REDUNDANCY AND LOCALIZED FRAMES

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OVERTURE: REDUNDANCY

What is redundancy?

How can we quantify redundancy?

Which elements may be removed?

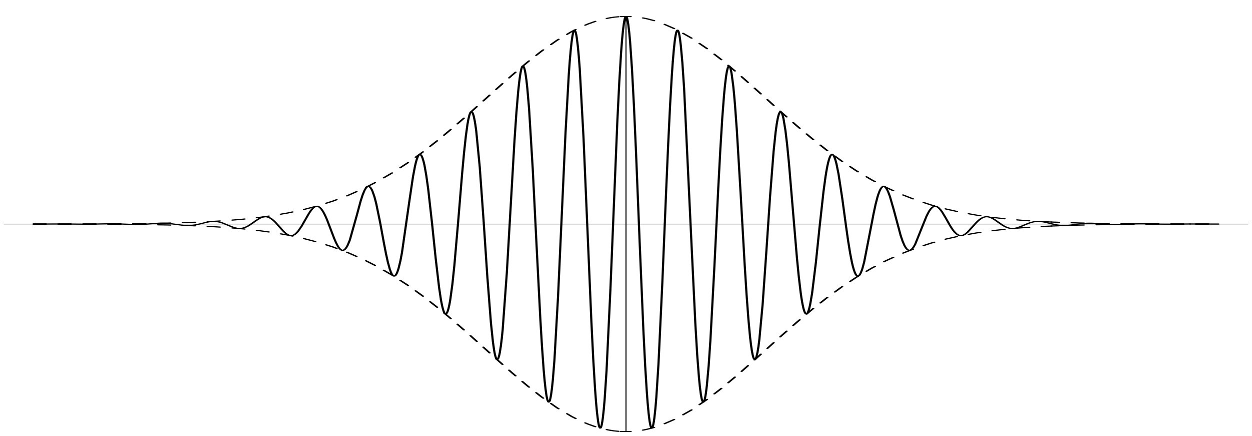
What is the effect of removing/losing elements?

How can we recognize that a system is the union of finitely many nonredundant systems?

<u>Goal</u>

Music-like bases or frames for $L^2(\mathbf{R})$.

Model of a note at time α and frequency β:



$$e^{2\pi i\beta x}g(x-\alpha) \;=\; M_\beta T_\alpha g(x)$$

Gabor System (Weyl–Heisenberg system, coherent states)

$$\begin{aligned}\mathcal{G}(g,\Lambda) &= \left\{M_\beta T_\alpha g\right\}_{(\alpha,\beta)\in\Lambda} \\ &= \left\{e^{2\pi i\beta x} g(x-\alpha)\right\}_{(\alpha,\beta)\in\Lambda} \\ &= \left\{\pi(\alpha,\beta,1)g\right\}_{(\alpha,\beta)\in\Lambda}\end{aligned}$$

where π is the Schrödinger representation of the Heisenberg group on $\mathbf{R}^n$.

"Music"

$$f = \sum_{(\alpha,\beta)\in\Lambda} c_{\alpha,\beta}(f)\, M_\beta T_\alpha g$$

Desired Properties: Stability

(a) **Expansions** $f = \sum_{(\alpha,\beta)\in\Lambda} c_{\alpha,\beta}(f)\, M_\beta T_\alpha g$

should converge unconditionally in the norm of an appropriate Banach space X**.**

(b) **More than just** $c_{\alpha,\beta} \in X^*$**, given an associated sequence space** X_d**,**

$$\begin{aligned} T : X &\to X_d \\ f &\mapsto \{c_{\alpha,\beta}(f)\}_{(\alpha,\beta)\in\Lambda} \end{aligned}$$

should be continuously invertible on its range. In particular,

$$\|f\|_X \cong \big\|\{c_{\alpha,\beta}(f)\}\big\|_{X_d}$$

(c) **Simultaneous applicability: Both properties above should hold simultaneously in an entire** *family* **of Banach spaces (e.g.,** L^p**, Hölder, Sobolev, Besov, modulation spaces).**

Uniqueness??

