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## Preface

A basis for a Banach space is a set of elementary building blocks that can be put together in a unique way to obtain any given element of the space. Bases are central to the study of the geometry of Banach spaces, which is a rich and beautiful classical subject in analysis. Bases and their relatives are also of key importance in classical and applied harmonic analysis, where they are used for decomposing and manipulating functions, operators, signals, images, and other objects.

In 1986 I was a young graduate student at the University of Maryland, College Park, just beginning to learn about harmonic analysis from my advisor, John Benedetto. I was at that awkward point of realizing that he wanted me to call him by his first name, yet feeling uncomfortable using any address less formal than “Dr. Benedetto” (for a long time I avoided this issue by never speaking to John unless he was already looking at me). One day John returned from a conference with news about an exciting new development, called “wavelets.” Together with John’s other graduate students of the time (David Walnut, William Heller, Rodney Kerby, and Jean-Pierre Gabardo), we began learning and thinking about wavelets. Through John we obtained early preprints of many papers that ultimately had a deep impact on the field. I recall spending many hours studying a xerox copy of a long preprint by Ingrid Daubechies, some 50 pages of meticulously handwritten single-spaced mathematics and exposition that eventually became published as the paper [Dau90]. I was especially fascinated by the use of *frames* in this paper, both wavelet frames and time-frequency (Gabor) frames. Frames are basis-like systems, but they allow the possibility of nonunique representations, and hence can incorporate redundancy. While uniqueness seems at first glance to be a property that we cannot live without, redundancy can actually be a useful and even essential property in many settings.

While reading these early papers, it became clear to our group that it would be important to understand the precise relationships between bases and frames, and John assigned me the task of becoming our resident expert on basis theory. Thus began a journey into the classical field of the geome-

try of Banach spaces. My instructors were the beautiful but comprehensive (even encyclopedic) volumes on bases in Banach spaces by Singer [Sin70], Lindenstrauss and Tzafriri [LT77], [LT79], and Marti [Mar69] (and if I had been aware of it at the time, the text by Diestel [Die84] would also be on this list). Additionally, while not specifically a basis theory volume, the elegant text on nonharmonic Fourier analysis by Young (now published as a “Revised First Edition” [You01]) provided a deep yet gentle introduction to both bases and frames that perfectly complemented the basis theory books mentioned above. A 1987 handwritten survey of what I learned about bases and frames circulated for some years among John’s students and colleagues. This survey was finally typed in 1997 and was the original incarnation of “A Basis Theory Primer.” Over the years, John and many others have asked me to turn that survey into a proper book, and the present volume is the result. The core material of the first Basis Primer has been greatly expanded and polished (hopefully benefiting from some 20 years of reflection on these subjects). Many new topics have been added, including chapters on Gabor bases and frames, wavelet bases and frames, and Fourier series (which are bases of complex exponentials). Introductory chapters on Banach spaces and functional analysis have also been included, which make the text almost entirely self-contained. A solutions manual for this volume is also available for instructors upon request at the Birkhäuser website.

## Outline and Goals

A *primer* is a old-fashioned word for a school book,<sup>1</sup> and this is a text for learning the theory of bases and frames and some of their appearances in classical and applied harmonic analysis. Extensive exercises complement the text and provide opportunities for learning by doing (hints for selected exercises appear at the end of the volume).

The text is divided into four parts. Part I reviews the functional analysis that underlies most of the concepts presented in the later parts of the text. Part II presents the abstract theory of bases and frames in Banach and Hilbert spaces. It begins with the classical topics of convergence, Schauder bases, biorthogonal systems, and unconditional bases, and concludes with more “modern” topics, such as Riesz bases and frames in Hilbert spaces. Part III is devoted to a study of concrete systems that form frames or bases for various Hilbert spaces. These include systems of weighted exponentials, systems of translations, Gabor systems, and wavelets. Each of these play important roles both in mathematics and in applications such as digital signal processing. It has become common to refer to these ideas as being part of the field of “applied harmonic analysis.” Part IV is concerned with the theory of Fourier series, which is usually considered to be part of “classical harmonic analysis,” although it is also widely applicable. Our presentation emphasizes

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<sup>1</sup>The correct pronunciation is *prim-er*, not *prime-er*.

the role played by bases, which is a different viewpoint than is taken in most discussions of Fourier series.

In summary, Parts I and II deal with the abstract development of bases and frames, while Parts III and IV apply these concepts to particular situations in applied and classical harmonic analysis. The mathematical tools needed to understand the abstract theory are the basic ideas of functional analysis and operator theory, which are presented in Part I. Measure theory makes only limited appearances in Parts I and II, and is used in those parts mostly to give examples. However, the theory of Lebesgue measure and Lebesgue integration is needed throughout Parts III and IV, and therefore a short review of basic ideas from measure theory is given in Appendix A. A small number of proofs and exercises in the text use the concepts of compact operators or tensor products of Hilbert spaces, and these topics are briefly reviewed in Appendix B.

