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A Basis Theory Primer: Errata

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Errata

I. Scientific/Factual Errors

1. Page 10, in the line following Notation 1.11, replace "$3' = 4/3$" by "$3' = 3/2$".

2. Page 13, Exercise 1.11. This problem is only valid for indices p in the range $1 < p < \infty$.

3. Page 16, Remark 1.20. Change "$C_b(E)$" to "$C_b(\mathbf{R})$" on line 4 of the remark, and change "$L^\infty(E)$" to "$L^\infty(\mathbf{R})$" on line 5 of the remark.

4. Page 21, item (d) of Definition 1.25. Replace "$\{x_n\}$ is *complete*" with "S is *complete*".

5. Page 31, Exercise 1.30. Replace "$Ax \cdot x > 0$ for all $x \in \mathbf{F}^n$" with "$Ax \cdot x > 0$ for all nonzero $x \in \mathbf{F}^n$".

6. Page 54, final displayed equation (2 lines from bottom of page). Replace "$T(ax + by)$" with "$T(ay + bz)$".

7. Page 67, line 14, the displayed equation that gives the inductive definition of T_{n+1}. Replace "$n \geq 2$" with "$n \geq 1$".

8. Page 74, first line after equation (2.5). Replace "$\|x\|_X < r = s/2$" with "$\|x\|_Y < r = s/2$".

9. Page 74, last line of the proof of Lemma 2.26. Replace "$\|y\| < 1$" with "$\|z\| < 1$".

10. Page 99, line preceding equation (3.3). Replace "$F_0 \subseteq N$" with "$F_0 \subseteq \mathbf{N}$".

11. Page 100, statement (c) of Exercise 3.8. Replace the hypothesis "$(c_n) \in \ell^1$" with "$(c_n) \in \ell^\infty$".

12. Page 100, Notation 3.12. In the definitions of $\mathcal{R}$, $\mathcal{R}_\mathcal{E}$, and $\mathcal{R}_\Lambda$, replace "$F \subseteq N$" with "$F \subseteq \mathbf{N}$".

13. Page 110, line 14 (last line of first paragraph after the proof of Theorem 3.24). Replace "$\sum_{n=1}^N \|e_n\|_{L^2}^2$" with "$\sum_{n=1}^N \|f_n\|_{L^2}^2$".

14. Page 155, last sentence of "$\Leftarrow$" direction of Lemma 5.4. Replace "$m \in N$" with "$m \in \mathbf{N}$".

15. Page 155, Theorem 5.6. Add hypothesis so that the integers n_k tend to infinity. Specifically, replace the first sentence of the theorem by "Let $0 \le n_1 \le n_2 \le \cdots$ be an increasing sequence of nonnegative integers *such that* $n_k \to \infty$."

16. Page 175, line 2. Change plus sign between summations to minus. That is, replace

$$0 \;=\; x - Tx \;=\; \sum_n \langle x, a_n\rangle\, x_n + \sum_n \langle x, a_n\rangle\,(x_n - y_n) \;=\; \sum_n \langle x, a_n\rangle\, y_n,$$

with

$$0 \;=\; x - Tx \;=\; \sum_n \langle x, a_n\rangle\, x_n - \sum_n \langle x, a_n\rangle\,(x_n - y_n) \;=\; \sum_n \langle x, a_n\rangle\, y_n,$$

17. Page 178, line 10 of Notation 6.3. There is a missing subscript, change "$|\lambda| \le 1$" to "$|\lambda_n| \le 1$".

18. Page 183, 9 lines from bottom. Change "(d) $\Rightarrow$ (c)" to "(d) $\Rightarrow$ (b)".

19. Page 183, 6 lines from bottom. Change "Hence statement (c) holds" to "Hence statement (b) holds".

20. Page 201, Exercise 7.19. Change "Let $\{x_n\}$ be an orthonormal basis" to "Let $\{e_n\}$ be an orthonormal basis".

21. Page 208, Example 8.6(e). The frame bounds of $\{2e_1, e_2, e_3, \dots\}$ are $A = 1$, $B = 4$ (not $A = 1$, $B = 2$). A different frame that has frame bounds $A = 1$, $B = 2$ is $\{e_1, e_1, e_2, e_3, \dots\}$.

22. Page 214, Exercise 8.10. In line 2 of the exercise, change "$\{x_n\}$ to "$\{f_n\}$", and change "$\{y_n\}$" to "$\{g_n\}$".

23. Page 218, Proof of Lemma 8.17. The directions of the arrows should be reversed: replace "$\Leftarrow$" with "$\Rightarrow$", and replace "$\Rightarrow$" with "$\Leftarrow$".