Frames in Hilbert Spaces

$\{f_n\}_{n\in\mathbf{N}}$ **is a frame for a Hilbert space** H **if there exist** A, $B > 0$ **such that**

$$\forall\, f \in H, \quad A\,\|f\|^2 \;\le\; \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \;\le\; B\,\|f\|^2$$

Finite dimensions: Frames = spanning sets

Expansions follow for free:

Analysis operator:

$$Tf \;=\; \bigl\{\langle f, f_n\rangle\bigr\}_{n\in\mathbf{N}} \quad \textbf{maps } H \to \ell^2$$

Frame operator:

$$Sf \;=\; T^*Tf \;=\; \sum_{n=1}^{\infty} \langle f, f_n\rangle\, f_n \quad \textbf{is positive definite}$$

Expansions:

$$f \;=\; SS^{-1}f \;=\; \sum_{n=1}^{\infty} \langle S^{-1}f, f_n\rangle\, f_n \;=\; \sum_{n=1}^{\infty} \langle f, S^{-1}f_n\rangle\, f_n$$

Duality: $\bigl\{\tilde{f}_n\bigr\}_{n\in\mathbf{N}} \;=\; \{S^{-1}f_n\}_{n\in\mathbf{N}}$ **is the** *dual frame*, **and**

$$\forall\, f \in H, \quad \frac{1}{B}\,\|f\|^2 \;\le\; \sum_{n=1}^{\infty} |\langle f, \tilde{f}_n\rangle|^2 \;\le\; \frac{1}{A}\,\|f\|^2$$

Remark: **Frame + basis** $\iff$ **Riesz basis**

(image of ONB under a continuous, invertible map)

Examples

(a) $\mathcal{G}(\chi_{[0,1)}, \mathbf{Z} \times \mathbf{Z}) = \{e^{2\pi inx}\chi_{[k,k+1)}(x)\}_{k,n\in\mathbf{Z}}$ **is an ONB for** $L^2(\mathbf{R})$**.**

(b) **Gabor's original system** $\mathcal{G}(e^{-x^2}, \mathbf{Z} \times \mathbf{Z})$ **is overcomplete but not a basis or frame.**

(c) $\mathcal{G}(e^{-x^2}, \mathbf{Z} \times \mathbf{Z} \setminus (0,0))$ **is complete and minimal but not a basis or frame.**

(d) $\mathcal{G}(e^{-x^2}, \alpha\mathbf{Z} \times \beta\mathbf{Z})$ **is a frame (overcomplete) exactly when** $0 < \alpha\beta < 1$ **(via Bargmann transform/analyticity).**

(e) **If supp**$(g) \subset [0, \frac{1}{\beta}]$ **and**

$$0 \;<\; C_1 \;\leq\; \sum_{k\in\mathbf{Z}} |g(x-k\alpha)|^2 \;\leq\; C_2 \;<\; \infty$$

then $\mathcal{G}(g, \alpha\mathbf{Z} \times \beta\mathbf{Z})$ **is a frame for** $L^2(\mathbf{R})$**. For this case:**

$\alpha\beta > 1$**: not possible**

$\alpha\beta = 1$**: basis, but** g **is discontinuous**

$\alpha\beta < 1$**: not a basis, but** g **can be smooth**

Remark

If g **is nice, then frame expansions extend to the entire family of** *modulation spaces***.**

Balian–Low Theorems $\quad \mathcal{G}(g, \alpha\mathbf{Z} \times \beta\mathbf{Z}) \;=\; \left\{e^{2\pi i\beta nx} g(x-\alpha k)\right\}_{k,n\in\mathbf{Z}}$

(a) **Classical BLT [Balian/Low]: If $\mathcal{G}(g, \alpha\mathbf{Z} \times \beta\mathbf{Z})$ is a Riesz basis for $L^2(\mathbf{R})$, then**

$$\left(\int_{-\infty}^{\infty} |tg(t)|^2 \, dt\right) \left(\int_{-\infty}^{\infty} |\omega\hat{g}(\omega)|^2 \, d\omega\right) = \infty.$$

(b) **Amalgam BLT [H.]: If $\mathcal{G}(g, \alpha\mathbf{Z} \times \beta\mathbf{Z})$ is a Riesz basis for $L^2(\mathbf{R})$, then g, $\hat{g} \notin W(C_0, \ell^1)$, where**

$$W(C_0, \ell^1) \;=\; \left\{\textbf{continuous } f : \sum_{k=-\infty}^{\infty} \|f \cdot \chi_{[k,k+1]}\|_{\infty} < \infty\right\}.$$

Open Questions

(a) **Classical BLT for general lattices in higher dimensions (yes if symplectic: Gröchenig/Han/H./Kutyniok, Benedetto/Czaja/Ya)**

