Sobolev Regularity for Refinement Equations via Ergodic Theory

Christopher Heil and David Colella

Abstract. The refinement equation $f(x) = \sum_{k=0}^{N} c_k\, f(2x-k)$ plays a key role in wavelet theory and in subdivision schemes in approximation theory. This paper explores the relationship of the refinement equation to the mapping $\tau(x) = 2x \bmod 1$. A simple necessary condition for the existence of an integrable solution to the refinement equation is obtained by considering the periodic cycles of τ. Another simple necessary condition for the existence of an integrable solution satisfying $(1+|\gamma|^2)^{s/2}\, \hat{f}(\gamma) \in L^p(\mathbf{R})$ is obtained by considering the ergodic property of τ. In particular, for $p = 2$ this is a necessary condition for f to lie in the Sobolev space H^s.

§1. Introduction

A *refinement equation*, *dilation equation*, or *two-scale difference equation* is a functional equation of the form

$$f(x) = \sum_{k=0}^{N} c_k\, f(2x-k). \tag{1}$$

Such equations play a key role in wavelet theory [4] and in subdivision schemes in approximation theory [1]. A nonzero solution f is called a *scaling function* or a *refinable function*.

A major problem is the determination of properties of f from the values of $c_0, \dots, c_N$. This has been approached in a wide variety of ways. We cite a few examples, but note that this list is *far from complete*. Necessary and/or sufficient conditions for Hölder continuity of f can be found in [5],

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[3], [10], [14]. Sobolev regularity is explored in [2], [7], [11]. L^p conditions are given in [9], [6], [14]. Besov characterizations appear in [12].

Our purpose in this note is to explore the relationship of the refinement equation to the celebrated mapping $\tau(x) = 2x \bmod 1$. The periodic cycles of τ have been applied to the refinement equation in [2], [12], and elsewhere. The ergodic property of τ has been used in [1]. We will briefly illustrate these techniques through results which we obtained independently, but which might possibly be inferred from other references.

In Section 2 we give a simple necessary but not sufficient condition for the existence of an integrable solution to the refinement equation, based on the periodic orbits of τ. In Section 3 we use the ergodic property of τ to derive a necessary but still not sufficient condition for the existence of an integrable solution f satisfying $(1+|\gamma|^2)^{s/2}\,\hat{f}(\gamma) \in L^p(\mathbf{R})$. In particular, for $p = 2$ this is a necessary condition for f to lie in the Sobolev space

$$H^s = \{f \in \mathcal{S}'(\mathbf{R}) : (1+|\gamma|^2)^{s/2}\,\hat{f}(\gamma) \in L^2(\mathbf{R})\}.$$

Here $\hat{f}(\gamma) = \int f(x)\, e^{-2\pi i\gamma x}\, dx$ is the Fourier transform of f, $\mathcal{S}(\mathbf{R})$ is the Schwartz class of infinitely differentiable functions decaying rapidly at infinity, and $\mathcal{S}'(\mathbf{R})$ is its topological dual, the space of tempered distributions.

We begin by reviewing some basic facts. The seminal work of Daubechies and Lagarias [5] established that if an integrable solution f to the refinement equation (1) exists then it is unique up to scalar multiples. Moreover, integrable solutions can exist only when $\Delta = \frac{1}{2}\sum c_k = 2^n$ for some integer $n \geq 0$. A solution with $n > 0$ is the nth derivative of a solution for the refinement equation determined by the coefficients $\{2^{-n}c_k\}$. Hence, refinement equations satisfying $\Delta = 1$ are fundamental, and we will impose this condition throughout.

If we define the *symbol* of the refinement equation to be the trigonometric polynomial $M(\gamma) = \frac{1}{2}\sum c_k\, e^{-2\pi i k\gamma}$, then equation (1) is equivalent to

$$\hat{f}(\gamma) = M(\gamma/2)\,\hat{f}(\gamma/2). \tag{2}$$

Since $\Delta = M(0) = \frac{1}{2}\sum c_k = 1$, the infinite product $P(\gamma) = \prod_{j=1}^{\infty} M(2^{-j}\gamma)$ converges uniformly on compact sets to a continuous function with polynomial growth at infinity. Considering (2), any integrable solution f to (1) must satisfy $\hat{f}(\gamma) = P(\gamma)\,\hat{f}(0)$, explaining why integrable solutions to (1) are unique up to scalar multiples. Note, however, that even though an integrable solution f may not exist, there will exist a solution in the sense of distributions: $\hat{f}(\gamma) = P(\gamma)$ defines a tempered distribution f which satisfies (2).

