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Duals of Weighted Exponential Systems

Gang Joon Yoon · Christopher Heil

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Abstract The paper considers the basis and frame properties of the system of weighted exponentials $\mathcal{E}(g, \mathbb{Z}\backslash F) = \{e^{2\pi inx}g(x)\}_{n\in\mathbb{Z}\backslash F}$ in $L^2(\mathbb{T})$, where $g \in L^2(\mathbb{T})\backslash\{0\}$ and $F \subset \mathbb{Z}$. It is shown that many of the frame properties of $\mathcal{E}(g, \mathbb{Z}\backslash F)$ are affected by the cardinalities of F and the behavior of the zeros of g.

Keywords Frames · Gabor systems · overcompleteness · Riesz bases · weighted exponentials

Mathematics Subject Classification (2000) 42C15 · 46C99

1 Introduction

Given a function $g \in L^2(\mathbb{T})$, where $\mathbb{T} = [0, 1]$, and given a set of frequencies $S \subseteq \mathbb{Z}$, the system of *weighted exponentials generated by* g is the collection $\mathcal{E}(g, S) = \{e^{2\pi inx}g(x)\}_{n\in S}$. If $g = 1$ and $S = \mathbb{Z}$ then this is the usual trigonometric system, which is an orthonormal basis for $L^2(\mathbb{T})$. Systems of weighted exponentials play an important role in many areas, such as sampling theory. For example, if we embed $\mathcal{E}(g, S)$ into $L^2(\mathbb{R})$ by extending functions by zero outside of $\mathbb{T}$, then we can create sampling theorems based on the properties

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G.J. Yoon
National Institute of Mathematical Sciences, Daejeon 305-811, Korea
E-mail: ykj@nims.re.kr

C. Heil
School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160 USA
E-mail: heil@math.gatech.edu

of $\{\varphi_n\}_{n\in S}$ where φ_n is the inverse Fourier transform of $e^{2\pi inx}g(x)\,\chi_{\mathbb{T}}(x)$. For details on sampling and nonharmonic Fourier series (where S is allowed to contain noninteger frequencies), we refer to the text by Young [25], and for connections to the theory of reconstruction from irregular samples see [1], [13].

In this paper we are interested in the completeness or overcompleteness properties of $\mathcal{E}(g,S)$, and in the question of whether $\mathcal{E}(g,S)$ can be a *frame* for $L^2(\mathbb{T})$. In general, a complete system need not be a frame. Frames have the advantage that they provide unconditionally convergent, basis-like representations, though these representations are generally not unique. Frames appear in a wide range of areas and applications, including sampling theory [1], sparse representations [7], operator theory [14], wavelet theory [10], wireless communications [23], data transmission with erasures [12], signal processing [4], and quantum computing [11].

One of the earliest results on the properties of weighted exponentials is due to Boas and Pollard [6]. In their classic paper on multiplicative completion, Boas and Pollard showed if F is any a finite subset of $\mathbb{Z}$, then there exists a function g such that $\mathcal{E}(g,\mathbb{Z}\backslash F)=\{e^{2\pi inx}g(x)\}_{n\notin F}$ is complete in $L^2(\mathbb{T})$. There are a surprising number of equivalent reformulations and interesting related results, for which we refer to the paper by Kazarian and Zink [21] and the references contained therein.

Completeness is a fairly weak condition; in many situations we would like to know if we have a Schauder basis or other basis-like properties, such as being a frame. Kazarian has shown, for example, that if finitely many elements are removed from the trigonometric system, then the resulting set $\{e^{2\pi inx}\}_{n\notin F}$ cannot be a Schauder basis for $L^p(\mu)$, where μ is a bounded Radon measure [18], [19], and in [20] he studied systems $\{e^{2\pi inx}\}_{n\notin F}$ in $L^p(\mu)$ where F is a finite sequence of consecutive integers. In this paper we address the basis and frame properties of sets of weighted exponentials of the form $\{e^{2\pi inx}g(x)\}_{n\notin F}$ in $L^2(\mathbb{T})$ where F is an arbitrary subset of $\mathbb{Z}$, with special interest in the case $g(x)=x^N$.

To motivate our results, consider the full *lattice system of weighted exponentials* $\mathcal{E}(g,\mathbb{Z})=\{e^{2\pi inx}g(x)\}_{n\in\mathbb{Z}}$. The basis and frame properties of this system in $L^2(\mathbb{T})$ are summarized in the following theorem (terminology is defined precisely in Section 2.1). For a complete proof of this result, see Theorem 10.10 in the text [15].

Theorem 1 *Let $Z_g=\{x\in[0,1]:g(x)=0\}$ denote the zero set of $g\in L^2(\mathbb{T})$.*

(a) *$\mathcal{E}(g,\mathbb{Z})$ is complete in $L^2(\mathbb{T})$ if and only if $g\neq 0$ a.e.*

(b) *$\mathcal{E}(g,\mathbb{Z})$ is minimal in $L^2(\mathbb{T})$ if and only if $1/g\in L^2(\mathbb{T})$. Moreover, in this case it is exact and the biorthogonal system is $\mathcal{E}(\widetilde{g},\mathbb{Z})$ where $\widetilde{g}=1/\overline{g}$.*

(c) *With respect to the ordering $\mathbb{Z}=\{0,-1,1,-2,2,\dots\}$, $\mathcal{E}(g,\mathbb{Z})$ is a Schauder basis for $L^2(\mathbb{T})$ if and only if $|g|^2$ belongs to the Muckenhaupt weight class $\mathcal{A}_2(\mathbb{T})$.*

(d) *$\mathcal{E}(g,\mathbb{Z})$ is a Bessel sequence in $L^2(\mathbb{T})$ if and only if $g\in L^\infty[0,1]$. Moreover, in this case $|g(x)|^2\le B$ a.e. where B is a Bessel bound.*

(e) *$\mathcal{E}(g,\mathbb{Z})$ is a frame sequence in $L^2(\mathbb{T})$ if and only if there exist A, $B > 0$ such that $A \le |g(x)|^2 \le B$ for a.e. $x \notin Z_g$. In this case, the closed span of $\mathcal{E}(g,\mathbb{Z})$ is $H_g = \{f \in L^2(\mathbb{T}) : f = 0 \text{ a.e. on } Z_g\}$, and A, B are frame bounds for $\mathcal{E}(g,\mathbb{Z})$ as a frame for H_g.*

(f) *$\mathcal{E}(g,\mathbb{Z})$ is an unconditional basis for $L^2(\mathbb{T})$ if and only if there exist A, $B > 0$ such that $A \le |g(x)|^2 \le B$ for a.e. x, and in this case it is a Riesz basis for $L^2(\mathbb{T})$.*

(g) *$\mathcal{E}(g,\mathbb{Z})$ is an orthonormal basis for $L^2(\mathbb{T})$ if and only if $|g(x)| = 1$ for a.e. x.*

A much more difficult question is the basis and frame properties of the "irregular" system $\mathcal{E}(g,\Lambda) = \{e^{2\pi i\lambda x} g(x)\}_{\lambda\in\Lambda}$ when Λ is an arbitrary countable sequence in $\mathbb{R}$. For background and references on this subject, we refer to the paper [17]. Even more difficult are the cases of "irregular" Gabor systems $\{e^{2\pi i\beta x} g(x-\alpha)\}_{(\alpha,\beta)\in\Gamma}$ and wavelet systems $\{a^{1/2}\psi(ax-b)\}_{(a,b)\in\Gamma}$ in $L^2(\mathbb{R})$. Typically, in each of these cases necessary conditions for the system to be a frame can be formulated in terms of the Beurling densities of the index set. For example, weighted exponentials and Gabor systems both exhibit a Nyquist density cutoff density [2], [3], [22], whereas the situation for wavelet systems is much more subtle [5], [16], [24].

