FRAMES AND TIME-FREQUENCY ANALYSIS

LECTURE 2: GABOR FRAMES

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READING

For background on Hilbert spaces and operator theory:

Chapters 1–2 in C. Heil, *A Basis Theory Primer*, Birkhäuser, Boston, 2011.

For background on the Fourier transform:

Chapter 9 in C. Heil, *A Basis Theory Primer*, Birkhäuser, Boston, 2011.

Today's lecture is based upon:

Chapter 11 (Sections 11.1–11.5 and 11.9) in C. Heil, *A Basis Theory Primer*, Birkhäuser, Boston, 2011.

Also see:

C. E. Heil and D. F. Walnut, *Continuous and discrete wavelet transforms*, **SIAM Review, 31 (1989), pp. 628–666.**

For further reading:

K. Gröchenig, *Foundations of Time-Frequency Analysis*, Birkhäuser, Boston, 2001.

C. Heil, *History and evolution of the Density Theorem for Gabor frames*, **J. Fourier Anal. Appl., 13 (2007), pp. 113–166.**

BAD EXAMPLE

Let $\chi = \chi_{[0,1]}$. **Then**

$$\left\{e^{2\pi inx}\chi(x)\right\}_{n\in\mathbb{Z}}$$

is an ONB for $L^2[0,1]$. **Likewise**

$$\left\{e^{2\pi inx}\chi(x-k)\right\}_{n\in\mathbb{Z}}$$

is an ONB for $L^2[k,k+1]$. **Since the intervals** $[k,k+1]$ **are nonoverlapping,**

$$\left\{e^{2\pi inx}\chi(x-k)\right\}_{k,n\in\mathbb{Z}}$$

is an ONB for $L^2(\mathbb{R})$.

Disadvantage 1: χ **is discontinuous (but it does decay *quickly*).**

Disadvantage 2: $\widehat{\chi}(\xi) = e^{-\pi i\xi}\,\dfrac{\sin \pi\xi}{\pi\xi}$ **decays on the order of** $1/\xi$ **(but it is *smooth*).**

Question: Can we somehow generate an ONB for $L^2(\mathbb{R})$ **based on a single function** g **that is both *smooth* and *decays quickly*?**

Remark: Smoothness and decay are interchanged under the Fourier transform.

Translation: $(T_a f)(x) = f(x-a), \qquad a \in \mathbb{R}.$

Modulation: $(M_b f)(x) = e^{2\pi i b x} f(x), \qquad b \in \mathbb{R}.$

Dilation: $D_\lambda f(x) = \lambda^{1/2} f(\lambda x), \qquad \lambda > 0.$

Involution: $\widetilde{f}(x) = \overline{f(-x)}.$

Time-frequency shifts **are** $M_b T_a$ **and** $T_a M_b$ **(note** $T_a M_b = e^{-2\pi i a b} M_b T_a$**)**

FIGURE 1. The Gaussian window $\phi(x) = e^{-\pi x^2}$ and the real part of the time-frequency shift $M_3 T_5 \phi$.

(Regular or Lattice) Gabor (Gah-bor) System:

$$\mathcal{G}(g,a,b) = \{M_{bn} T_{ak} g\}_{k,n\in\mathbb{Z}} = \{e^{2\pi i b n x} g(x-ak)\}_{k,n\in\mathbb{Z}}.$$

g **is the** ***atom*** **and** $a\mathbb{Z} \times b\mathbb{Z}$ **is the** ***lattice.***

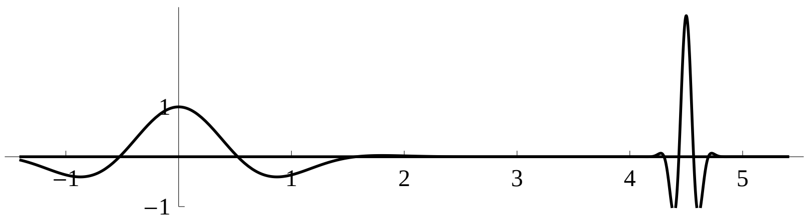


FIGURE 2. The function $\psi(x) = e^{-\pi x^2}\cos 3x$ and a time-scale shift $D_{2^3}T_{36}\psi(x) = 8^{1/2}\psi(8x-36)$.