We discuss each part of the text in more detail below.

### **Part 1: A Primer on Functional Analysis**

This part presents the background material that is needed to develop the theory of bases, frames, and applications in the remainder of the text. Most of this background is drawn from functional analysis and operator theory. Chapter 1 introduces Banach and Hilbert spaces and basic operator theory, while Chapter 2 presents more involved material from functional analysis, such as the Hahn–Banach Theorem and the Uniform Boundedness Principle. Proofs of most results are included, as well as extensive exercises.

### **Part II: Bases and Frames**

Part II develops the abstract theory of bases and frames in Banach and Hilbert spaces. Chapter 3 begins with a detailed account of convergence of infinite series in Banach spaces, focusing especially on the meaning of unconditional convergence. Chapter 4 defines bases and derives their basic properties, one of the most important of which is the fact that the coefficient functionals associated with a basis are automatically continuous. The Schauder, Haar, and trigonometric systems are studied as concrete examples of bases in particular Banach spaces. Generalizations of bases that allow weak or weak\* convergence instead of norm convergence are also discussed.

Chapter 5 continues the development of basis theory by examining the subtle distinctions between true bases, which provide unique infinite series representations of vectors in Banach spaces, and exact systems, which possess the minimality and completeness properties enjoyed by bases but which do not yield the infinite series representations that bases provide. This understanding allows us to extend the basis properties of the Haar system to  $L^p[0, 1]$ , to derive results on the stability of bases under perturbations, and to characterize the

basis properties of the sequence of coefficient functionals associated with a basis.

In Chapter 6 we consider bases that have the important extra property that the basis expansions converge unconditionally, i.e., regardless of ordering. We see that the Schauder system, which was proved by Schauder to be a basis for the space  $C[0, 1]$  of continuous functions on  $[0, 1]$ , fails to be an unconditional basis for that space, as does the Haar system in  $L^1[0, 1]$ .

Chapter 7 turns to the study of bases in the setting of Hilbert spaces. Unlike generic Banach spaces, Hilbert spaces have a notion of orthogonality, and we can take advantage of this additional structure to derive much more concrete results. We consider orthonormal bases, Riesz bases, unconditional bases, and the more general concept of Bessel sequences in Hilbert spaces. A key role is played by the *analysis operator*, which breaks a vector into a sequence of scalars that (we hope) captures all the information about a vector in a discrete fashion, much as a musical score is a discrete representation of a symphony as a “sequence of notes.” Conversely, the *synthesis operator* forms (infinite) linear combinations of special vectors (perhaps basis vectors), much as the musicians in an orchestra create the symphony by superimposing their musical notes. When these special vectors form a basis, we have unique representations. Analysis and synthesis are injective in this case, and their composition is the identity operator.

Chapter 8 is inspired by a simple question: Why do we need *unique* representations? In many circumstances it is enough to know that we have a set of vectors that we can use as building blocks in the sense that any vector in the space can be represented as *some* suitable superposition of these building blocks. This leads us to define and study frames in Hilbert spaces, which provide such nonunique representations. Moreover, frames also have important “stability” properties such as unconditional convergence and the existence of an equivalent discrete-type norm for the space via the analysis operator. Frames provide a mathematically elegant means for dealing with nonunique or redundant representations, and have found many practical applications both in mathematics and engineering.

### **Part III: Bases and Frames in Applied Harmonic Analysis**

In Part III of the text, we focus on concrete systems that form bases or frames in particular Hilbert spaces. The material in this part, and also in Part IV, does require more fluency with the tools of measure theory than was needed in Parts I or II. Appendix A contains a brief review, without proofs, of the main results from measure theory that are used in the text, such as the Lebesgue Dominated Convergence Theorem and Fubini’s Theorem.

Much of what we do in Part III is related to what is today called “applied harmonic analysis.” The Fourier transform on the real line is a fundamental tool for the analysis of many of these systems, and therefore Chapter 9 presents

a short review of this topic. For the purposes of this volume, the most essential facts about the Fourier transform are that:

- (i) the Fourier transform is a unitary mapping of  $L^2(\mathbf{R})$  onto itself, and
- (ii) the operations of translation and modulation are interchanged when the Fourier transform is applied.