Our results are based on the following iterated version of (2):

$$\hat{f}(2^n\gamma) = P_n(\gamma)\,\hat{f}(\gamma), \tag{3}$$

where $P_n(\gamma) = \prod_{j=1}^{n-1} M(2^j\gamma)$.

In addition to the fundamental hypothesis $\Delta = 1$, additional *sum rule* or *Strang–Fix* conditions are often imposed on the coefficients. These have the form

$$\sum_k (-1)^k \, k^j \, c_k = 0 \qquad \text{for } j = 0, \ldots, L. \tag{4}$$

Equivalently, $M^{(j)}(1/2) = 0$ for $j = 0, \ldots, L$. This implies that the integer translates of the scaling function f can exactly reproduce the polynomials $1, x, \ldots, x^L$. From (4), M factors as

$$M(\gamma) = \left(\frac{1 + e^{2\pi i\gamma}}{2}\right)^{L+1} \tilde{M}(\gamma), \tag{5}$$

with $\tilde{M}(1/2) \neq 0$ if L is maximal.

§2. Periodic Cycles

By considering equation (3) applied to the periodic cycles of $\tau(\gamma) = 2\gamma \bmod 1$, we obtain the following necessary condition for the existence of an integrable scaling function. This result was announced in [8] without proof.

Theorem 1. *Assume $c_0, \ldots, c_N$ satisfy $\Delta = 1$. Let $\{\gamma_0, \ldots, \gamma_{n-1}\}$ be a periodic cycle in $[0, 1)$ with $n > 1$, meaning that $\{\gamma_0, \ldots, \gamma_{n-1}\}$ is invariant under τ. If:*

(a) $P(\gamma_0) \neq 0$, *and*

(b) $|M(\gamma_0) \cdots M(\gamma_{n-1})| \geq 1$,

then there is no integrable solution to the refinement equation (1). *Hypothesis* (a) *is equivalent to the following:*

(a') $M(\gamma_0/2^j) \neq 0$ *for all* $j > 0$.

Proof: Assume (a) and (b) hold, and set $R = M(\gamma_0) \cdots M(\gamma_{n-1})$. Let $\gamma_0, \ldots, \gamma_{n-1}$ be indexed so that $\gamma_{k+1} = 2\gamma_k \bmod 1$ for $k = 0, \ldots, n-2$ and $\gamma_0 = 2\gamma_{n-1} \bmod 1$. Then $M(2^{kn+r}\gamma_0) = M(2^r\gamma_0)$ for all k, $r \geq 0$, so $P(2^{kn}\gamma_0) = \big(M(2^{n-1}\gamma_0) \cdots M(\gamma_0)\big)^k \, P(\gamma_0) = R^k \, P(\gamma_0)$. Since $|R| \geq 1$, we conclude that P does not vanish at infinity, and therefore cannot be the Fourier transform of any integrable function.

To see that (a) and (a') are equivalent, assume that $P(\gamma_0) = 0$ but that $M(2^{-j}\gamma_0) \neq 0$ for all $j > 0$. Then the fact that

$$P(\gamma_0) = M(2^{-1}\gamma_0) \cdots M(2^{-n}\gamma_0) \, P(2^{-n}\gamma_0),$$

implies that $P(2^{-n}\gamma_0) = 0$ for all $n > 0$. Yet $P(0) = 1$ since $M(0) = 1$, contradicting the fact that P is continuous. ■

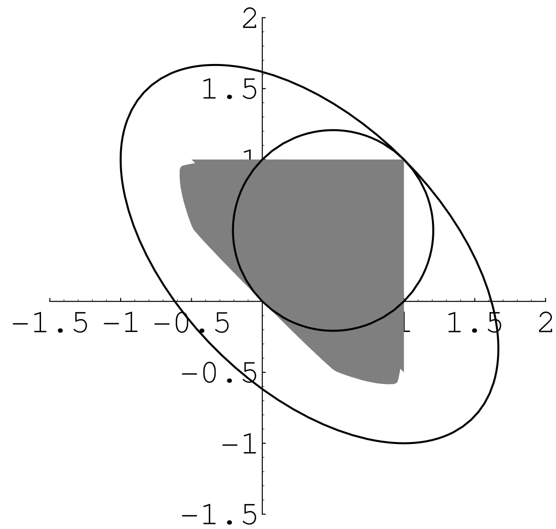

Figure 1. The (c_0, c_3)-plane.