(b) **Amalgam BLT for general lattices in all dimensions.**

(c) **BLT for non-lattices in all dimensions.**

(d) **BLT for bases that are not Riesz bases.**

<u>Nyquist Density Theorem</u> for $\mathcal{G}(g, \alpha\mathbf{Z} \times \beta\mathbf{Z})$

(a) **Frame** $\implies 0 < \alpha\beta \le 1$**.**

(b) **Riesz basis** $\implies \alpha\beta = 1$**.**

(c) $\alpha\beta > 1 \implies$ **incomplete.**

<u>Techniques</u>

Baggett, Rieffel: von Neumann algebra generated by T_α, M_β

Daubechies: Zak Transform

Janssen: Wexler–Raz relations

All these tools are useless for general $\mathcal{G}(g, \Lambda)$.

Beurling Densities of Λ

$$D^-(\Lambda) \;=\; \liminf_{r\to\infty} \inf_{z\in\mathbf{R}^2} \frac{\#(\Lambda\cap Q_r(z))}{r^2},$$

$$D^+(\Lambda) \;=\; \limsup_{r\to\infty} \sup_{z\in\mathbf{R}^2} \frac{\#(\Lambda\cap Q_r(z))}{r^2},$$

where $Q_r(z)$ is the square centered at z with side lengths r.

Example: $\quad D^-(\alpha\mathbf{Z}\times\beta\mathbf{Z}) \;=\; D^+(\alpha\mathbf{Z}\times\beta\mathbf{Z}) \;=\; \dfrac{1}{\alpha\beta}$

Nyquist Density for $\mathcal{G}(g,\Lambda)$ [Ramanathan/Steger]

(a) **Frame** $\implies 1 \le D^-(\Lambda) \le D^+(\Lambda) < \infty$.

(b) **Riesz basis** $\implies D^-(\Lambda) = D^+(\Lambda) = 1$.

Remarks

- **Irregular Gabor systems can be complete (but not frames) even if they are very sparse [Walnut/H.]**

- $\exists$ **(very) irregular Gabor ONB [Y. Wang]**

Definitions/Facts

(a) **A frame is** *redundant* **or** *overcomplete* **if it is not a basis.**

(b) **If a frame is a basis then it is a** *Riesz basis* **(the image of an ONB under a continuous invertible map).**

(c) **A** *near-Riesz basis* **is a Riesz basis plus finitely many elements.**

(d) **A frame** $\mathcal{F} = \{f_n\}_{n\in\mathbf{N}}$ **is** *bounded* **if** $\inf \|f_n\| > 0$**.**

Q. Does every bounded frame contain a basis?

A. No [Casazza/Christensen, Seip].

Theorem [Duffin/Schaeffer]

If $\mathcal{F}$ **is an overcomplete frame then at least finitely many elements can be removed yet still leave a frame.**

Q. Aside from near-Riesz bases, can infinitely many elements be removed yet leave a frame?

A. No [Balan/Casazza/H./Landau, with characterization].

Examples

(a) $\mathcal{G}(\chi_{[0,1)}, \mathbf{Z} \times \mathbf{Z})$ **is an ONB.**

(b) $\mathcal{G}(\chi_{[0,1)}, \frac{1}{N}\mathbf{Z} \times \mathbf{Z})$ **is the union of N ONBs.**

(c) $\mathcal{G}(e^{-x^2}, \frac{1}{N}\mathbf{Z} \times \mathbf{Z})$ **is not the union of N ONBs, but is the union of N minimal systems plus N more elements.**

Q. Given a frame, how can you recognize that it is a union of finitely many ON sequences? Or finitely many Riesz sequences (Riesz bases for their closed spans)?

Q. Is frame (c) above a union of finitely many Riesz sequences?
A. Yes [B./C./H./L.]

Feichtinger Conjecture

Every bounded frame is a union of finitely many Riesz sequences.

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Kadison–Singer Conjecture (Paving Conjecture) [1959]

$\forall\, \varepsilon > 0,\ \exists\, M$ **such that** $\forall\, n$**,** $\forall\, n \times n$ **matrices** S **having zero diagonal,** $\exists$ **partition** $\{\sigma_j\}_{j=1}^{M}$ **of** $\{1, \dots, n\}$ **such that**

$$\|P_{\sigma_j} S P_{\sigma_j}\| \le \varepsilon \, \|S\|, \quad j = 1, \dots, M,$$

where P_I **is the orthogonal projection onto span**$\{e_i\}_{i \in I}$**.**

Conjectured Generalization of Bourgain–Tzafriri R.I.T.

