms of fibers.** A useful tool in the theory of shift-invariant spaces is based on early work of Helson [Hel64]. An $L^2(\mathbb{R})$ function is decomposed into "fibers." This produces a characterization of SIS in terms of closed subspaces of $\ell^2(\mathbb{Z})$ (the fiber spaces). For a detailed description of this approach, see [Bow00] and the references therein.

Definition 4.8. Given $f \in L^2(\mathbb{R})$ and $\omega \in [0,1)$, the *fiber* $\widehat{f}_\omega$ of f at ω is the sequence

$$\widehat{f}_\omega = \big\{\widehat{f}(\omega+k)\big\}_{k\in\mathbb{Z}}.$$

If f is in $L^2(\mathbb{R})$, then the fiber $\widehat{f}_\omega$ belongs to $\ell^2(\mathbb{Z})$ for almost every $\omega \in [0,1)$.

Definition 4.9. Given a closed subspace V of $L^2(\mathbb{R})$ and $\omega \in [0,1)$, the *fiber space* of V at ω is

$$\mathcal{J}_V(\omega) = \overline{\text{span}}\big\{\widehat{f}_\omega : f \in V \text{ and } \widehat{f}_\omega \in \ell^2(\mathbb{Z})\big\},$$

where the closure is taken in the norm of $\ell^2(\mathbb{Z})$.

For proof that $\mathcal{J}_V(\omega)$ is a well-defined closed subspace of $\ell_2(\mathbb{Z})$ for almost every ω, see [Bow00], [Hel64].

We will need the following two results.

Proposition 4.10 ([Hel64])**.** *If S is a SIS, then*

$$S = \big\{f \in L^2(\mathbb{R}) : \widehat{f}_\omega \in \mathcal{J}_S(\omega) \text{ for a.e. } \omega\big\}.$$

Proposition 4.11. *Let S_1 and S_2 be SISs. If $S = S_1 \dot{\oplus} S_2$, then*

$$\mathcal{J}_S(\omega) = \mathcal{J}_{S_1}(\omega) \dot{\oplus} \mathcal{J}_{S_2}(\omega), \quad \text{a.e. } \omega.$$

The converse of Proposition 4.11 is also true, but will not be needed.

Combining Theorem 4.4 with Proposition 4.10 yields the following characterization of $\frac{1}{n}$-invariance in terms of the fiber spaces.

Theorem 4.12. *Let S be a SIS. Then the following statements are equivalent.*

(a) *S is $\frac{1}{n}\mathbb{Z}$-invariant.*

(b) *$\mathcal{J}_{U_k}(\omega) \subseteq \mathcal{J}_S(\omega)$ for almost every ω and each $k = 0, \dots, n-1$.*

For the finitely generated case we can obtain a slightly simpler characterization of $\frac{1}{n}\mathbb{Z}$-invariance.

Theorem 4.13. *If S is a finitely generated SIS, then the following statements are equivalent.*

(a) *S is $\frac{1}{n}\mathbb{Z}$-invariant.*

(b) *For almost every $\omega \in [0,1)$,*

$$\dim\big(\mathcal{J}_S(\omega)\big) = \sum_{k=0}^{n-1} \dim\big(\mathcal{J}_{U_k}(\omega)\big).$$

Proof. (a) $\Rightarrow$ (b). If S is $\frac{1}{n}\mathbb{Z}$-invariant then $S = \dot{\bigoplus}_{k=0}^{n-1} U_k$. This is an orthogonal direct sum, so Proposition 4.11 implies that $\mathcal{J}_S(\omega) = \dot{\bigoplus}_{k=0}^{n-1} \mathcal{J}_{U_k}(\omega)$ for a.e. ω, with this sum also orthogonal. The equality of dimensions in statement (b) therefore holds.

(b) $\Rightarrow$ (a). Suppose that statement (b) holds. It is clear that the inclusion $\mathcal{J}_S(\omega) \subseteq \dot{\bigoplus}_{k=0}^{n-1} \mathcal{J}_{U_k}(\omega)$ holds for a.e. ω. Since the spaces U_k are orthogonal, Proposition 4.11 implies that, for a.e. ω, the spaces $\mathcal{J}_{U_k}(\omega)$ are also orthogonal. Counting dimensions and applying statement (b), we conclude that $\mathcal{J}_S(\omega) = \dot{\bigoplus}_{k=0}^{n-1} \mathcal{J}_{U_k}(\omega)$ for a.e. ω.