Example 1. Consider the particular case of four-coefficient refinement equations. We impose the condition $\Delta = 1$, or $c_0 + c_1 + c_2 + c_3 = 2$, and a single, zero-order sum rule, $c_0 + c_2 = c_1 + c_3$. Then $c_1 = 1 - c_3$ and $c_2 = 1 - c_0$, so this family has two free parameters. Choosing these as c_0 and c_3, we identify the class of all such refinement equations with the (c_0, c_3) plane. This plane is shown in Figure 1.

Now consider the short invariant cycle $\{1/3, 2/3\}$ in $[0,1)$. We compute

$$|M(1/3)\,M(2/3)| = \frac{1}{4}\left(1 + 3c_0c_3 - 3c_0(1-c_0) - 3c_3(1-c_3)\right).$$

Set

$$\begin{aligned} E &= \{(c_0,c_3) : |M(1/3)\,M(2/3)| \geq 1\} \\ &= \{(c_0,c_3) : c_0c_3 - c_0(1-c_0) - c_3(1-c_3) \geq 1\}. \end{aligned}$$

Then E is the boundary and exterior of the ellipse shown in Figure 1. We apply Theorem 1 to show that there are no integrable solutions of the refinement equation corresponding to the point (c_0, c_3) if $(c_0, c_3) \in E \setminus \{(1,1)\}$. Note that there is an integrable solution for $(c_0, c_3) = (1,1)$, namely $f = \chi_{[0,3]}$. This is the celebrated "stretched Haar" example.

A simple computation yields the fact that M has a zero if and only if $c_0 = c_3 > 1/4$. In this case, M has a unique zero in the interval $[0, 1/2)$, at the point $\zeta = \frac{1}{2\pi} \arccos\left(\frac{2c_0-1}{2c_0}\right)$. Theorem 1 therefore implies that there are no integrable solutions to (1) if $(c_0, c_3) \in E \backslash S$, where S is the countable collection of points

$$\begin{aligned} S &= \{(c_0,c_3) : c_0 = c_3 = \frac{1}{2 - 2\cos(2^{-j}\pi/3)},\ j \geq 0\} \\ &= \{(1,1),\ (2+\sqrt{3}, 2+\sqrt{3}), \ldots\}. \end{aligned}$$

Label the points in S as $(c_0^{(j)}, c_3^{(j)})$ for $j = 0, 1, 2, \ldots$, and let M_j denote the corresponding symbols The points $(c_0^{(j)}, c_3^{(j)})$ lie increasingly far from the origin as j increases. Theorem 1 fails to apply to the point $(c_0^{(j)}, c_3^{(j)})$ because $M_j(2^{-(j+1)}\gamma_0) = 0$, where $\gamma_0 = 1/3$.

It remains only to show that there are no integrable solutions corresponding to the points $(c_0^{(j)}, c_3^{(j)})$ with $j > 0$. Consider the length-3 cycle $\{1/7, 2/7, 4/7\}$. It is easy to check that $|M_j(1/7)\, M_j(2/7)\, M_j(4/7)| \geq 1$ for each $(c_0^{(j)}, c_3^{(j)})$ with $j > 0$. Moreover, with $\gamma_0 = 1/7$, we already know that $M_j(2^{-k}\gamma_0) \neq 0$ for every $k > 0$ since $M_j(\zeta) = 0$ in the interval $[0, 1/2)$ only when $\zeta = (1/3)\, 2^{-(j+1)}$. Therefore we can apply Theorem 1 using this length-3 cycle to any point $(c_0^{(j)}, c_3^{(j)})$ with $j > 0$, and conclude that no integrable scaling functions exist for of these points.