$\forall\, B,\ \exists\, M, A$ **such that** $\forall\, n \times n$ **matrices** T **such that** $\|Te_i\| = 1$ **and** $\|T\| \le \sqrt{B}$**,** $\exists$ **partition** $\{I_j\}_{j=1}^{M}$ **of** $\{1, \dots, n\}$ **such that** $\forall\, \{a_i\}_{i \in I_j}$**,**

$$\Big\| \sum_{i \in I_j} a_i T e_i \Big\|^2 \ \ge\ A \sum_{i \in I_j} |a_i|^2, \quad j = 1, \dots, M.$$

Theorem [Casazza/Christensen/Lindner/Vershynin]

Kadison–Singer $\implies$ **Feichtinger** $\iff$ **Bourgain–Tzafriri**

Redundancy is not a "local" issue

A Gabor frame $\mathcal{G}(g, \frac{1}{N}\mathbf{Z} \times \mathbf{Z})$ seems to be "N times overcomplete." Yet, every finite subset is (probably) independent.

The following conjecture is known to hold for many special cases, but is open in the generality stated.

Conjecture (H./Ramanathan/Topiwala)

If $g \in L^2(\mathbf{R})$, $g \neq 0$, and $\Lambda = \{(\alpha_k, \beta_k)\}_{k=1}^{N}$ are distinct points in $\mathbf{R}^2$, then $\mathcal{G}(g, \Lambda) = \{e^{2\pi i \beta_k x} g(x - \alpha_k)\}_{k=1}^{N}$ is linearly independent.

Open HRT Subconjectures

If $g \in L^2(\mathbf{R})$ is continuous and nonzero then the following sets are independent:

(a) $\{g(x),\ g(x-1),\ e^{2\pi i x} g(x),\ e^{2\pi i \sqrt{2} x} g(x-\sqrt{2})\}$

(b) $\{g(x),\ g(x-1),\ g(x-\pi),\ e^{2\pi i x} g(x)\}$

Remarks

(a) **This is the "Zero Divisor Conjecture" for the case of the Heisenberg group.**

(b) **The analogous conjecture for the affine group is false. Moreover, the construction of wavelet ONBs depends crucially on linear** *dependence*.

Dual Frames

The dual frame of

$$\mathcal{G}(g, \alpha\mathbf{Z} \times \beta\mathbf{Z}) \;=\; \big\{M_{\beta n} T_{\alpha k}\, g\big\}_{k,n\in\mathbf{Z}}$$

has the form

$$\mathcal{G}(\tilde{g}, \alpha\mathbf{Z} \times \beta\mathbf{Z}) \;=\; \big\{M_{\beta n} T_{\alpha k}\, \tilde{g}\big\}_{k,n\in\mathbf{Z}}$$

because

$$S M_{\beta n} T_{\alpha k} \;=\; M_{\beta n} T_{\alpha k} S$$

Theorem **[Gröchenig/Leinert, via C^* algebras]**
$g \in M^1 \implies \tilde{g} \in M^1$

Fundamental Problem **[Open until B./C./H./L.]**
If Λ is not a lattice, what does the dual frame of

$$\mathcal{G}(g, \Lambda) \;=\; \big\{M_\beta T_\alpha\, g\big\}_{(\alpha,\beta)\in\Lambda}$$

look like??

Definition: Localized Frames

Given $\mathcal{F} = \{f_i\}_{i\in I}$, $\mathcal{E} = \{e_j\}_{j\in G}$, **and** $a\colon I \to G$.

(a) **($\mathcal{F}$, a, $\mathcal{E}$) is** ℓ^p*-localized* **if** $\exists\, r = (r_k)_{k\in G} \in \ell^p(G)$ **such that**