Chapter 10 explores several important frames that are related to the trigonometric system  $\{e^{2\pi inx}\}_{n \in \mathbf{Z}}$ . The trigonometric system is an orthonormal basis for the Hilbert space  $L^2[0, 1]$ , but by embedding it into  $L^2(\mathbf{R})$  and applying the Fourier transform we magically obtain a fundamental result in signal processing known as the Classical Sampling Theorem: A bandlimited function  $f \in L^2(\mathbf{R})$  with  $\text{supp}(\hat{f}) \subseteq [-\frac{1}{2}, \frac{1}{2}]$  can be recovered from its sample values  $\{f(n)\}_{n \in \mathbf{Z}}$ . Moreover, the fact that  $\{e^{2\pi inbx}\}_{n \in \mathbf{Z}}$  is a frame for  $L^2[0, 1]$  when  $0 < b < 1$  translates into a statement about stable reconstruction by oversampling. Also discussed in Chapter 10 are two closely related types of systems, namely systems of weighted exponentials  $\{e^{2\pi inx}\varphi(x)\}_{n \in \mathbf{Z}}$  in  $L^2[0, 1]$  and systems of translates  $\{g(x - k)\}_{k \in \mathbf{Z}}$  in  $L^2(\mathbf{R})$ .

In Chapter 11 we analyze *Gabor systems* in  $L^2(\mathbf{R})$ . These are generated by simple *time-frequency shifts* of a single function, and have the form  $\{e^{2\pi ibnx}g(x - ak)\}_{k, n \in \mathbf{Z}}$  where  $g \in L^2(\mathbf{R})$  and  $a, b > 0$  are fixed. Thus a Gabor system incorporates features from both systems of weighted exponentials and systems of translates. The elements  $e^{2\pi ibnx}g(x - ak)$  of a Gabor system are much like notes of different frequencies played at different times that are superimposed to create a symphony. The theory of Gabor frames and bases, which is named in honor of the Nobel prize winner Dennis Gabor, is not only mathematically beautiful but has a great utility in mathematics, physics, and engineering. We will see that the mathematical formulation of the quantum mechanical uncertainty principle forces us to rely on Gabor frames that are *not* bases—redundancy is essential to these frames. There do exist Gabor systems that are orthonormal or Riesz bases for  $L^2(\mathbf{R})$ , but the generator  $g$  of such a system cannot be a very “nice” function. Such a generator cannot simultaneously be continuous and have good decay at infinity; more precisely, the Heisenberg product  $\|xg(x)\|_{L^2} \|\xi\hat{g}(\xi)\|_{L^2}$  that appears in the uncertainty principle must be infinite.

Wavelet systems, which are the topic of Chapter 12, are also simply generated from a single function, but through *time-scale* shifts instead of time-frequency shifts. A wavelet system has the form  $\{a^{n/2}\psi(a^n x - mbk)\}_{k, n \in \mathbf{Z}}$ , where  $\psi \in L^2(\mathbf{R})$ ,  $a > 1$ , and  $b > 0$  are fixed. The Haar system is an example of a wavelet orthonormal basis, and has been known since 1910. Unfortunately, the Haar system, which is the wavelet system with  $a = 2$ ,  $b = 1$ , and  $\psi = \chi_{[0, 1/2)} - \chi_{[1/2, 1]}$ , is generated by a discontinuous function. The “wavelet revolution” of the 1980s began with the discovery of orthonormal wavelet bases generated by very nice functions (in striking contrast to the nonexistence of “nice” Gabor bases). In particular, we will encounter generators  $\psi$  that are either  $m$ -times differentiable and compactly supported, or are infinitely differentiable and have compactly supported Fourier transforms.

## Part IV: Fourier Series

In Part IV of this volume, which consists of Chapters 13 and 14, we see how basis theory relates to that part of classical harmonic analysis that deals with Fourier series. The main goal of these chapters is to prove that the trigonometric system  $\{e^{2\pi inx}\}_{n \in \mathbf{Z}}$  is a basis for  $L^p[0, 1]$  for each  $1 < p < \infty$ , although we will see that this basis is conditional when  $p \neq 2$ .

The basis properties of the trigonometric system are a beautiful application of the machinery developed earlier in Part II. However, to properly work with Fourier series we need an additional set of tools that were not required in the preceding chapters. This is one reason that Fourier series have been placed into a separate portion of the text. The tools we need, including convolution, approximate identities, and Cesàro summation, are quite elegant in themselves, not to mention very important in digital signal processing as well as mathematics. Chapter 13 is devoted to developing these tools, which then provide the foundation for our analysis of the basis properties of the trigonometric system in Chapter 14.

## Course Outlines

This text is a learning tool, suitable for independent study or as the basis for an advanced course. There are several options for building a course around this text, two of which are listed below.