Suppose now that $f \in U_k$. Then $\widehat{f}_\omega \in \mathcal{J}_{U_k}(\omega) \subseteq \mathcal{J}_S(\omega)$ for a.e. ω, so Proposition 4.10 implies that $f \in S$. Thus $U_k \subseteq S$ for each k, so it follows from Theorem 4.4 that S is $\frac{1}{n}\mathbb{Z}$-invariant. □

4.6. **The Bownik decomposition and $\frac{1}{n}$-invariance.** In [Bow00], Bownik obtained a decomposition for general shift-invariant spaces, extending the earlier works [dBVR94a] and [dBVR94b], which applied to the finitely generated case. We will apply this decomposition to shift-invariant spaces that are $\frac{1}{n}\mathbb{Z}$-invariant.

Theorem 4.14 (Bownik)**.** *Let $S \subseteq L^2(\mathbb{R})$ be a SIS. Then for each $j \in \mathbb{N}$ we can find a function $\varphi_j \in L^2(\mathbb{R})$ such that $\mathcal{T}_\mathbb{Z}(\varphi_j)$ is a Parseval frame for $\mathfrak{S}(\varphi_j)$, and furthermore*

$$S = \dot{\bigoplus_{j\in\mathbb{N}}} \mathfrak{S}(\varphi_j).$$

Note that a consequence of this theorem is that every SIS has always a set of generators whose integer translates form a Parseval frame of the SIS.

By applying Theorem 4.14 to each space U_k, we obtain the following result.

Theorem 4.15. *Let S be a SIS that is $\frac{1}{n}\mathbb{Z}$-invariant. Then there exist functions $\varphi_{k,j} \in L^2(\mathbb{R})$ such that*

$$S = \dot{\bigoplus_{k=0}^{n-1}} \dot{\bigoplus_{j\in\mathbb{N}}} \mathfrak{S}(\varphi_{k,j}),$$

with the following properties holding.

(a) $\mathcal{T}_{\mathbb{Z}}(\varphi_{k,j})$ *is a Parseval frame for* $\mathfrak{S}(\varphi_{k,j})$.

(b) $\mathfrak{S}(\varphi_{k,j}) \subseteq U_k$ *for each* $j \in \mathbb{N}$, *and*

$$U_k = \dot{\bigoplus_{j\in\mathbb{N}}} \mathfrak{S}(\varphi_{k,j}).$$

(c) *Each space* $\mathfrak{S}(\varphi_{k,j})$ *is* $\frac{1}{n}\mathbb{Z}$*-invariant.*

5. Finitely Generated Shift-Invariant Spaces and $\frac{1}{n}$-Invariance

In this section we will apply some of the general results obtained so far to the particular case of finitely generated shift-invariant spaces.

5.1. Characterization of $\frac{1}{n}$-invariance in terms of the Grammian.

Definition 5.1. Let $\Phi = \{\varphi_1, \dots, \varphi_m\}$ be a collection of finitely many functions in $L^2(\mathbb{R})$. Then the *Grammian* G_Φ of Φ is the $m \times m$ matrix of $\mathbb{Z}$-periodic functions

$$[G_\Phi(\omega)]_{ij} = \Big\langle (\widehat{\varphi}_i)_\omega, (\widehat{\varphi}_j)_\omega \Big\rangle = \sum_{k\in\mathbb{Z}} \widehat{\varphi}_i(\omega+k)\,\overline{\widehat{\varphi}_j(\omega+k)}, \qquad \omega \in \mathbb{R}, \tag{4}$$

where $(\widehat{\varphi}_j)_\omega$ is the fiber of φ_j at ω.

We consider now the SIS $S = \mathfrak{S}(\Phi)$ generated by the set $\Phi = \{\varphi_1, \dots, \varphi_m\}$. It is known [dBVR94b] that if $f \in \mathfrak{S}(\Phi)$, then there exist $\mathbb{Z}$-periodic functions $a_1, \dots, a_m$ such that

$$\widehat{f}(\omega) = \sum_{j=1}^{m} a_j(\omega)\,\widehat{\varphi}_j(\omega), \qquad \text{a.e. } \omega.$$

This implies that the fiber spaces $\mathcal{J}_S(\omega)$ are generated by the fibers of the generators of S at ω (see also [Bow00]). That is, for almost every ω we have that

$$\mathcal{J}_S(\omega) = \operatorname{span}\big\{(\widehat{\varphi}_j)_\omega : j = 1, \dots, m\big\}.$$

Therefore

$$\dim\big(\mathcal{J}_S(\omega)\big) = \operatorname{rank}[G_\Phi(\omega)]$$

for almost every ω.