Our focus in this paper is the case where the index set Λ is a subset of the integers. For motivation, set

$$e_n(x) \;=\; e^{2\pi i n x}$$

and consider the weighting function $g(x) = x$. The system $\{x e_n(x)\}_{n\ne 0}$ is exact in $L^2(\mathbb{T})$, and therefore has a unique biorthogonal dual system, but because the index set $\mathbb{Z}\backslash\{0\}$ is not a lattice, Theorem 1 does not apply, and in fact the dual system is not a system of weighted exponentials. Instead, we have the following result.

Theorem 2 *The sequence $\{x e_n(x)\}_{n\ne 0}$ is exact in $L^2(\mathbb{T})$, and its biorthogonal sequence is $\{\widetilde{e}_n\}_{n\ne 0} = \left\{\frac{e^{2\pi i n x}-1}{x}\right\}_{n\ne 0}$. This dual system is exact in $L^2(\mathbb{T})$ but not bounded above in norm, and therefore $\{x e_n(x)\}_{n\ne 0}$ is not a Schauder basis for $L^2(\mathbb{T})$.*

Proof The biorthogonality follows from a direct calculation. To show completeness, suppose that $f \in L^2(\mathbb{T})$ satisfies $\langle f(x), x e_n(x)\rangle = 0$ for every $n \ne 0$. Then the function $x f(x)$ belongs to $L^2(\mathbb{T})$ and is orthogonal to e_n for every $n \ne 0$, so we must have $x f(x) = c$ a.e. where c is a constant. If $c \ne 0$ then $f(x) = c/x \notin L^2(\mathbb{T})$, which is a contradiction. Therefore $c = 0$, so $f = 0$ a.e. and $\{x e_n(x)\}_{n\ne 0}$ is complete. A direct computation shows that

$$\|\widetilde{e}_n\|_2^2 \;=\; 4\pi n \int_0^{2\pi n} \frac{\sin x}{x}\, dx \;<\; \infty,$$

so $\widetilde{e}_n \in L^2(\mathbb{T})$. Since $\int_0^{2\pi n} \frac{\sin x}{x}\, dx \to \frac{\pi}{2}$ as $n \to \infty$, we see that $\{\widetilde{e}_n\}_{n\ne 0}$ is not bounded above in norm. However, $\{x e_n(x)\}_{n\ne 0}$ is bounded above and below in norm, so if it was a Schauder basis then its biorthogonal system must would also be bounded above and below in norm (see [15, Sect. 5.6]). Therefore $\{xe_n(x)\}_{n\neq 0}$ cannot be a Schauder basis for $L^2(\mathbb{T})$.

To complete the proof, it remains to show that $\{\widetilde{e}_n\}_{n\neq 0}$ is complete in $L^2(\mathbb{T})$. Suppose that $h \in L^2(\mathbb{T})$ satisfies $\langle h, \widetilde{e}_n\rangle = 0$ for $n \neq 0$. If we define $\widetilde{e}_0(t) = \frac{e^{-2\pi i 0\cdot t}-1}{t} = 0$, then $\langle h, \widetilde{e}_n\rangle = 0$ for all $n \in \mathbb{Z}$. Set $g(t) = h(t)\,\frac{e^{2\pi i t}-1}{t} \in L^2(\mathbb{T})$. Then for every $m \in \mathbb{Z}$ we have

$$\begin{aligned}\langle g, e_m\rangle &= \int_0^1 g(t)\, e^{-2\pi i m t}\, dt \\ &= \int_0^1 h(t)\, \frac{e^{-2\pi i (m-1)t} - 1 + 1 - e^{-2\pi i m t}}{t}\, dt \\ &= \langle h, \widetilde{e}_{m-1}\rangle - \langle h, \widetilde{e}_m\rangle = 0.\end{aligned}$$

Therefore $g = 0$ a.e., which implies $h = 0$ a.e. □

The main results obtained in this paper can be summarized as follows. For $g(x) = x^N$, we will show in Theorem 3 that if $\{x^N e_n(x)\}_{n\neq F}$ is complete then $|F| \leq N$, and if $\{x^N e_n(x)\}_{n\neq F}$ is minimal then $|F| \geq N$. Theorem 4 proves the equivalence that $\{x^N e_n(x)\}_{n\neq F}$ is exact in $L^2(\mathbb{T})$ if and only if $|F| = N$. With minor appropriate changes to the proofs, the same results hold for the weights $(x-a)^N$. We focus in this paper on integer powers of x, but an interesting topic for future research is to consider the extent to which the results carry over, or fail to carry over, to weights x^α with α noninteger.

Then we consider the general case of $g \in L^2(\mathbb{T})$ and F finite. We define a terminology of L^2-zeros that measures the degree of singularity of g in L^2-norm (Definition 1), and investigate the relationship between minimality or completeness of $\mathcal{E}(g, \mathbb{Z}\backslash F)$ and properties of g and F. We show in Theorem 8 that $\mathcal{E}(g, \mathbb{Z}\backslash F)$ is exact in $L^2(\mathbb{T})$ if and only if there exist a finite number of points $\alpha_1, \dots, \alpha_k$ and positive integers $n_1, \dots, n_k$ such that $n_1 + \cdots + n_k \geq |F|$ and g has an L^2-zero at α_i of order n_i for each $i = 1, \dots, k$. In this case, we show that $\mathcal{E}(g, \mathbb{Z}\backslash F)$ cannot be a frame for $L^2(\mathbb{T})$.

Finally, we prove in Theorem 9 that $\mathcal{E}(g, \mathbb{Z}\backslash F)$ can never be a frame for $L^2(\mathbb{T})$ when F is nonempty by showing the nonexistence of a lower frame bound.

2 Preliminaries

2.1 Terminology

We use standard notations for frames, Riesz bases, and related concepts, as found in the texts [9], [10], [15], [25], or the research-tutorial [8]. We outline some particular terminology and facts that we will need in this section.

Let $\{f_i\}_{i\in\mathbb{N}}$ be a sequence in a Hilbert space H. We say $\{f_i\}_{i\in\mathbb{N}}$ is *complete* if its finite linear span is dense in H. It is *minimal* if there exists a sequence

$\{\widetilde{f}_i\}_{i\in\mathbb{N}}$ in H that is biorthogonal to $\{f_i\}_{i\in\mathbb{N}}$, i.e., $\langle f_i, \widetilde{f}_j \rangle = \delta_{ij}$ for $i,\, j \in \mathbb{N}$. Equivalently, $\{f_i\}_{i\in\mathbb{N}}$ is minimal if $f_j \notin \overline{\mathrm{span}}\{f_i\}_{i\neq j}$ for each $j \in \mathbb{N}$. A sequence that is both minimal and complete is called *exact*. In this case the biorthogonal sequence is unique.

We say $\{f_i\}_{i\in\mathbb{N}}$ is a *Schauder basis* if for each $f \in H$ there exist unique scalars c_i such that $f = \sum_{i=1}^{\infty} c_i f_i$. Every Schauder basis is exact, and the biorthogonal sequence $\{\widetilde{f}_i\}_{i\in\mathbb{N}}$ is also a basis, called the *dual Schauder basis*. Further, we have

$$\forall\, f \in H, \quad f \;=\; \sum_{i=1}^{\infty} \langle f, \widetilde{f}_i \rangle\, f_i \;=\; \sum_{i=1}^{\infty} \langle f, f_i\rangle\, \widetilde{f}_i, \tag{1}$$

with uniqueness of the scalars in these expansions. If $0 < \inf \|f_i\| \le \sup \|f_i\| < \infty$, then $0 < \inf \|\widetilde{f}_i\| \le \sup \|\widetilde{f}_i\| < \infty$.