24. Page 220, Exercise 8.15. (a) Add assumption that U and V are nonnegative. That is, change "$U \le V$" to "$0 \le U \le V$".
(b) Replace
$$0 \;\le\; I - \frac{2}{A+B}\,S \;\le\; \frac{B-A}{B+A}\,I,$$
by
$$\frac{A-B}{B+A}\,I \;\le\; I - \frac{2}{A+B}\,S \;\le\; \frac{B-A}{B+A}\,I.$$
The conclusion of the problem remains unchanged.

25. Page 229, last line of the proof of the "Rightarrow" direction of Corollary 8.30. Replace "UR" with "RU" twice, so that the line reads "Therefore $RU\colon H \to H$ is surjective and satisfies $RUe_n = x_n$."

26. Page 230, first line of the page. Replace "for a d-dimensional" with "for an n-dimensional".

27. Page 231, first line after Corollary 8.34. Replace "Corollary 8.33 is so named" with "Corollary 8.34 is so named".

28. Page 246, Exercise 8.40. In line 10 of the exercise, replace "$L(A/B)-\varepsilon$, 1" with "$L(A/B)-\varepsilon$, B". In line 12 of the exercise, replace "$L-\varepsilon$, 1" with "$L-\varepsilon$, B".

29. Page 251, equation (9.4). Replace "$\frac{1}{r}\widehat{f}(\xi/r)$" with "$\frac{1}{r^{1/2}}\widehat{f}(\xi/r)$".

30. Page 257, 9 lines from bottom. Replace "$e_\xi(t)$" with $e_\xi(x)$".

31. Page 267, last line of the page. The independent variable on the left-hand side of the equation should be ξ instead of x. That is, replace "$d_\pi(x)$" with "$d_\pi(\xi)$".

32. Page 267, Example 10.3. In both the first and seventh line of the example, replace "$d_\pi(x)$" with "$d_\pi(\xi)$".

33. Page 267, 6 lines from bottom. Replace "$L^2_{[\frac{1}{2},\frac{1}{2}]}$" with "$L^2_{[-\frac{1}{2},\frac{1}{2}]}$" (two times).

34. Page 270, Definition 10.2. Replace "$[\frac{1}{2},\frac{1}{2}]$" with "$[-\frac{1}{2},\frac{1}{2}]$" (two times).

35. Page 270, Example 10.3. In the next to last line of the example, replace "Theorem 9.3" with "Theorem 9.5(a)".

36. Page 270, Example 10.3. In the last line of the example, the independent variable in the final term should be ξ, and a minus sign is missing in the exponent. That is, replace

$$e^{2\pi iax}\chi_{[-\frac{1}{2},\frac{1}{2}]}(x) \qquad \text{with} \qquad e^{-2\pi ia\xi}\chi_{[-\frac{1}{2},\frac{1}{2}]}(\xi).$$

37. Page 272, 3 lines from bottom. Replace "$L^2_{[\frac{1}{2},\frac{1}{2}]}$" with "$L^2_{[-\frac{1}{2},\frac{1}{2}]}$".

38. Page 273, line 2. Replace "part (d)" with "part (e)".

39. Page 275, equation (10.3). The independent variable on the right-hand side of this equation should be x instead of ξ. That is, replace

$$f(x) \;=\; b\sum_{n\in\mathbf{Z}} f(bn)\,\frac{\sin\pi(\xi-bn)}{\pi(\xi-bn)}$$

with

$$f(x) \;=\; b\sum_{n\in\mathbf{Z}} f(bn)\,\frac{\sin\pi(x-bn)}{\pi(x-bn)}.$$

40. Page 276, 4 lines from bottom. Change inverse Fourier transform to Fourier transform. That is, replace

$$\widetilde{s}_n \;=\; \big(\widetilde{e}_n\,\chi_{[-\frac{1}{2},\frac{1}{2}]}\big)^{\vee} \qquad \text{with} \qquad \widetilde{s}_n \;=\; \big(\widetilde{e}_n\,\chi_{[-\frac{1}{2},\frac{1}{2}]}\big)^{\wedge}.$$

41. Page 285, Definition 10.17. In part (b), a subspace is *translation-invariant* if it is invariant under *every real translate*. Therefore, in part (b) replace "for every $a \in \mathbf{Z}$" with "for every $a \in \mathbf{R}$".

42. Page 287, 4 lines from the bottom. The index for the series on this line is k and not n. That is, "$\sum_{n\in\mathbf{Z}} c_k\,T_k g$" should be replaced with "$\sum_{k\in\mathbf{Z}} c_k\,T_k g$".