In the same way, since the SIS U_k is generated by $\Phi^k = P_k\Phi = \{\varphi_1^k, \dots, \varphi_m^k\}$, where $\varphi_j^k = P_k\varphi_j$, we have for almost every ω that the fiber spaces $\mathcal{J}_{U_k}(\omega)$ satisfy

$$\mathcal{J}_{U_k}(\omega) \;=\; \operatorname{span}\Big\{\Big(\widehat{\varphi_j^k}\Big)_\omega : j = 1, \dots, m\Big\}.$$

Let us denote by G_{Φ^k} the Grammian matrix associated with the generators of U_k. Then, as above we have that $\dim(\mathcal{J}_{U_k}(\omega)) = \operatorname{rank}[G_{\Phi^k}(\omega)]$ for almost every ω and $k = 0, \dots, n-1$. Now Theorem 4.13 can be restated in the following way.

Theorem 5.2. *If $S = \mathfrak{S}(\Phi)$ is the SIS generated by $\Phi = \{\varphi_1, \dots, \varphi_m\}$, then the following statements are equivalent.*

(a) *S is $\frac{1}{n}\mathbb{Z}$-invariant.*

(b) *For almost every $\omega \in [0,1)$ we have*

$$\operatorname{rank}[G_\Phi(\omega)] \;=\; \sum_{k=0}^{n-1} \operatorname{rank}[G_{\Phi^k}(\omega)].$$

5.2. **Implications for frequency support.** As a consequence of Theorem 5.2 we deduce an interesting result about the supports of the Fourier transforms of the generators of a SIS.

Theorem 5.3. *Let $S = \mathfrak{S}(\Phi)$ be the SIS generated by $\Phi = \{\varphi_1, \dots, \varphi_m\}$, and define*

$$E_j \;=\; \big\{\omega \in [0,1) : \operatorname{rank}[G_\Phi(\omega)] = j\big\}, \quad j = 0, \dots, m.$$

If S is $\frac{1}{n}\mathbb{Z}$-invariant for some integer $n > m$, then for each interval $I \subseteq \mathbb{R}$ of length n we have for each $h = 1, \dots, m$ that

$$\big|\big\{\omega \in I : \widehat{\varphi}_h(\omega) = 0\big\}\big| \;\ge\; \sum_{j=0}^{m} (n-j)\,|E_j| \;\ge\; n - m.$$

Proof. The measurability of the sets E_j follows from the results of Helson [Hel64], e.g., see [BK06] for an argument of this type.

It is enough to prove the theorem for the interval $I_0 = [0, n)$. We note that the set

$$K \;=\; \{\omega \in [0,1) : (\widehat{\varphi}_h)_\omega \in \ell^2(\mathbb{Z}) \text{ for } h = 1, \dots, m\}$$

has full measure. Therefore

$$I \;=\; \bigcup_{k=0}^{n-1} (K + k)$$

is a subset of I_0 of measure n.

Fix any particular $j \in \{0, \dots, m\}$. By Theorem 5.2,

$$\operatorname{rank}[G_\Phi(\omega)] \;=\; \sum_{k=0}^{n-1} \operatorname{rank}[G_{\Phi^k}(\omega)], \qquad \text{a.e. } \omega. \tag{5}$$

Therefore, if $\omega \in E_j$ then at least $n-j$ terms on the right-hand side of equation (5) must vanish. That is, given such an ω, there exists a subset $\sigma \subseteq \{0, \dots, n-1\}$ with at least $n-j$ elements such that

$$\operatorname{rank}[G_{\Phi^k}(\omega)] = 0, \qquad k \in \sigma.$$

Let Σ denote the set of subsets of $\{0, \dots, n-1\}$. For each $\sigma \in \Sigma$ define

$$E_j^\sigma \;=\; \bigl\{\omega \in E_j : \operatorname{rank}[G_{\Phi^k}(\omega)] = 0 \text{ if and only if } k \in \sigma\bigr\}.$$

Then $E_j^\sigma \cap E_j^{\sigma'} = \emptyset$ if $\sigma \neq \sigma'$, and $E_j = \cup_{\sigma \in \Sigma} E_j^\sigma$. Also, $E_j^\sigma = \emptyset$ if $\#\sigma < n-j$.