A *Riesz basis* is the image of an orthonormal basis for H under a continuously invertible linear mapping of H onto itself.

We say $\{f_i\}_{i\in\mathbb{N}}$ is a *frame* for H if there exist constants $A,\, B > 0$, called *frame bounds*, such that

$$\forall\, f \in H, \quad A\,\|f\|^2 \;\le\; \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \;\le\; B\,\|f\|^2. \tag{2}$$

All Riesz bases are frames, but not conversely.

If $\{f_i\}_{i\in\mathbb{N}}$ satisfies at least the second inequality in (2) then we say that $\{f_i\}_{i\in\mathbb{N}}$ is a *Bessel sequence* or that it *possesses an upper frame bound*, and we call B a *Bessel bound*. Likewise if at least the first inequality in (2) is satisfied then we say that $\{f_i\}_{i\in\mathbb{N}}$ *possesses a lower frame bound*.

A sequence $\{f_i\}_{i\in\mathbb{N}}$ is Bessel if and only if the *analysis operator* $Cf = \{\langle f, f_i\rangle\}_{i\in\mathbb{N}}$ is a bounded mapping $C\colon H \to \ell^2$. If $\{f_i\}_{i\in\mathbb{N}}$ is a frame then the *frame operator* $Sf = C^*Cf = \sum \langle f, f_i\rangle\, f_i$ is a bounded, positive definite, invertible map of H onto itself. Every frame $\mathcal{F} = \{f_i\}_{i\in\mathbb{N}}$ has a *canonical dual frame* $\widetilde{\mathcal{F}} = \{\widetilde{f}_i\}_{i\in\mathbb{N}}$ given by $\widetilde{f}_i = S^{-1} f_i$ where S is the frame operator. In particular,

$$\forall\, f \in H, \quad f \;=\; \sum_{i=1}^{\infty} \langle f, \widetilde{f}_i \rangle\, f_i \;=\; \sum_{i=1}^{\infty} \langle f, f_i\rangle\, \widetilde{f}_i, \tag{3}$$

and furthermore the series in (3) converges unconditionally for every f (so any countable index set can be used to index a frame). In general, for a frame the coefficients in (3) need not be unique. In fact, uniqueness holds if and only if $\mathcal{F} = \{f_i\}_{i\in\mathbb{N}}$ is a Riesz basis.

2.2 Lemmas

We will need the following lemmas, which follow by using Taylor series and properties of Vandermonde determinants.

Lemma 1 *Given $N \ge 1$, $m \in \mathbb{Z}$, and distinct integers $\{n_k\}_{k=1}^N$, there exist unique scalars $\{c_k\}_{k=1}^N$ such that*

$$e_m(x) \;=\; \sum_{k=1}^{N} c_k e_{n_k}(x) \,+\, O(x^N) \quad \text{as } x \to 0. \tag{4}$$

Proof The Taylor series for e_m and e_{n_k} are

$$e_m(x) \;=\; \sum_{\ell=0}^{\infty} \frac{(2\pi i m)^\ell}{\ell!} x^\ell \quad \text{and} \quad e_{n_k}(x) \;=\; \sum_{\ell=0}^{\infty} \frac{(2\pi i n_k)^\ell}{\ell!} x^\ell, \quad k = 1, \dots, N.$$

Since the integers n_k are distinct, by using properties of the determinants of Vandermonde matrices, we see that the $N \times N$ matrix

$$A \;=\; \begin{pmatrix} 1 & 1 & \cdots & 1 \\ n_1 & n_2 & \cdots & n_N \\ \vdots & \vdots & \ddots & \vdots \\ n_1^{N-1} & n_2^{N-1} & \cdots & n_N^{N-1} \end{pmatrix}$$

is invertible and there exist unique constants c_k satisfying the equations

$$m^\ell \;=\; \sum_{k=1}^{N} c_k\, n_k^\ell, \quad \ell = 0, 1, \dots, N-1.$$

Then for $\ell = 0, \dots, N-1$, the coefficients of x^ℓ in the Taylor series of $e_m(x)$ and the trigonometric polynomial $\sum_{k=1}^N c_k e_{n_k}(x)$ are the same, so

$$e_m(x) - \sum_{k=1}^{N} c_k e_{n_k}(x) \;=\; x^N \sum_{\ell=N}^{\infty} \frac{(2\pi i)^\ell}{\ell!} \left(m^\ell - \sum_{k=1}^{N} c_k\, n_k^\ell \right) x^{\ell - N}.$$

This shows that the condition given in equation (4) holds. □

Note that when m is distinct from the integers $n_1, \dots, n_N$, we have $e_m - \sum_{k=1}^N c_k e_{n_k} \neq 0$ no matter which constants $\{c_k\}_{k=1}^N$ that we choose. This is a consequence of the fact that the complex exponentials $e^{2\pi i n x}$ are linearly independent.

Lemma 2 *Given $N \ge 1$, let $\{n_k\}_{k=1}^N$ be distinct integers. If*

$$p(x) \;=\; \sum_{k=1}^{N} c_k e^{2\pi i n_k x}$$

is a trigonometric polynomial such that $x^{-N} p(x) \in L^2(\mathbb{T})$, then $c_1 = c_2 = \cdots = c_N = 0$.

Proof Suppose $x^{-N}p(x) \in L^2(\mathbb{T})$, and define

$$f(x) \;=\; \sum_{m\geq N}\sum_{k=1}^{N} c_k \,\frac{(2\pi i n_k)^m}{m!}\, x^{m-N}, \qquad x\in[0,1].$$

Then f is continuous on $[0,1]$, so $g(x) = x^{-N}p(x) - f(x) \in L^2(\mathbb{T})$. On the other hand, using the Taylor series of $p(x)$ at $x = 0$ we can rewrite g as

$$g(x) \;=\; \sum_{m=0}^{N-1}\biggl(\sum_{k=1}^{N} c_k \,\frac{(2\pi i n_k)^m}{m!}\biggr)x^{m-N}.$$

Since $g \in L^2(\mathbb{T})$ but $x^{m-N} \notin L^2(\mathbb{T})$ for $m = 0, \dots, N-1$, this implies that

$$\sum_{k=1}^{N} c_k \,\frac{(2\pi i n_k)^m}{m!} \;=\; 0, \qquad m = 0, 1, \dots, N-1.$$

Again using properties of the determinants of Vandermonde matrices, we conclude that $c_1 = c_2 = \cdots = c_N = 0$. □

In fact, this proof shows that Lemma 2 holds for distinct real numbers $\{n_k\}_{k=1}^N$.

3 Main Results

3.1 Exactness of $\{x^N e^{2\pi i n x}\}_{n\in\mathbb{Z}\setminus F}$ for $F \subset \mathbb{Z}$

We will consider the sequence

$$\mathcal{E}(x^N, \mathbb{Z}\backslash F) \;=\; \{x^N e^{2\pi i n x}\}_{n\in\mathbb{Z}\backslash F} \;=\; \{x^N e_n(x)\}_{n\in\mathbb{Z}\backslash F}$$

in $L^2(\mathbb{T})$, where F is a subset of $\mathbb{Z}$. The following result applies to all subsets F of $\mathbb{Z}$.

Theorem 3 *Fix $F \subset \mathbb{Z}$ (where F may be infinite).*

(a) *If $\mathcal{E}(x^N, \mathbb{Z}\backslash F)$ is complete, then $|F| \leq N$.*
(b) *If $\mathcal{E}(x^N, \mathbb{Z}\backslash F)$ is minimal, then $|F| \geq N$.*

Proof (a) Assume $|F| > N$ and let $m, n_1, \dots, n_N$ be distinct integers in F. Then Lemma 1 implies that there exist constants $c_1, \dots, c_N$ such that the function

$$f(x) \;:=\; \biggl(e_m(x) - \sum_{k=1}^{N} c_k e_{n_k}(x)\biggr)x^{-N}$$

is not identically zero, belongs to $L^2(\mathbb{T})$, and is orthogonal to $x^N e_n(x)$ for each $n \in \mathbb{Z}\backslash F$. Therefore $\mathcal{E}(x^N, \mathbb{Z}\backslash F)$ is incomplete.