$$|\langle f_i, e_j\rangle| \;\le\; r_{a(i)-j}$$

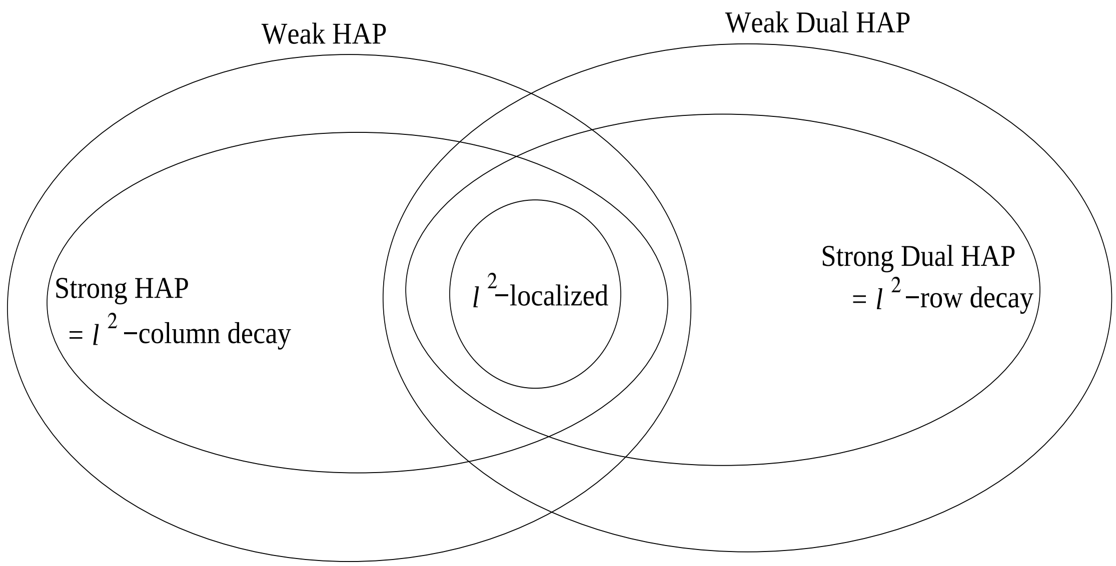
(b) $(\mathcal{F}, a, \mathcal{E})$ **has** ℓ^p*-column decay* **if** $\forall\, \varepsilon > 0$, $\exists\, N_\varepsilon > 0$ **such that**

$$\forall\, j \in G, \qquad \sum_{i\in I\setminus I_{N_\varepsilon}(j)} |\langle f_i, e_j\rangle|^p \;<\; \varepsilon$$

(c) $(\mathcal{F}, a, \mathcal{E})$ **has** ℓ^p*-row decay* **if** $\forall\, \varepsilon > 0$, $\exists\, N_\varepsilon > 0$ **such that**

$$\forall\, i \in I, \qquad \sum_{j\in G\setminus S_{N_\varepsilon}(a(i))} |\langle f_i, e_j\rangle|^p \;<\; \varepsilon$$

Relations among localization and HAP properties



Theorem **[B./C./H./L.]**

If

(a) $\mathcal{F} = \{f_i\}_{i \in I}$ **and** $\mathcal{E} = \{e_j\}_{j \in G}$ **are frame sequences,**

(b) $D^+(I) < \infty$**, and**

(c) $(\mathcal{F}, a, \mathcal{E})$ **has both** ℓ^2**-column and row decay,**

then

$$D(I) \cdot \mathcal{M}_{\mathcal{E}}(\mathcal{F}) \;=\; \mathcal{M}_{\mathcal{F}}(\mathcal{E})$$

where

$$\mathcal{M}_{\mathcal{E}}(\mathcal{F}) \;=\; \begin{cases} \textbf{Limits of averages of diagonal} \\ \textbf{elements of } \left[\langle P_{\mathcal{E}} f_i, \tilde{f}_j \rangle\right]_{i,j \in I} \end{cases}$$

and

$$\mathcal{M}_{\mathcal{F}}(\mathcal{E}) \;=\; \begin{cases} \textbf{Limits of averages of diagonal} \\ \textbf{elements of } \left[\langle P_{\mathcal{F}} e_i, \tilde{e}_j \rangle\right]_{i,j \in G} \end{cases}$$

Remark

"Limits" include Beurling-type upper and lower limits as well as ultrafilter limits.