43. Page 307, 6 lines from the bottom. A square is missing on the norm: replace $\|f\cdot T_{ak}\bar{g}\|_{L^2(I_k)}$ with $\|f\cdot T_{ak}\bar{g}\|^2_{L^2(I_k)}$.

44. Page 315, Exercise 11.12. Replace "$\{T_k g\}_{k\in\mathbf{Z}}$" with "$\{T_{ak} g\}_{k\in\mathbf{Z}}$".

45. Page 316, line 4. Replace $\mathcal{G}(g,1,1)$ with $L^2(\mathbf{R})$, so that it reads "it follows that $\mathcal{G}(g,1,1)$ is also an orthonormal basis for $L^2(\mathbf{R})$".

46. Page 316, Definition 11.12. Replace "consists of those functions $f \in L^p(\mathbf{R})$" with "consists of those functions $f\colon \mathbf{R} \to \mathbf{C}$".

47. Page 318, Lemma 11.15. Replace "bounded" with "essentially bounded".

48. Page 320, line following equation (11.15). Replace "$W(L^\infty, L^1)$" with "$W(L^\infty, \ell^1)$".

49. Page 321, line 2 of the proof of Theorem 11.8. An "f" is missing in the series on the right-hand side. That is, the displayed equation should read

$$Lf \;=\; b^{-1} \sum_{n\in\mathbf{Z}} T_{\frac{n}{b}} f \cdot G_n$$

50. Page 321, line 4 of the proof of Theorem 11.8. A subscript L^2 is missing from the norm of Lf. That is, the displayed equation should read

$$\|Lf\|_{L^2} \;\le\; b^{-1} \sum_{n\in\mathbf{Z}} \|T_{\frac{n}{b}} f\|_{L^2}\, \|G_n\|_{L^\infty} \;\le\; B\, \|f\|_{L^2}$$

51. Page 323, line 8. Replace "Since T is bounded" with "Since L is bounded".

52. Page 328, two lines before Theorem 11.26. Change "algebraic geometry" to "algebraic topology".

53. Page 330, 8 lines from bottom. Replace "e^{-r}" with "$e^{-\pi r}$".

54. Page 331, Exercise 11.22. The first sentence of the exercise should read

    "If $f \in L^1(\mathbf{R}) = W(L^1, \ell^1)$ then $Zf \in L^1(Q)$ by Theorem 11.22."

    Also, in the statement of part (d), replace "$Z^{-1}\colon L^1(Q) \to L^1(\mathbf{R})$" with "range$(Z)\colon L^1(Q) \to L^1(\mathbf{R})$".

55. Page 333, third line of proof of part (b) of Theorem 11.31, replace "$\widetilde{g} \in L^2(Q)$" with "$\widetilde{g} \in L^2(\mathbf{R})$".

56. Page 349, first line of Exercise 11.41. Replace "$\Lambda = \{(p_k, q_k)\}_{k=1}^N$" with "$\Lambda = \{(\alpha_k, \beta_k)\}_{k=1}^N$".

57. Page 354, last line of Exercise 12.2. Replace "Heisenberg affine group" with "Heisenberg group".

58. Page 364, line 6. Replace "$W_0(a\xi) = W_0(\xi)$" with "$W(a\xi) = W(\xi)$".

59. Page 367, two lines after equation (12.17). Change "equations (12.16) and (12.17)" to "equations (12.15) and (12.17)".

60. Page 372, change "$f \in V_n$" to "$f \in V_0$" in the statement of part (a) of Lemma 12.10, so that it reads:

    (a) $V_n = D_{2^n}(V_0) = \{f(2^n x) : f \in V_0\}$.

61. Page 374, in equation (12.22) the characteristic function should be on the interval $\big[\frac{k}{2^n}, \frac{k+1}{2^n}\big)$. That is, equation (12.22) should read:

$$c_{k,n} \;=\; 2^n \,\langle f, \chi_{[\frac{k}{2^n},\frac{k+1}{2^n})}\rangle \;=\; 2^n \int_{k/2^n}^{(k+1)/2^n} f(x)\,dx \qquad (12.22)$$

62. Page 378, five lines from bottom of page. Change the exponent of the exponential from "$-2\pi i(n+k)\xi$" to "$-2\pi i n(\xi+k)$", so that this line reads

$$= \int_0^1 \sum_{k\in\mathbf{Z}} |\widehat{\varphi}(\xi+k)|^2\, e^{-2\pi i n(\xi+k)}\, d\xi$$

63. Page 397, first two lines of the proof of part (b) of Corollary 12.26. Replace "$P \in L^2(\mathbf{T})$" with "$P \in L^2(\mathbf{R})$", and replace "$\varphi = \check{P} \in L^2(\mathbf{T})$" with "$\varphi = \check{P} \in L^2(\mathbf{R})$".