(b) There is nothing to prove if $|F| = \infty$, so assume that $F = \{n_k\}_{k=1}^{M}$ is finite. If $\mathcal{E}(x^N, \mathbb{Z}\backslash F)$ is minimal, then there exists a biorthogonal sequence $\{f_m\}_{m\in\mathbb{Z}\backslash F}$ in $L^2(\mathbb{T})$. Given $m \in \mathbb{Z}\backslash F$, we have $\langle x^N f_m, e_n\rangle = \delta_{m,n}$ for all $n \neq n_k$, so there exist constants $c_{m,1}, \dots, c_{m,M}$ such that

$$f_m(x) \;=\; \biggl(e_m(x) + \sum_{k=1}^{M} c_{m,k} e_{n_k}(x)\biggr) x^{-N}.$$

In fact, $\sum_{k=1}^{M} c_{m,k} e_{n_k}(x)$ is the Fourier expansion of $x^N f_m(x) - e_m$. Lemma 2 therefore implies that $M \geq N$. □

Now we consider the exactness of $\mathcal{E}(x^N, \mathbb{Z}\backslash F)$ when F is finite.

Theorem 4 *Let F be a subset of $\mathbb{Z}$. Then $\mathcal{E}(x^N, \mathbb{Z}\backslash F)$ is exact in $L^2(\mathbb{T})$ if and only if $|F| = N$.*

Proof Theorem 3 shows that if $\mathcal{E}(x^N, \mathbb{Z}\backslash F)$ is exact then $|F| = N$.

For the converse, fix $F = \{n_k\}_{k=1}^{N}$ and suppose $f \in L^2(\mathbb{T})$ is orthogonal to $x^N e_n(x)$ for all $n \in \mathbb{Z}\backslash F$. Then $x^N f(x)$ is orthogonal to e_n for $n \in \mathbb{Z}\backslash F$, so

$$x^N f(x) \;=\; \sum_{k=1}^{N} c_k e_{n_k}(x)$$

for some constants $c_1, \dots, c_N$. Lemma 2 therefore implies that each $c_k = 0$, so $\mathcal{E}(x^N, \mathbb{Z}\backslash F)$ is complete.

Next, Lemma 1 shows that for each integer $m \in \mathbb{Z}\backslash F$ there exist constants $c_{m,1}, \dots, c_{m,N}$ such that the function

$$\widetilde{g}_m(x) \;:=\; \biggl(e_m(x) - \sum_{k=1}^{N} c_{m,k} e_{n_k}(x)\biggr) x^{-N}$$

belongs to $L^2(\mathbb{T})$ and is not identically zero. For $m \in \mathbb{Z}\backslash F$ we have

$$\bigl\langle \widetilde{g}_m(x),\, x^N e_m(x)\bigr\rangle \;=\; \biggl\langle e_m - \sum_{k=1}^{N} c_{m,k} e_{n_k},\, e_m \biggr\rangle \;=\; 1,$$

and similarly $\bigl\langle \widetilde{g}_m(x),\, x^N e_n(x)\bigr\rangle = 0$ for all $n \in \mathbb{Z}\backslash F$ with $n \neq m$. Therefore $\mathcal{E}(x^N, \mathbb{Z}\backslash F)$ has a biorthogonal sequence, so it is minimal. □

3.2 Completeness and Exactness of $\mathcal{E}(g, \mathbb{Z}\backslash F)$

In this part we will show that the assumption that $\mathcal{E}(g, \mathbb{Z}\backslash F)$ is complete or minimal implies some constraints on the function g.

Theorem 5 *Suppose $g \in L^2(\mathbb{T})\backslash\{0\}$ and $F \subset \mathbb{Z}$ is finite and nonempty.*

(a) *If* $\mathcal{E}(g, \mathbb{Z}\backslash F)$ *is complete, then* $g(x) \neq 0$ *a.e. in* $[0,1]$ *and* $1/g \notin L^2(\mathbb{T})$.

(b) *Suppose* $\mathcal{E}(g, \mathbb{Z}\backslash F)$ *is minimal and* $\{\widetilde{g}_m\}_{m \in \mathbb{Z}\backslash F}$ *is a biorthogonal sequence in* $L^2(\mathbb{T})$. *Then for each* $m \notin F$, *the function* $p_m = \widetilde{g}_m \, \overline{g}$ *is a trigonometric polynomial of the form*

$$p_m \;=\; e_m + \sum_{k \in F} c_{m,k} e_k. \tag{5}$$

(c) $\mathcal{E}(g, \mathbb{Z}\backslash F)$ *is exact if and only if* $g(x) \neq 0$ *a.e. in* $[0,1]$, $1/g \notin L^2(\mathbb{T})$, *and for* $m \in \mathbb{Z}\backslash F$ *there exists a unique trigonometric polynomial* p_m *of the form given in equation* (5) *such that* $p_m/\overline{g} \in L^2(\mathbb{T})$.

Proof (a) If $Z = \{x \in [0,1] : g(x) = 0\}$ has positive measure then χ_Z is a nontrivial function that is orthogonal to $\mathcal{E}(g, \mathbb{Z}\backslash F)$, so $\mathcal{E}(g, \mathbb{Z}\backslash F)$ is incomplete whenever g vanishes on a set of positive measure.

If $1/g \in L^2(\mathbb{T})$ then for each $m \in F$ the function $e_m/\overline{g} \in L^2(\mathbb{T})$ is orthogonal to $\mathcal{E}(g, \mathbb{Z}\backslash F)$, so $\mathcal{E}(g, \mathbb{Z}\backslash F)$ is incomplete in this case.

(b) If $\mathcal{E}(g, \mathbb{Z}\backslash F)$ is minimal with biorthogonal sequence $\{\widetilde{g}_m\}_{m \in \mathbb{Z}\backslash F}$ then

$$\big\langle \widetilde{g}_m \, \overline{g},\, e_n \big\rangle \;=\; \big\langle \widetilde{g}_m,\, g e_n \big\rangle \;=\; \delta_{m,n}, \qquad m, n \in \mathbb{Z}\backslash F.$$

Since $\widetilde{g}_m \, \overline{g} \in L^1(\mathbb{T})$ and integrable functions are uniquely determined by their Fourier coefficients, there exist constants $c_{m,k}$ such that

$$\widetilde{g}_m \, \overline{g} \;=\; e_m + \sum_{k \in F} c_{m,k} e_k.$$

(c) Suppose that $\mathcal{E}(g, \mathbb{Z}\backslash F)$ is exact. Then $\mathcal{E}(g, \mathbb{Z}\backslash F)$ is complete, so by statement (a) we have $g \neq 0$ a.e. and $1/g \notin L^2(\mathbb{T})$. Suppose $\{p_m\}_{m \in \mathbb{Z}\backslash F}$ is any sequence of trigonometric polynomials of the form given in equation (5) such that $p_m/\overline{g} \in L^2(\mathbb{T})$ for each $m \in \mathbb{Z}\backslash F$. Then for m, $n \in \mathbb{Z}\backslash F$ we have

$$\big\langle p_m/\overline{g},\, g e_n \big\rangle \;=\; \langle p_m, e_n \rangle \;=\; \langle e_m, e_n \rangle \;=\; \delta_{mn}. \tag{6}$$

Thus $\{p_m/\overline{g}\}_{m \in \mathbb{Z}\backslash F}$ is biorthogonal to $\mathcal{E}(g, \mathbb{Z}\backslash F)$. However, since $\mathcal{E}(g, \mathbb{Z}\backslash F)$ is exact it has a *unique* biorthogonal sequence $\{\widetilde{g}_m\}_{g \in \mathbb{Z}\backslash F}$. Hence $p_m = \widetilde{g}_m \, \overline{g}$ for each $m \in \mathbb{Z}\backslash F$, and thus the sequence $\{p_m\}_{m \in \mathbb{Z}\backslash F}$ is unique.