Suppose now that $\sigma \in \Sigma$ and $\omega \in E_j^\sigma$. Given $k \in \sigma$, we therefore have that $\operatorname{rank}[G_{\Phi^k}(\omega)] = 0$. Hence $G_{\Phi^k}(\omega)$ is the zero matrix, so its diagonal entries are zero:

$$\Bigl\|\Bigl(\widehat{\varphi_h^k}\Bigr)_\omega\Bigr\|_2 \;=\; 0, \qquad h = 1, \dots, m.$$

Therefore $\widehat{\varphi_h^k}(\omega+\ell) = 0$ for each $\ell \in \mathbb{Z}$. Since the functions $\widehat{\varphi}_h$ and $\widehat{\varphi_h^k}$ are equal on the interval $[k, k+1)$, we conclude that $\widehat{\varphi}_h(\omega+k) = 0$. Equivalently,

$$k \in \sigma,\, \omega \in (E_j^\sigma + k) \qquad \Longrightarrow \qquad \widehat{\varphi}_h(\omega) = 0 \text{ for } h = 1, \dots, m.$$

Therefore, since j was arbitrarily chosen, we have for each $h = 1, \dots, m$ that

$$\bigl\{\omega \in I : \widehat{\varphi}_h(\omega) = 0\bigr\} \;\supseteq\; \bigcup_{j=0}^{m} \bigcup_{\#\sigma \geq n-j} \bigcup_{k \in \sigma} (E_j^\sigma + k),$$

and furthermore the sets in this union are disjoint. Hence,

$$\begin{aligned}
\bigl|\bigl\{\omega \in I : \widehat{\varphi}_h(\omega) = 0\bigr\}\bigr| &\geq \sum_{j=0}^{m} \sum_{r=n-j}^{n} \sum_{\#\sigma=r} \sum_{k \in \sigma} |(E_j^\sigma + k)| \\
&= \sum_{j=0}^{m} \sum_{r=n-j}^{n} \sum_{\#\sigma=r} r\, |E_j^\sigma| \\
&= \sum_{j=0}^{m} \sum_{r=n-j}^{n} r \left| \bigcup_{\#\sigma=r} E_j^\sigma \right| \\
&\geq \sum_{j=0}^{m} (n-j) \sum_{r=n-j}^{n} \left| \bigcup_{\#\sigma=r} E_j^\sigma \right| \\
&= \sum_{j=0}^{m} (n-j) \left| \bigcup_{\sigma \in \Sigma} E_j^\sigma \right| \\
&= \sum_{j=0}^{m} (n-j)\, |E_j|.
\end{aligned}$$

Finally, since the sets E_j are disjoint,

$$\sum_{j=0}^{m} (n-j)\, |E_j| \;\geq\; \sum_{j=0}^{m} (n-m)\, |E_j| \;=\; (n-m). \qquad \square$$

Note that if S is a principal SIS, say $S = \mathfrak{S}(\varphi)$, then the Grammian is scalar-valued:

$$G_\varphi(\omega) \,=\, \bigl\langle \widehat{\varphi}_\omega,\, \widehat{\varphi}_\omega \bigr\rangle \,=\, \sum_{k\in\mathbb{Z}} |\widehat{\varphi}(\omega+k)|^2.$$

Applying Theorem 5.3 to this case, we obtain the following corollary.

Corollary 5.4. *Let $\varphi \in L^2(\mathbb{R})$ be given. If the SIS $\mathfrak{S}(\varphi)$ is $\frac{1}{n}\mathbb{Z}$-invariant for some $n > 1$, then $\widehat{\varphi}$ must vanish on a set of infinite Lebesgue measure. Furthermore, for each interval $I \subseteq \mathbb{R}$ of length n, we have that*

$$\bigl|\bigl\{\omega \in I : \widehat{\varphi}(\omega) = 0\bigr\}\bigr| \,\ge\, n\,|E_0| + (n-1)\,|E_1| \,\ge\, n-1,$$

where $E_j = \{\omega \in [0,1) : G_\varphi(\omega) = j\}$, $\;j = 0,\, 1$.

This yields the following fact regarding the order of invariance of a principal SIS generated by a compactly supported function.

Proposition 5.5. *If a nonzero function $\varphi \in L^2(\mathbb{R})$ has compact support, then $\mathfrak{S}(\varphi)$ has invariance order one. That is $\mathfrak{S}(\varphi)$ is not $\frac{1}{n}\mathbb{Z}$-invariant for any $n > 1$.*

Proof. Because of Corollary 5.4, if $\mathfrak{S}(\varphi)$ is $\frac{1}{n}\mathbb{Z}$-invariant with $n > 1$, then $\widehat{\varphi}$ must vanish on a set of positive measure. Since φ has compact support, the Paley–Wiener Theorem implies that $\varphi = 0$ a.e. □

It is not difficult to construct a function $\varphi \in L^2(\mathbb{R})$ such that $\widehat{\varphi}$ is compactly supported in frequency yet the SIS $\mathfrak{S}(\varphi)$ is not translation-invariant. In fact, we have the following consequence of Corollary 5.4.