Example

If $\mathcal{F}$ **is a frame and** $\mathcal{E}$ **is a Riesz basis then** $\mathcal{M}_{\mathcal{F}}(\mathcal{E}) = M(\mathcal{E}) = 1$**.**

Approximate Definition

$M^1 \;\approx\; \big\{f \in L^2 : f, \hat{f} \in L^1\big\}$

Theorem

(a) **If** $g \in L^2$ **and** $\varphi \in M^1$ **then**

$$\big(\mathcal{G}(g,\Lambda),\;\; a,\;\; \mathcal{G}(\varphi, \alpha\mathbf{Z} \times \beta\mathbf{Z})\big)$$

is ℓ^2**-localized.**

(b) **If** $g \in M^1$ **and** $\varphi \in M^1$ **then**

$$\big(\mathcal{G}(g,\Lambda),\;\; a,\;\; \mathcal{G}(\varphi, \alpha\mathbf{Z} \times \beta\mathbf{Z})\big)$$

is ℓ^1**-localized.**

Here $a(\lambda)$ = **closest point in** $\alpha\mathbf{Z} \times \beta\mathbf{Z}$

<u>Application 1: Necessary Density Conditions</u>

Let $\mathcal{G}(g,\Lambda) = \{M_\beta T_\alpha g\}_{(\alpha,\beta)\in\Lambda}$ be Gabor frame for $L^2(\mathbf{R})$. Then:

(a) $D^{\pm}(\Lambda) \;=\; \dfrac{1}{\mathcal{M}^{\mp}(\mathcal{G}(g,\Lambda))}$**.**

(b) $D^{-}(\Lambda) \geq 1$**.**

(c) **Riesz basis** $\implies D^{-}(\Lambda) = D^{+}(\Lambda) = 1$**.**

<u>Application 2: Relations between Density and Frame Bounds</u>

Let $\mathcal{G}(g,\Lambda) = \{M_\beta T_\alpha g\}_{(\alpha,\beta)\in\Lambda}$ be Gabor frame for $L^2(\mathbf{R})$ with frame bounds A, B. Then:

(a) $A \;\leq\; \|g\|_2^2\, D^{-}(\Lambda) \;\leq\; \|g\|_2^2\, D^{+}(\Lambda) \;\leq\; B$**.**

(b) **Tight frame ($A = B$)** $\implies D^{-}(\Lambda) = D^{+}(\Lambda)$**.**

<u>Application 3: Quantifying Excess; Feichtinger Conjecture</u>

Let $\mathcal{G}(g,\Lambda) = \{M_\beta T_\alpha g\}_{(\alpha,\beta)\in\Lambda}$ be Gabor frame for $L^2(\mathbf{R})$ with $g \in M^1$. Then:

(a) **If $D^{-}(\Lambda) > 1$, then there exists $J \subset \Lambda$ with $D^{-}(J) = D^{+}(J) > 0$ such that $\mathcal{G}(g,\Lambda\setminus J)$ is a frame for $L^2(\mathbf{R})$.**

(b) **$\mathcal{G}(g,\Lambda)$ can be written as a finite union of Riesz sequences.**

Application 4: Structure of the Dual Frame

Let $\mathcal{G}(g,\Lambda) = \{M_\beta T_\alpha g\}_{(\alpha,\beta)\in\Lambda}$ be Gabor frame for $L^2(\mathbf{R})$ with $g \in M^1$. Then:

(a) **The dual frame $\tilde{\mathcal{G}} = \{\tilde{g}_{\alpha,\beta}\}_{(\alpha,\beta)\in\Lambda}$ is also contained in M^1.**

(Gröchenig/Leinert is for $\Lambda =$ lattice only.)

(b) **The dual frame $\tilde{\mathcal{G}} = \{\tilde{g}_{\alpha,\beta}\}_{(\alpha,\beta)\in\Lambda}$ is a set of *Gabor molecules*, i.e., $\exists\, F \in L^1(\mathbf{R}^2)$ such that**

$$|V_\varphi(\tilde{g}_{\alpha,\beta})(x,\omega)| \;\le\; F(x-\alpha,\omega-\beta).$$

Compare:

$$|V_\varphi(M_\beta T_\alpha g)(x,\omega)| \;=\; |V_\varphi g(x-\alpha,\omega-\beta)|.$$

Remarks

(a) **Applications 1–4 continue to hold (with minor changes) if the Gabor frame $\mathcal{G}(g,\Lambda)$ is replaced by a frame of Gabor molecules $\{g_{\alpha,\beta}\}_{(\alpha,\beta)\in\Lambda}$.**

(b) **Applications 1–4 are only special cases of results for general localized frames.**

FINALE: CONCLUSIONS AND RELATED TOPICS

a. Localization is a powerful tool for dealing with frames which possess modest amounts of structure but are largely "irregular."

b. Insights into relations among density, redundancy, frame properties, the structure of the dual frame, ...

c. Extensions to families of associated spaces.

d. Insights and contrasts with wavelets.