For the converse, suppose that $g \neq 0$ a.e., $1/g \notin L^2(\mathbb{T})$, and unique trigonometric polynomials p_m of the given form exist. As in equation (6), it follows that $\{p_m/\overline{g}\}_{m \in \mathbb{Z}\backslash F}$ is biorthogonal to $\mathcal{E}(g, \mathbb{Z}\backslash F)$, so $\mathcal{E}(g, \mathbb{Z}\backslash F)$ is minimal. Now let $\{\widetilde{g}_m\}_{m \in \mathbb{Z}\backslash F}$ be an arbitrary biorthogonal sequence to $\mathcal{E}(g, \mathbb{Z}\backslash F)$. Then by statement (b), the functions $\widetilde{g}_m \, \overline{g}$ are trigonometric polynomials of the form given in equation (5), so we must have $\widetilde{g}_m \, \overline{g} = p_m$. As g is nonzero almost everywhere, this implies that $\widetilde{g}_m = p_m/\overline{g}$. Hence the biorthogonal system to $\mathcal{E}(g, \mathbb{Z}\backslash F)$ is unique, which implies that $\mathcal{E}(g, \mathbb{Z}\backslash F)$ is exact. □

For example, $\mathcal{E}(\sqrt{2}\chi_{[0,1/2]}, 2\mathbb{Z})$ is minimal in $L^2(\mathbb{T})$ since it is biorthogonal to itself, but it is not complete.

3.3 Properties of atoms g

Let $g \in L^2(\mathbb{T})\backslash\{0\}$ be such that $\mathcal{E}(g, \mathbb{Z}\backslash F)$ is exact. In this case, we have shown that for each $m \in \mathbb{Z}\backslash F$ there exists a unique trigonometric polynomial $p_m = e_m + \sum_{k\in F} c_{m,k} e_k$ such that $p_m/\overline{g} \in L^2(\mathbb{T})$, even though $1/g \notin L^2(\mathbb{T})$. In this part we will characterize such functions g.

We will need the following independence lemma.

Lemma 3 *Given $\alpha_1, \dots, \alpha_k$ in $[0,1]$ and distinct positive integers $n_1, \dots, n_k$, let $q(x) = (x-\alpha_1)^{n_1} \cdots (x-\alpha_k)^{n_k}$, so q is a polynomial of degree*

$$N \;:=\; n_1 + \cdots + n_k.$$

Let

$$m(x) \;=\; \sum_{\gamma=1}^{N} c_\gamma e^{2\pi i \lambda_\gamma x}$$

be a trigonometric polynomial, where $\lambda_1, \dots, \lambda_N$ are distinct integers. For $j = 0, 1, \dots, n_\ell - 1$ and $\ell = 1, 2, \dots, k$ define

$$z_{j,\ell} \;=\; \bigl(\lambda_1^j e^{2\pi i \lambda_1 \alpha_\ell}, \dots, \lambda_N^j e^{2\pi i \lambda_N \alpha_\ell}\bigr) \;\in\; \mathbb{C}^N. \tag{7}$$

If the N vectors $\{z_{j,\ell}\}$ are linearly independent, then $m/q \in L^2(\mathbb{T})$ if and only if $m = 0$.

Proof Suppose that $m/q \in L^2(\mathbb{T})$. For $\ell = 1, \dots, k$, let I_ℓ be an interval containing α_ℓ and away from any point α_j for $j \neq \alpha$. That is, for each $\ell = 1, \dots, k$ we require that

$$\inf\{|x - \alpha_i| : x \in I_\ell \text{ and } i \neq \ell\} \;>\; 0.$$

This implies that the absolute value of the polynomial

$$p_\ell(x) \;=\; \prod_{i \neq \ell} (x - \alpha_i)^{n_i} \;=\; \frac{q(x)}{(x-\alpha_\ell)^{n_\ell}}$$

is bounded below on I_ℓ, and for any $j \geq 1$ we have

$$\frac{1}{(x-\alpha_\ell)^j} \;\notin\; L^2(I_\ell).$$

Since $m/q \in L^2(\mathbb{T})$, it is obvious that

$$\frac{m(x)}{(x-\alpha_\ell)^{n_\ell}} \;=\; \frac{m(x)\, p_\ell(x)}{q(x)} \;\in\; L^2(I_\ell).$$

Let m_ℓ be the Taylor polynomial for m of degree $n_\ell - 1$ about the point α_ℓ, i.e.,

$$m_\ell(x) \;=\; \sum_{j=0}^{n_\ell - 1} \frac{m^{(j)}(\alpha_\ell)}{j!}\,(x-\alpha_\ell)^j.$$

The remainder term $R_\ell = m - m_\ell$ satisfies

$$\frac{R_\ell(x)}{(x-\alpha_\ell)^{n_\ell}} \; = \; \sum_{j=n_\ell}^{\infty} \frac{m^{(j)}(\alpha_\ell)}{j!}\,(x-\alpha_\ell)^{j-n_\ell} \in L^2(I_\ell).$$

Note that the convergence radius of the series on the right side of the equation above is ∞, so $\frac{R_\ell(x)}{(x-\alpha_\ell)^{n_\ell}}$ is bounded on I_ℓ. Therefore

$$\sum_{j=0}^{n_\ell-1} \frac{m^{(j)}(\alpha_\ell)}{j!}\,(x-\alpha_\ell)^{j-n_\ell} \; = \; \frac{m(x)-R_\ell(x)}{(x-\alpha_\ell)^{n_\ell}} \; \in \; L^2(I_\ell),$$

which implies that $m^{(j)}(\alpha_\ell) = 0$ for $j = 0,\dots,n_\ell - 1$. Thus m satisfies the equations

$$m^{(j)}(\alpha_\ell) \; = \; 0, \quad j = 0,\dots,n_\ell - 1, \; \ell = 1,\dots,k.$$

Equivalently,

$$\sum_{\gamma=1}^{N} c_\gamma\,(2\pi i\lambda_\gamma)^j\, e^{2\pi i\lambda_\gamma\alpha_\ell} \; = \; 0, \quad j = 0,\dots,n_\ell - 1, \; \ell = 1,\dots,k.$$

The independence assumption on the vectors $z_{j,\ell}$ given in equation (7) therefore implies that $c_\gamma = 0$ for $\gamma = 1,\dots,N$, and therefore $m = 0$. □

Lemma 3 can be generalized by using the following terminology.

Definition 1 Given a function $f \in L^2(\mathbb{T})$ and a point α in $[0,1]$, we say that f has an *L^2-zero at α of order $k \in \mathbb{N}$* if there exists a nondegenerate interval I having α as a limit point such that

$$f_k(x) \; := \; \frac{(x-\alpha)^k}{f(x)}\chi_I(x) \in L^2(\mathbb{T})$$

but

$$f_i(x) \; := \; \frac{(x-\alpha)^i}{f(x)}\chi_I(x) \notin L^2(\mathbb{T}), \quad i = 0,\dots,k-1.$$

Note that, in this case, it follows that if $J \subset I$ is an interval having α as a limit point then

$$f_k(x) \; := \; \frac{(x-\alpha)^k}{f(x)}\chi_J(x) \in L^2(\mathbb{T})$$

but

$$f_i(x) \; := \; \frac{(x-\alpha)^i}{f(x)}\chi_J(x) \notin L^2(\mathbb{T}), \quad i = 0,\dots,k-1.$$

A nonzero function may have infinitely many L^2-zeros. For example, $f(x) = x \sin\frac{1}{x}$ is continuous but $1/f \notin L^2(\mathbb{T})$, so f has L^2-zeros of order 1 at $x = 1/(n\pi)$ for all integers $n > 0$.