Thus, there are no integrable solutions to the refinement equation by any point (c_0, c_3) on or outside the ellipse shown in Figure 1, with the single exception of the point $(1, 1)$. For comparison, the shaded region in Figure 1 is a numerical approximation of the set of points (c_0, c_3) for which a continuous integrable solution exists, and the circle is the set of points (c_0, c_3) such that the integer translates $f(x - k)$ of f are orthogonal. Such scaling functions with orthogonal translates are used in the construction of *orthogonal wavelets*, *i.e.*, functions g such that $\{2^{n/2} g(2^n x - k)\}_{n,k \in \mathbf{Z}}$ forms an orthonormal basis for $L^2(\mathbf{R})$. The shaded region was computed using the joint spectral radius approach of [5], [3].

§3. Ergodic Theory

In this section we use the ergodic property of τ to obtain a necessary condition for the existence of an integrable scaling function f such that $(1 + |\gamma|^2)^{s/2}\, \hat{f}(\gamma) \in L^p(\mathbf{R})$. We note that Villemoes [12] has an interesting viewpoint, in which our Theorem 2 for the Sobolev case $p = 2$ is obtained as a limit of invariant cycle results via a Riemann sum.

Theorem 2. *Assume $c_0, \ldots, c_N$ satisfy $\Delta = 1$, and set*

$$\alpha = \int_0^1 \log_2 |M(\gamma)|\, d\gamma. \tag{6}$$

If there exists an integrable solution f to the refinement equation (1) *such that $(1 + |\gamma|^2)^{s/2}\, \hat{f}(\gamma) \in L^p(\mathbf{R})$, then $s \leq -\frac{1}{p} - \alpha$.*

Proof: Assume that f is integrable and that $(1 + |\gamma|^2)^{s/2}\, \hat{f}(\gamma) \in L^p(\mathbf{R})$. Then since $\hat{f}$ is continuous and $\hat{f}(0) \neq 0$, we can find a set $U \subset [0, 1)$ with positive measure such that $m = \inf_{\gamma \in U} |\hat{f}(\gamma)| > 0$. Define $L(\gamma) =$

$\log_2 |M(\gamma)|$. Because M is a trigonometric polynomial of degree $N+1$, its zeros have order at most $N+1$. Therefore L is integrable on $[0,1)$, so α is a finite number. Also, L is 1-periodic, and $\log_2 |P_n(2^n\gamma)| = \sum_{j=0}^{n-1} L(2^j\gamma)$.

Now, τ is an ergodic mapping of $[0,1)$ onto itself, so the Birkhoff Ergodic Theorem [13, Theorem 1.14] implies that

$$\lim_{n\to\infty} \log_2 |P_n(2^n\gamma)|^{1/n} = \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} L(2^j\gamma) = \alpha \qquad \text{for a.e. } \gamma. \tag{7}$$

In particular, (7) holds for a.e. $\gamma \in U$. By Egoroff's Theorem, we can find a set $E \subset U \subset [0,1)$ with positive measure such that the convergence in (7) is uniform for $\gamma \in E$. That is, for each $\varepsilon > 0$ there is an $n_0 > 0$ such that $\log_2 |(|P_n(2^n\gamma)|^{1/n}) - \alpha| < \varepsilon$ for all $n \geq n_0$ and $\gamma \in E$. Therefore $\log_2 |P_n(2^n\gamma)|^{1/n} \geq \alpha - \varepsilon$, or $|P_n(2^n\gamma)| \geq 2^{n(\alpha-\varepsilon)}$, for $n \geq n_0$ and $\gamma \in E$. Since $E \subset U$, we conclude that

$$|\hat{f}(2^n\gamma)| = |P_n(2^n\gamma)|\,|\hat{f}(\gamma)| \geq m\,2^{n(\alpha-\varepsilon)} \qquad \text{for } n \geq n_0 \text{ and } \gamma \in E.$$