Corollary 5.6. *If $\varphi \in L^2(\mathbb{R})$ and $\mathfrak{S}(\varphi)$ is translation-invariant, then* $|\mathrm{supp}(\widehat{\varphi})| \le 1$.

5.3. **Application to multiresolution analysis.** For definitions and details on wavelets and multiresolution analysis, see [Dau92] or [Mal89].

Suppose that $\{V_j\}_{j\in\mathbb{Z}}$ is a multiresolution analysis (MRA) of $L^2(\mathbb{R})$. By definition, $V_0 = \mathfrak{S}(\varphi)$ for some $\varphi \in L^2(\mathbb{R})$, called the *scaling function*, and V_j is the image of V_0 under the unitary operator $D_{2^j} f(x) = 2^{j/2}\, f(2^j x)$.

The preceding results imply that if φ is compactly supported (as is the case for the Daubechies scaling functions, for example), then the SIS V_0 has invariance order exactly 1. Thus V_0 is invariant only under integer translations. The same remarks apply to the associated wavelet ψ and wavelet SIS $W_0 = \mathfrak{S}(\psi)$ if ψ is compactly supported.

Further, if φ is compactly supported then at resolution level j, the subspace V_j is invariant exactly under translations $2^j\mathbb{Z}$, and similarly for the wavelet space W_j if ψ is compactly supported. This includes all the spaces associated with the Daubechies scaling functions and wavelets, for example.

References

[Bow00] M. Bownik, *The structure of shift-invariant subspaces of $L^2(\mathbb{R}^n)$*, J. Funct. Anal., **177**(2) (2000), 282–309.

[BK06] M. Bownik and N. Kaiblinger, *Minimal generator sets for finitely generated shift-invariant subspaces of $L^2(\mathbb{R}^n)$*, J. Math. Anal. Appl., **313**(1) (2006), 342–352.

[CS03] C. K. Chui and Q. Sun, *Tight frame oversampling and its equivalence to shift-invariance of affine frame operators*, Proc. Amer. Math. Soc., **131**(5) (2003), 1527–1538.

[Dau92] I. Daubechies, *Ten Lectures on Wavelets*, SIAM, Philadelphia, 1992.

[dBVR94a] C. de Boor, R. De Vore, and A. Ron, *Approximation from shift-invariant subspaces of $L_2(\mathbf{R}^d)$*, Trans. Amer. Math. Soc., **341**(2) (1994), 787—806.

[dBVR94b] C. de Boor, R. De Vore, and A. Ron, *The structure of finitely generated shift-invariant subspaces of $L_2(\mathbb{R}^d)$*, J. Funct. Anal., **119**(1) (1994), 37–78.

[Grö01] K. Gröchenig, *Foundations of Time-Frequency Analysis*, Birkhäuser, Boston, 2001.

[Hel64] H. Helson, *Lectures on Invariant Subspaces*, Academic Press, New York–London, 1964.

[HW96] E. Hernández and G. Weiss, *A First Course on Wavelets*, CRC Press, Boca Raton, FL, 1996.

[HL01] J. A. Hogan, and J. Lakey, *Sampling and aliasing without translation-invariance*, Proc. Fourth Int. Conf. on Sampling Theory and Applications (SampTA'01), Orlando, FL, 2001, 61–66.

[Mal89] S. G. Mallat, *Multiresolution approximations and wavelet orthonormal bases of $L^2(\mathbf{R})$*, Trans. Amer. Math. Soc., **315**(1) (1989), 69–87.

[Mal98] S. Mallat, *A Wavelet Tour of Signal Processing*, Academic Press, San Diego, CA, 1998.

[Web00] E. Weber, *On the translation invariance of wavelet subspaces*, J. Fourier Anal. Appl., **6**(5) (2000), 551–558.

(A. Aldroubi) Department of Mathematics, Vanderbilt University, Nashville, Tennessee 37240-0001 USA

E-mail address: `aldroubi@math.vanderbilt.edu`

(C. Cabrelli) Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Ciudad Universitaria, Pabellón I, 1428 Buenos Aires, Argentina and CONICET, Consejo Nacional de Investigaciones Científicas y Técnicas, Argentina

E-mail address: `cabrelli@dm.uba.ar`

(C. Heil) School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332-0160 USA

E-mail address: `heil@math.gatech.edu`

(K. Kornelson) Department of Mathematics, Grinnell College, Grinnell, Iowa 50112 USA

E-mail address: `kornelso@math.grinnell.edu`

(U. Molter) Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Ciudad Universitaria, Pabellón I, 1428 Buenos Aires, Argentina and CONICET, Consejo Nacional de Investigaciones Científicas y Técnicas, Argentina

E-mail address: `umolter@dm.uba.ar`