Using the notion of L^2-zeros, we see that Lemma 3 holds if we replace the polynomial $q(x) = (x-\alpha_1)^{n_1} \dots (x-a_k)^{n_k}$ by a function $g \in L^2(\mathbb{T})$ that has an L^2-zero at α_i of order n_i for each $i = 1, \dots, k$. To be precise, we have the following lemma, which can be proved using an argument similar to the one used to prove Lemma 3.

Lemma 4 *Fix points $\alpha_1, \dots, \alpha_k \in [0,1]$ and positive integers $n_1, \dots, n_k$. Assume that $g \in L^2(\mathbb{T}) \backslash \{0\}$ satisfies:*

(a) *g has an L^2-zero at α_i of order n_i for $i = 1, \dots, k$, and*

(b) *$(1/g)\, \chi_I \in L^2(\mathbb{T})$ for any interval $I \subset [0,1]$ that is away from the constants $\alpha_1, \dots, \alpha_k$, i.e., I satisfies*

$$\mathit{dist}(I, \{\alpha_i\}_{i=1}^k) \;:=\; \inf\{|x-\alpha_i| : x \in I, i = 1, \dots, k\} \;>\; 0.$$

Let $m(x) = \sum_{\gamma=1}^N c_\gamma e^{2\pi i \lambda_\gamma x}$ be a trigonometric polynomial with $N := n_1 + \dots + n_k$, where $\lambda_1, \dots, \lambda_N$ are distinct integers. If the N vectors

$$\bigl(\lambda_1^j e^{2\pi i \lambda_1 \alpha_\ell}, \dots, \lambda_N^j e^{2\pi i \lambda_N \alpha_\ell}\bigr), \quad j = 0, \dots, n_\ell - 1,\ \ell = 1, \dots, k, \tag{8}$$

are linearly independent in $\mathbb{C}^N$ then $m/g \notin L^2(\mathbb{T})$, otherwise $m = 0$.

Proof Suppose that $m/g \in L^2(\mathbb{T})$. For $\ell = 1, \dots, k$, let I_ℓ be an interval containing α_ℓ and away from any point α_i with $i \neq \ell$. Let m_ℓ be the Taylor polynomial for m of degree $n_\ell - 1$ about the point α_ℓ, i.e.,

$$m_\ell(x) \;=\; \sum_{j=0}^{n_\ell - 1} \frac{m^{(j)}(\alpha_\ell)}{j!}\, (x-\alpha_\ell)^j.$$

From the proof of Lemma 3, we can see that for the remainder function $R_\ell = m - m_\ell$, the function defined by

$$\frac{R_\ell(x)}{(x-\alpha_\ell)^{n_\ell}} \;=\; \sum_{j=n_\ell}^{\infty} \frac{m^{(j)}(\alpha_\ell)}{j!}\, (x-\alpha_\ell)^{j-n_\ell}$$

is bounded for any $x \in \mathbb{T}$, so $\frac{R_\ell(x)}{(x-\alpha_\ell)^{n_\ell}} \in L^2(I_\ell)$. Therefore

$$\frac{1}{g(x)} \sum_{j=n_\ell}^{\infty} \frac{m^{(j)}(\alpha_\ell)}{j!}\, (x-\alpha_\ell)^j \;=\; \frac{(x-\alpha_\ell)^{n_\ell}}{g(x)} \frac{R_\ell(x)}{(x-\alpha_\ell)^{n_\ell}} \;\in\; L^2(I_\ell).$$

Combining this with the assumption that $m/g \in L^2(\mathbb{T})$, we obtain

$$\begin{aligned} \frac{m(x) - R_\ell(x)}{g(x)} \;&=\; \frac{1}{g(x)} \sum_{j=0}^{n_\ell - 1} \frac{m^{(j)}(\alpha_\ell)}{j!}\, (x-\alpha_\ell)^j \\ &=\; \sum_{j=0}^{n_\ell - 1} \frac{m^{(j)}(\alpha_\ell)}{j!}\, \frac{(x-\alpha_\ell)^j}{g(x)} \;\in\; L^2(I_\ell). \end{aligned}$$

Inductively, for $j = 1, \dots, n_\ell - 1$, multiplying by $(x-\alpha_\ell)^{n_\ell - j}$, we see that

$$\frac{m^{(j)}(\alpha_\ell)(x-\alpha_\ell)^j}{g(x)} \;\in\; L^2(I_\ell), \quad j = 0, \dots, n_\ell - 1.$$

This implies that $m^{(j)}(\alpha_\ell) = 0$ for $j = 0, \dots, n_\ell - 1$. Thus m satisfies the equations

$$m^{(j)}(\alpha_\ell) \;=\; 0, \quad j = 0, \dots, n_\ell - 1,\; \ell = 1, \dots, N.$$

Equivalently,

$$\sum_{\gamma=1}^{N} c_\gamma\, (2\pi i \lambda_\gamma)^j\, e^{2\pi i \lambda_\gamma \alpha_\ell} \;=\; 0, \quad j = 0, \dots, n_\ell - 1,\; \ell = 1, \dots, N.$$

The independence assumption on the vectors $z_{j,\ell}$ given in equation (8) therefore implies that $m = 0$. □

Theorem 6 *Let $g \in L^2(\mathbb{T})$ be such that $1/g \notin L^2(\mathbb{T})$. Let $\{p_m\}_{m\in\mathbb{Z}\setminus F}$ be a sequence of trigonometric polynomials p_m of the form given in equation (5). Then $p_m/g \in L^2(\mathbb{T})$ for each $m \in \mathbb{Z}\setminus F$ if and only if there exist a finite number of points $\alpha_1, \dots, \alpha_k$ and positive integers $n_1, \dots, n_k$ such that*

(a) *α_i is an L^2-zero of g of order n_i for each $i = 1, \dots, k$,*

(b) *for any closed interval I not containing $\alpha_1, \dots, \alpha_k$ we have $(1/g)\,\chi_I \in L^2(\mathbb{T})$, and*

(c) *for every $m \in \mathbb{Z}\setminus F$, $p_m^{(j)}(\alpha_i) = 0$ for $i = 1, \dots, k$ and $j = 0, \dots, n_i - 1$.*

Proof Assume that $p_m/g \in L^2(\mathbb{T})$ for each $m \in \mathbb{Z}\setminus F$. Suppose there was an $m \in \mathbb{Z}\setminus F$ such that $p_m(x) \neq 0$ for all $x \in [0,1]$. Then $|p_m(x)| \geq r > 0$ on $[0,1]$, and since $p_m/g \in L^2(\mathbb{T})$, this implies $1/g \in L^2(\mathbb{T})$, which is a contradiction.

Therefore, each p_m for $m \in \mathbb{Z}\setminus F$ has zeros in $[0,1]$, say $\{\beta_{m,k}\}_{k=1}^{n_m}$. Let $n_{m,k}$ be the order of the zero at $\beta_{m,k}$, i.e., the Taylor series of p_m about $x = \beta_{m,k}$ has the form

$$p_m(x) \;=\; (x-\beta_{m,k})^{n_{m,k}}\, q_{m,k}(x),$$

where

$$q_{m,k}(x) \;=\; \sum_{j=n_{m,k}}^{\infty} \frac{p_m^{(j)}(\beta_{m,k})}{j!}\, (x-\beta_{m,k})^{j-n_{m,k}}$$

belongs to $L^2(\mathbb{T})$ and $q_{m,k}(\beta_{m,k}) \neq 0$.