Now, there must be some $k > 0$ such that $F = E \cap [2^{-k}, 2^{-k+1}]$ has positive measure. Then

$$\begin{aligned}
\infty &> \int_{-\infty}^{+\infty} |\hat{f}(\gamma)|^p\,(1+|\gamma|^2)^{ps/2}\,d\gamma \\
&\geq \sum_{n=n_0}^{\infty} \int_{2^{n-k}}^{2^{n-k+1}} |\hat{f}(\gamma)|^p\,(1+|\gamma|^2)^{ps/2}\,d\gamma \\
&= \sum_{n=n_0}^{\infty} 2^n \int_{2^{-k}}^{2^{-k+1}} |\hat{f}(2^n\gamma)|^p\,(1+|2^n\gamma|^2)^{ps/2}\,d\gamma \\
&\geq \sum_{n=n_0}^{\infty} 2^n \int_F |\hat{f}(2^n\gamma)|^p\,(1+|2^n\gamma|^2)^{ps/2}\,d\gamma \\
&\geq \sum_{n=n_0}^{\infty} 2^n \int_F (m\,2^{n(\alpha-\varepsilon)})^p\,(2^n\gamma)^{ps}\,d\gamma \\
&\geq \sum_{n=n_0}^{\infty} 2^n\,|F|\,m^p\,2^{np(\alpha-\varepsilon)}\,2^{nps}\,2^{-kps} \\
&= 2^{-kps}\,|F|\,m^p \sum_{n=n_0}^{\infty} 2^{n(p(\alpha-\varepsilon+s)+1)}.
\end{aligned}$$

Therefore, we must have $p(\alpha+s-\varepsilon)+1 < 0$. Since this must be true for every $\varepsilon > 0$, we have $p(\alpha+s)+1 \leq 0$, or $s \leq -\frac{1}{p} - \alpha$. ■

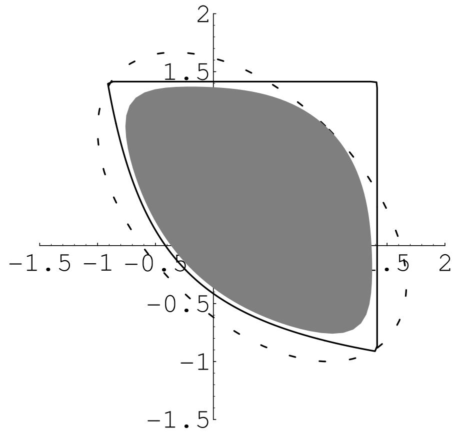

Figure 2. Regions for L^2 solutions in the (c_0, c_3)-plane.

Recall that if the coefficients $c_0, \ldots, c_N$ satisfy the sum rules in (4) then M factors as in (5). Since

$$\int_0^1 \log_2 \left| \frac{1 + e^{2\pi i \gamma}}{2} \right| \, d\gamma = \int_0^1 \log_2 |\cos \pi\gamma| \, d\gamma = -1,$$

the value of α in (6) is then $\alpha = -L + \int_0^1 \log_2 |\tilde{M}(\gamma)| \, d\gamma$. Thus, increasing the number of sum rules "tends to" increase the smoothness of the scaling function, although this effect can be offset by the behavior of the remainder $\tilde{M}$.

Example 2. Return to the situation of four-coefficient refinement equations discussed in Example 1. Here $L = 1$. Consider the problem of determining L^2 solutions to the refinement equation. Since scaling functions are compactly supported, we know that any L^2 solution will be integrable. Now, $L^2 = H^0$, so we apply Theorem 2 with $p = 2$ and $s = 0$. An L^2 solution therefore requires $0 \le -(1/2) - \alpha$, or

$$\int_0^1 \log_2 |\tilde{M}(\gamma)| \, d\gamma \le \frac{1}{2}. \tag{8}$$

The interior of the solid curve in Figure 2 is an approximation to the set of coefficients which satisfy (8). We note that an *exact* calculation of L^2 solutions is possible [7], [9], [11], and that L^2 solutions occur precisely in the shaded region shown in Figure 2. The ellipse of Figure 1 is also shown (dashed) for comparison.

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Christopher Heil
School of Mathematics
Georgia Institute of Technology
Atlanta, Georgia 30332-0160
heil@math.gatech.edu

David Colella
The MITRE Corporation
McLean, Virginia 22102
colella@mitre.org

Authors' Note*

The techniques used in this paper are actually quite similar to some of those employed by Cohen in his paper [Coh90], which was published in French in 1990. The authors were unaware of this fact until the first author edited the volume [HW06], which includes an English translation of Cohen's article. In particular, Theorem 2 of the present article can be compared to Proposition 2 of the translation of Cohen's article in [HW06].

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* This authors' note is not included in the published version of this paper.