64. Page 402, part (a) of Exercise 12.26. Change "$e^{\pi i\xi}$" to "$e^{-\pi i\xi}$", so that statement (a) reads

    (a) Show that $\widehat{\chi}(\xi) = e^{-\pi i\xi}\,\frac{\sin \pi\xi}{\pi\xi}$.

65. Page 406, statement of Theorem 12.30. The hypothesis "$\widehat{\varphi}(0) = 1$" needs to be added to the theorem. To do this, replace "If we set" with "If $\widehat{\varphi}(0) = 1$, and we set".

66. Page 413, line 11. Replace "$[\widehat{f}, \widehat{g}]$" in the left-hand side of the displayed equation with "$[\widehat{f}, \widehat{\varphi}]$".

67. Page 415, displayed equation in the statement of part (b) of Theorem 12.33, replace "φ" with "$\widehat{\varphi}$", so the equation reads

$$\widehat{\psi}(\xi) \;=\; m_1(\xi/2)\,\widehat{\varphi}(\xi/2) \quad \text{where} \quad m_1(\xi) \;=\; e^{-2\pi i\xi}\,\overline{m_0(\xi+\tfrac{1}{2})}.$$

68. Page 415, last 2 lines on page. Replace "ξ" with "$\xi/2$" four times, so that the last two lines of the page read

$$= \; |m_1(\tfrac{\xi}{2})|^2 + |m_1(\tfrac{\xi}{2}+\tfrac{1}{2})|^2$$

$$= \; |m_0(\tfrac{\xi}{2}+\tfrac{1}{2})|^2 + |m_0(\tfrac{\xi}{2})|^2 \;=\; 1 \quad \text{a.e.} \qquad (12.63)$$

69. Page 434, Exercise 13.3. The left-hand side of the displayed equation is missing one absolute value symbol, it should read:

$$|\widehat{f}(n)| \;\le\; \frac{\|f'\|_{L^1}}{2\pi\,|n|}, \qquad n \neq 0.$$

70. Page 437, Theorem 13.10. The statement of part (b) of this theorem should read as follows:

    (b) If $c,\, d \in \ell^1(\mathbf{Z})$ then $(c*d)^\wedge$ is the function

$$(c*d)^\wedge(x) \;=\; \widehat{c}(x)\,\widehat{d}(x).$$

71. Page 440, Definition 13.12. Replace "dt" in part (a) with "dx", so that the equation reads "$\int_0^1 k_N(x)\,dx = 1$ for every N".

72. Page 454, part (a) of Problem 13.26. Replace "$L^1(\mathbf{T})$" with "$\ell^1(\mathbf{Z})$".

73. Page 463, last line of displayed equation immediately preceding *Step 3*. Replace "$(C+1)$" with "$2C$".

74. Page 463, line 4 of *Step 3*. Replace "$(2C+2)$" with "$4C$".

75. Page 464, 3 lines before the Exercises for Section 14.2. Replace "$\mathbf{Z} = \{0,-1,1,2,-2,\dots\}$" with "$\mathbf{Z} = \{0,-1,1,-2,2,\dots\}$".

76. Page 488, line 2 of Exercise B.8. Replace "$e_m \otimes e_n$" with "$e_m \otimes f_n$".

77. Page 488, line 2 of part (e) of Exercise B.9. Replace "$L^2(R)$" with "$L^2(F)$".

78. Page 489, line 5 of the page (first line of the proof of Theorem B.13). Change "Fix $k \in L^2(\mathbf{R}^2)$" to "Fix $k \in L^2(E \times F)$".

79. Page 489, lines 14, 17, and last line of the page. Replace "$e_m \otimes e_n$" with "$e_m \otimes f_n$".

80. Page 490, lines 13 and 14. Replace "$e_m \otimes e_n$" with "$e_m \otimes f_n$".

81. Page 490, line 18. Replace

$$L_k f \;=\; \sum_{m,n} \langle Te_m, f_n\rangle\, \langle f, e_m\rangle\, e_n$$

    with

$$L_k f \;=\; \sum_{m,n} \langle Te_m, f_n\rangle\, \langle f, e_m\rangle\, f_n$$

82. Page 499, hint for Exercise 8.19. For the implication (b) $\implies$ (a), replace "$RV^* = C^*V^* = (CV)^* = I$" with "$RV^* = C^*V^* = (VC)^* = I$".