Choose nondegenerate intervals $I_{m,1}, \dots, I_{m,n_m}$ such that

(1) $\overline{I_{m,i}} \cap \overline{I_{m,j}} = \emptyset$ for $i \neq j$, and

(2) $\overline{I_{m,k}}$ contains $\beta_{m,k}$ but no point $\beta_{m,j}$ with $j \neq k$.

Here $\overline{I}$ denotes the closure of I. Since $|p_m|$ is bounded below on the set $D := [0,1] \setminus \cup_{i=1}^{n_m} I_{m,i}$ and since $(p_m/g)\,\chi_D \in L^2(\mathbb{T})$, we infer that $(1/g)\,\chi_D \in L^2(\mathbb{T})$. If $1/g \in L^2(I_{m,k})$ for all $k = 1, \dots, n_m$, then

$$\frac{1}{g} \;=\; \sum_{i=1}^{n_m} \frac{1}{g}\,\chi_{I_{m,i}} + \frac{1}{g}\,\chi_D,$$

and therefore $1/g \in L^2(\mathbb{T})$, which is a contradiction.

Combining the relation

$$\frac{p_m}{g} \;=\; \sum_{i=1}^{n_m} \frac{p_m}{g}\,\chi_{I_{m,i}} + \frac{p_m}{g}\,\chi_D$$

with the assumption that $p_m/g \in L^2(\mathbb{T})$, there exists a nonempty subset $\{\alpha_{m,i}\}_{i=1}^{N_m}$ of $\{\beta_{m,k}\}_{k=1}^{n_m}$ such that g has an L^2-zero at $\alpha_{m,i}$ of order $\gamma_{m,k}$ for $k = 1, \dots, N_m$ and for any $\beta_{m,j} \in \{\beta_{m,k}\}_{k=1}^{n_m} \setminus \{\alpha_{m,i}\}_{i=1}^{N_m}$ we have $(1/g)\,\chi_{I_{m,j}} \in L^2(\mathbb{T})$.

Thus we have shown that for each $m \in \mathbb{Z}\backslash F$, g has L^2-zeros $A_m := \{\alpha_{m,i}\}_{i=1}^{N_m}$ such that

(i) for each $k = 1, \dots, N_m$, there exists an integer $\gamma_{m,k} \geq 1$ such that

$$\frac{(x-\alpha_{m,k})^{\gamma_{m,k}}}{g(x)}\,\chi_{I_{m,k}}(x) \in L^2(\mathbb{T}) \quad \text{but} \quad \frac{(x-\alpha_{m,k})^{i}}{g(x)}\,\chi_{I_{m,k}}(x) \notin L^2(\mathbb{T})$$

for $i = 0, 1, \dots, \gamma_{m,k} - 1$, where $I_{m,k}$ is an interval away from $\{\alpha_{m,i}\}_{i \neq k}$;

(ii) for any interval I away from $\{\alpha_{m,i}\}_{i=1}^{N_m}$ we have $\frac{1}{g}\,\chi_I \in L^2(\mathbb{T})$.

We will show that all the sets A_m are the same, that is, $A_i = A_j$ for every $i, j \in \mathbb{Z}\backslash F$. Suppose that there were i and j such that $A_i \neq A_j$. Then there exists a point $\alpha_{jk} \in A_j \setminus A_i$. Choose an interval I_α containing α_{jk} such that $I_\alpha \cap (A_i \cup \{\alpha_{jn}\}_{n \neq k}) = \emptyset$. Then from condition (i) with $m = j$ we have that $(1/g)\,\chi_{I_\alpha} \notin L^2(\mathbb{T})$, but condition (ii) with $m = i$ implies that $(1/g)\,\chi_{I_\alpha} \in L^2(\mathbb{T})$, which is a contradiction. So for all $m \in \mathbb{Z}\backslash F$ we have $A_m = \{\alpha_i\}_{i=1}^{k}$ where g has an L^2-zero of order n_i for $i = 1, \dots, k$. This proves the claims (a)–(c).

Conversely, assume that there exist a finite number of points $\alpha_1, \dots, \alpha_k$ and positive integers $n_1, \dots, n_k$ for which the conditions (a)–(c) hold. In this case, we can show easily that $p_m/g \in L^2(\mathbb{T})$ for each $m \in \mathbb{Z}\backslash F$. □

Theorem 7 *Let $g \in L^2(\mathbb{T})$ be such that $1/g \notin L^2(\mathbb{T})$. Assume there exist a finite number of points $\alpha_1, \dots, \alpha_k$ and positive integers $n_1, \dots, n_k$ such that g has an L^2-zero at α_i of order n_i for each $i = 1, \dots, k$. Assume that $(1/g)\,\chi_I \in L^2(\mathbb{T})$ for any interval I away from $\{\alpha_i\}_{i=1}^{k}$. If $n_1 + \cdots + n_k < |F|$, then $\mathcal{E}(g, \mathbb{Z}\backslash F)$ is incomplete.*

Proof Assume that $n_1 + \cdots + n_k < |F|$. Then we can find constants $\{c_k\}_{k\in F}$, not all zero, such that the trigonometric polynomial $p(x) = \sum_{k\in F} c_k e^{2\pi ikx}$ satisfies the $N := n_1 + \cdots + n_k$ equations

$$p^{(j)}(\alpha_i) = 0, \quad j = 0, \ldots, n_i - 1 \quad \text{and} \quad i = 1, \ldots, k. \tag{9}$$

This follows from the fact that any homogeneous system of linear equations with more unknowns than equations always has nontrivial solutions.

Then from the two assumptions that g has an L^2-zero at α_i of order n_i for each $i = 1, \ldots, k$ and $(1/g)\,\chi_I \in L^2(\mathbb{T})$ for any interval away from $\{\alpha_i\}_{i=1}^k$, we can see that $p/g \in L^2(\mathbb{T})$. On the other hand, the function $f = p/\overline{g}$ is orthogonal to $\mathcal{E}(g, \mathbb{Z}\backslash F)$:

$$\langle p/\overline{g},\, ge_m\rangle \;=\; \Big\langle \sum_{k\in F} c_k e_k,\, e_m\Big\rangle \;=\; 0, \quad m \in \mathbb{Z}\backslash F.$$

Thus, $\mathcal{E}(g, \mathbb{Z}\backslash F)$ is incomplete in $L^2(\mathbb{T})$, which completes the proof. □

In general, the converse of Theorem 7 does not hold.

Example 1 Set

$$g(x) \;=\; x\,(x - \tfrac{1}{4})\,(x - \tfrac{1}{2})\,(x - \tfrac{3}{4})$$

and $F = \{0, 4\}$. Then g has L^2 zeros of order 1 at $x = 0$, $1/4$, $1/2$, and $3/4$, and the function

$$f(x) \;:=\; \frac{e^{8\pi ix} - 1}{g(x)} \in L^2(\mathbb{T})$$

is orthogonal to $g(x)\,e^{2\pi ikx}$ for every $k \in \mathbb{Z}\backslash F$. Thus, $\mathcal{E}(g, \mathbb{Z}\backslash F)$ is incomplete in $L^2(\mathbb{T})$ even though $|F| < N$.

Also consider $g_0(x) = x\,(x - 1)$ and $F = \{0\}$. For $m \neq 0$,

$$f_m(x) \;:=\; \frac{e^{2\pi imx} - 1}{g_0(x)} \in L^2(\mathbb{T}),$$

and $\{f_m\}_{m\neq 0}$ is the unique sequence biorthogonal to $\mathcal{E}(g_0, \mathbb{Z}\backslash\{0\})$. Therefore $\mathcal{E}(g_0, \mathbb{Z}\backslash\{0\})$ is exact, but $N > |F|$.