**Course 1: Functional Analysis, Bases, and Frames.** A course on functional analysis, bases, and frames could focus on Parts I and II. This would be ideal for students who have not already had an in-depth course on functional analysis. Additionally, the material in Parts I and II does not require deep familiarity with Lebesgue measure or integration. Part I develops the most important tools of functional analysis and operator theory, and then Part II applies these tools to develop the theory of bases and frames.

**Course 2: Bases, Frames, Applied Harmonic Analysis, and Fourier Series.** A course for students already familiar with functional analysis can begin with Part II, and treat Chapters 1 and 2 as a quick reference guide on functional analysis and operator theory. This course would emphasize bases and frames and their roles in applied and classical harmonic analysis. The abstract theory of bases and frames contained in Part II of the text would not require expertise in measure theory, while the applications in Parts III and IV will require fluency with Lebesgue measure and integral on the part of the reader.

A solutions manual for instructors is available upon request; instructions for obtaining a copy are given on the Birkhäuser website.

## Further Reading

As the title emphasizes, this volume is a primer rather than an exhaustive treatment of the subject. There are many possible directions for the reader who wishes to learn more, including those listed below.

- *Functional Analysis*. Chapters 1 and 2 provide an introduction to operator theory and functional analysis. More detailed and extensive development of these topics is available in texts such as Conway [Con90], Folland [Fol99], Gohberg and Goldberg [GG01], and Rudin [Rud91], to name only a few.
- *Classical Basis Theory*. For the student interested in bases and the geometry of Banach spaces there are a number of classic texts available, including the volumes mentioned above by Singer [Sin70], Lindenstrauss and Tzafriri [LT77], [LT79], Marti [Mar69], and Diestel [Die84]. These books contain an enormous amount of material on basis theory. Most of the proofs on bases that we give in Chapters 3–6 are either adapted directly from or are inspired by the proofs given in these texts.
- *Frame Theory*. The recent text [Chr03] by Christensen provides a thorough and accessible introduction to both frames and Riesz bases. My own “Basis and Frame Primer” was the text by Young [You01], and his volume is still a gem that I highly recommend. Moreover, Young’s text is a standard reference on sampling theory and nonharmonic Fourier series, and it provides a wealth of fascinating and useful historical notes.
- *Gabor Systems and Time-Frequency Analysis*. Gabor systems form one part of the modern theory of time-frequency analysis, which is itself a part of applied harmonic analysis. For a much more extensive account of time-frequency analysis than appears in this volume I highly recommend the essential text [Grö01] by Gröchenig.
- *Wavelet Theory*. There are now many texts on wavelet theory, but the volume by Daubechies [Dau92] is a classic. We also recommend Hernández and Weiss [HW96], especially for wavelet theory in function spaces other than  $L^2(\mathbf{R})$ . The text [Wal02] by my mathematical sibling Walnut provides an introduction to wavelet theory and many of its applications. Moreover, Walnut’s text avoids measure theory and so is suitable for a course aimed at upper-level undergraduate students. For wavelet theory from the engineering point of view, we mention the texts by Mallat [Mal09], Strang and Nguyen [SN96], and Vetterli and Kovačević [VK95].
- *Fourier Series*. Chapters 13 and 14 delve into classical harmonic analysis, proceeding just far enough to develop the tools needed to understand the basis properties of the trigonometric system on  $[0, 1]$ , the one-dimensional torus. These same tools are the foundation of much of harmonic analysis, both on the torus, the real line, and abstractly. For further reading on harmonic analysis we suggest the volumes by Benedetto [Ben97], Katznelson [Kat04], Grafakos [Gra04], or the author’s forthcoming text [Heil].

### Acknowledgments

A text such as this is indebted to and has as its foundation many classic and recent volumes in Banach space theory, applied harmonic analysis, and classical harmonic analysis. Many of those volumes have been mentioned already in this preface, and others that have influenced the writing are listed in the references. A few results due to the author have been included in Chapters 8–12, but aside from these, this volume is an exposition and introduction to results due to many others. As my goal was to make a text to learn from, I have not attempted to provide detailed historical accounts or attributions of results.

Lost in the depths of time are the names of the many people who provided feedback on my original basis primer surveys, both the handwritten 1987 notes and the typed 1997 version. I recall helpful criticisms from Jae Kun Lim and Georg Zimmermann, and thank everyone who commented on those manuscripts.

It is a pleasure to thank some of the many people who helped me in the process of writing this new version of the basis primer. Amit Einav, Ole Christensen, and David Walnut provided extensive feedback and identified many typographical errors in various parts of the manuscript. Kathy Merrill rescued me from my limited fluency with Mathematica by providing the wedding cake wavelet set pictured in Figure 12.4. Discussions with Radu Balan and Alex Powell during a snowy walk in Banff, Canada influenced the writing of Chapter 4.

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