83. Page 500, hint for Exercise 8.40. On lines 3 and 5 of the hint, replace "$\{S^{-1}x_n\}$" with "$\{S^{1/2}x_n\}$".

84. Page 507, line 5 of the hint for Problem 12.30. Replace

$$h(x) = \frac{h(x)}{2}$$

with

$$h(x) = \frac{h(2x)}{2}$$

II. Other Errors

1. Page 163, 4 lines before Definition 5.14. Replace "by by" with "by".

2. Page 343, Conjecture 11.39. Conjecture should end with a "$\diamondsuit$" symbol.

III. Comments (not errors)

1. Page 211, Conjecture 8.9. Not errata, but an update: The Kadison–Singer Conjecture has been proved! The proof is in the paper

   A. Marcus, D. Spielman, and N. Srivastava, Interlacing Families II: Mixed Characteristic Polynomials and the Kadison-Singer Problem, *Annals of Mathematics (2)*, **182** (2015), 327-350.

   Casazza and Tremain had earlier proved that the Feichtinger Conjecture is equivalent to Kadison–Singer, see

   P. G. Casazza and J. C. Tremain, The Kadison–Singer problem in mathematics and engineering, *Proc. Natl. Acad. Sci. USA*, **103** (2006), 2032–2039.

   Therefore the Feichtinger Conjecture is now known to be true.

2. Page 221, Theorem 8.21. Just a clarification, not a typo. The displayed equation in the statement of the theorem holds true if (c_n) does not belong to ℓ^2. In this case $\sum |c_n|^2 = \infty$ but we know that $\sum |\langle x, \widetilde{x}_n \rangle|^2 < \infty$, so the combination of these two facts implies that $\sum |\langle x, \widetilde{x}_n \rangle - c_n|^2 = \infty$. Therefore equality does hold in the displayed equation when $(c_n) \notin \ell^2$, and likewise the conclusion that $(\langle x, \widetilde{x}_n \rangle)$ has minimal ℓ^2-norm among all sequences (c_n) that satisfy $\sum c_n x_n = x$ is correct even when we include those sequences (c_n) that do not belong to ℓ^2.

3. Page 310, the Remark to Exercise 11.5. Not errata, but an update: Xin-Rong Dai and Qiyu Sun have solved this! They completely characterize the set of points (a, b) such that $\mathcal{G}(\chi_{[0,1]}, a, b)$ is a frame in their paper

   X.-R. Dai and Q. Sun, The *abc*-problem for Gabor systems, *Memoirs of the American Mathematical Society*, **244** (2016), no. 1152.

   Also see their paper

   X.-R. Dai and Q. Sun, "The *abc*-problem for Gabor systems and uniform sampling in shift-invariant spaces," in: *Excursions in Harmonic Analysis, Vol. 3*, Birkhäuser/Springer, New York, 2015, pp. 177–194.

4. Page 312, line 2 of the proof of Corollary 11.7. The displayed equation is correct, but the second inequality can be replaced by an equality, so that it reads

$$Aab \;=\; \int_0^a Ab\,dx \;\leq\; \int_0^a \sum_{k\in\mathbf{Z}} |g(x-ak)|^2\,dx \;=\; \int_{-\infty}^{\infty} |g(x)|^2\,dx \;=\; \|g\|_{L^2}^2.$$

5. Page 343, Conjecture 11.39. Not errata, but an update: This *HRT Subconjecture* has been solved by Kasso Okoudjou and Vignon Oussa! Their paper is available on the Math arXiv:

   K. A. Okoudjou and V. Oussa, *The HRT conjecture for two classes of special configurations*, preprint (2021); arXiv:2110.04053.

   In fact, they prove that if $g \in L^2(\mathbf{R})$ is not the zero function, then

$$\{g(x),\; g(x-1),\; e^{2\pi i x} g(x),\; e^{2\pi i b x} g(x-a)\}$$

   is linearly independent for any choice of (a, b) distinct from $(0, 0)$, $(1, 0)$, and $(0, 1)$. The "simplest" currently open case appears to be the following.

   Conjecture (New HRT Subconjecture) If $g \in L^2(\mathbf{R}) \backslash \{0\}$ then

$$\{g(x),\; g(x-1/2),\; e^{2\pi i x} g(x),\; e^{2\pi i \sqrt{2} x} g(x-\sqrt{2})\}$$

   is linearly independent. $\diamondsuit$

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