Theorem 8 *Given $g \in L^2(\mathbb{T})\backslash\{0\}$ and a nonempty finite set $F \in \mathbb{Z}$, the following statements hold.*

(a) *$\mathcal{E}(g, \mathbb{Z}\backslash F)$ is minimal in $L^2(\mathbb{T})$ if and only if for each $k \in \mathbb{Z}\backslash F$ there exist constants $\{c_{k\ell}\}_{\ell\in F}$ such that $(e_k + \sum_{\ell\in F} c_{k\ell} e_\ell)/\overline{g} \in L^2(\mathbb{T})$.*

(b) *$\mathcal{E}(g, \mathbb{Z}\backslash F)$ is complete in $L^2(\mathbb{T})$ if and only if $g(x) \neq 0$ a.e. in $[0, 1]$ and $1/g \notin L^2(\mathbb{T})$ and for constants $\{a_\ell\}_{\ell\in F}$ we have $\sum_{\ell\in F} a_\ell e_\ell/\overline{g} \in L^2(\mathbb{T})$ only when $a_\ell = 0$ for all $\ell \in F$.*

(c) *$\mathcal{E}(g, \mathbb{Z}\backslash F)$ is exact in $L^2(\mathbb{T})$ if and only if there exist a finite number of points $\alpha_1, \ldots, \alpha_k$ and positive integers $n_1, \ldots, n_k$ such that*

(c-1) *g has an L^2-zero at α_i of order n_i for each $i = 1, \ldots, k$,*

(c-2) *$(1/g)\chi_I \in L^2(\mathbb{T})$ for any closed interval $I \subset \mathbb{T} \setminus \{\alpha_i\}_{i=1}^k$,* (c-3) $n_1 + \cdots + n_k \geq |F|$,

(c-4) *for each* $m \in \mathbb{Z}\backslash F$, *there exists a unique trigonometric polynomial* p_m *of the form*

$$p_m(x) \; = \; e_m(x) + \sum_{k\in F} c_{m,k} e_k(x),$$

such that

$$p_m^{(j)}(\alpha_i) = 0 \qquad \text{for } i = 1, \ldots, k \text{ and } j = 0, \ldots, n_i - 1.$$

In this case, $\mathcal{E}(g, \mathbb{Z}\backslash F)$ *cannot be a frame for* $L^2(\mathbb{T})$.

Proof We have shown statements (a) and (b) in the previous theorems. To prove statement (c), it suffices to show that $\mathcal{E}(g, \mathbb{Z}\backslash F)$ cannot be a frame for $L^2(\mathbb{T})$ since the other claims follow directly from Theorem 5(c) and Theorems 6 and 7.

To complete the proof, assume that $\mathcal{E}(g, \mathbb{Z}\backslash F)$ is a frame for $L^2(\mathbb{T})$ with lower frame bound $A > 0$. Then there exist a finite number of points $\alpha_1, \ldots, \alpha_k$ and positive integers $n_1, \ldots, n_k$ such that g has an L^2-zero at α_i of order n_i for each $i = 1, \ldots, k$. On the one hand, we have that for any $f \in L^2(\mathbb{T})$ that

$$A \, \|f\|_2^2 \; \leq \; \sum_{m\in\mathbb{Z}\backslash F} |\langle f, ge_m\rangle|^2,$$

and on the other hand we have the inequality

$$\sum_{m\in\mathbb{Z}\backslash F} |\langle f, ge_m\rangle|^2 \; \leq \; \sum_{m\in\mathbb{Z}} |\langle f, ge_m\rangle|^2 \; = \; \|f\overline{g}\|_2^2.$$

By the first inequality, we have for all $f \in L^2(\mathbb{T})$ that

$$A \, \|f\|_2^2 \; \leq \; \|f \, \overline{g}\|_2^2.$$

Equivalently, if $f \in L^2(\mathbb{T})$ then

$$\int_0^1 |f(x)|^2 \, (|\overline{g}(x)|^2 - A) \, dx \; \geq \; 0. \tag{10}$$

Since g has an L^2-zero at α_1, for any $\varepsilon > 0$ we can find a nondegenerate interval I_{α_1} such that $|g(x)| \leq \varepsilon$ for $x \in I_{\alpha_1}$. Taking $f = \chi_{I_{a_1}}$ in equation (10), we obtain a contradiction. As a consequence, $\mathcal{E}(g, \mathbb{Z}\backslash F)$ cannot be a frame for $L^2(\mathbb{T})$. □

Note that if g has an L^2-zero at α_i of order n_i for each $i = 1, \ldots, k$, and if $\alpha_1 = 0$ and $\alpha_k = 1$, then the inequality in statement (c-3) of Theorem 8 becomes

(c-3) $\max\{n_1, n_k\} + n_2 + \cdots + n_{k-1} \geq |F|$,

and equality holds when $n_i = 1$ for all $i = 1, \ldots, k$ or the integers in F are consecutive.

3.4 On the sequence $\mathcal{E}(g, \mathbb{Z}\backslash F)$ in $L^2(\mathbb{T})$

In this section, we will show that $\mathcal{E}(g, \mathbb{Z}\backslash F)$ cannot be a frame in $L^2(\mathbb{T})$ if F is not empty.

Lemma 5 *Let I be a bounded, nonempty interval in $\mathbb{R}$. If $g \in L^2(I)$ is such that $1/g \notin L^2(I)$, then for any $\varepsilon > 0$ there is a set $I_\varepsilon \subset I$ with measure $|I_\varepsilon| > 0$ such that $\chi_{I_\varepsilon} \in L^2(I)$ and $|g(x)| \le \varepsilon$ for all $x \in I_\varepsilon$.*

Proof Assume that I is a bounded interval and $1/g \notin L^2(I)$. If g vanishes on a subset of I that has positive measure then the lemma holds trivially. On the other hand, if $g = 0$ a.e. on I then it is not difficult to see that

$$\inf\bigl\{|g(x)| : g(x) \neq 0,\, x \in I\bigr\} \;=\; 0.$$

Given $\varepsilon > 0$, let $I_\varepsilon = \{x \in I : |g(x)| \le \varepsilon\}$. Since $1/g \notin L^2(I)$, we must have $|I_\varepsilon| > 0$. □

Theorem 9 *Let $g \in L^2(\mathbb{T})$ be given and let F be a nonempty subset of $\mathbb{Z}$. Then $\mathcal{E}(g, \mathbb{Z}\backslash F)$ cannot be a frame for $L^2(\mathbb{T})$.*

Proof Suppose that $\mathcal{E}(g, \mathbb{Z}\backslash F)$ was a frame for $L^2(\mathbb{T})$, and let $A > 0$ be a lower frame bound. Then $\mathcal{E}(g, \mathbb{Z}\backslash F)$ is complete in $L^2(\mathbb{T})$. Therefore $1/g \notin L^2(\mathbb{T})$, and for any $f \in L^2(\mathbb{T})$ we have

$$A\,\|f\|_2^2 \;\le\; \sum_{k \in \mathbb{Z}\backslash F} |\langle f, ge_k\rangle|^2 \;\le\; \sum_{k \in \mathbb{Z}} |\langle f\,\overline{g}, e_k\rangle|^2 \;=\; \|f\,\overline{g}\|_2^2. \tag{11}$$

On the other hand, Lemma 5 implies that for any $\varepsilon > 0$ there exists a set $I_\varepsilon \subset [0,1]$ with measure $|I_\varepsilon| > 0$ such that $|g(x)| \le \varepsilon$ for almost all $x \in I_\varepsilon$. Taking $f = \chi_{I_\varepsilon}$ in equation (11), we obtain a contradiction. □

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