(Regular) Wavelet (Wave-let) System:

$$\mathcal{W}(\psi,a,b) \;=\; \{D_{a^n}T_{bk}\psi\}_{k,n\in\mathbb{Z}} \;=\; \{a^{n/2}\psi(a^n x - bk)\}_{n\in\mathbb{Z}}.$$

Remark

The *Heisenberg group* is $\quad \mathbf{H} \;=\; \{e^{2\pi i t}T_aM_b\}_{a,b,t\in\mathbb{R}}.$

The *Affine group* is $\quad \mathbf{A} \;=\; \{D_aT_b\}_{a>0,b\in\mathbb{R}}.$

Both are nonabelian LCGs, but H is unimodular while A is not. $\mathbf{H} \cong \mathbb{T}\times\mathbb{R}\times\mathbb{R}$ **with group operation**

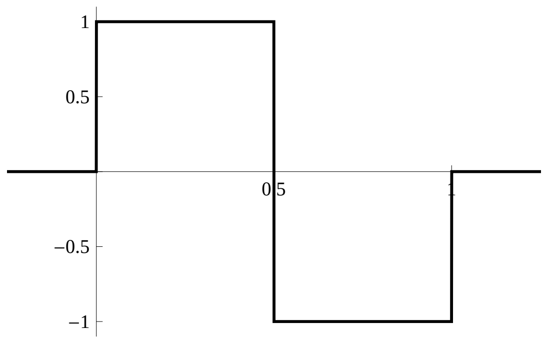
$$(z,a,b)\cdot(w,c,d) = (e^{-2\pi i a d}zw, a+c, b+d).$$

The *Schrödinger representation* is $(z,a,b)\mapsto zM_bT_a$.

TWO WAVELET ONBs

Haar System (1910): $\mathcal{W}(\psi, 2, 1) \;=\; \{2^{n/2}\psi(2^n x - k)\}_{n\in\mathbb{Z}}$ **with** $\psi = \chi_{[0,1/2]} - \chi_{[1/2,1]}$.

Daubechies W_4 System ($\approx$ 1988): $\mathcal{W}(W_4, 2, 1) \;=\; \{2^{n/2}W_4(2^n x - k)\}_{n\in\mathbb{Z}}$



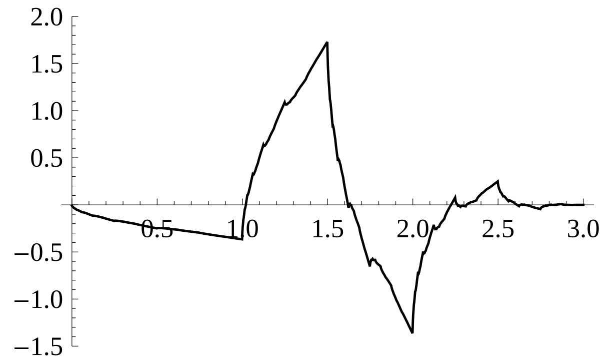


FIGURE 3. Left: Haar wavelet ψ. Right: Daubechies wavelet W_4.

Assuming

$$\mathcal{G}(g,a,b) \;=\; \{M_{bn}T_{ak}g\}_{k,n\in\mathbb{Z}} \;=\; \{e^{2\pi i bnx}g(x-ak)\}_{k,n\in\mathbb{Z}}$$

is a frame, its frame operator is

$$Sf \;=\; \sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \langle f, M_{bn}T_{ak}g\rangle\, M_{bn}T_{ak}g, \qquad f\in L^2(\mathbb{R}).$$

Assignment 1. Prove that the Gabor frame operator S commutes with M_{bn} and T_{ak}, and this implies that S^{-1} commutes with M_{bn} and T_{ak}. Therefore

$$S^{-1}(M_{bn}T_{ak}g) \;=\; M_{bn}T_{ak}(S^{-1}g). \qquad \diamondsuit$$

Since the canonical dual is the image of $\mathcal{G}(g,a,b)$ under S^{-1}, it has the form

$$\{S^{-1}(M_{bn}T_{ak}g)\}_{k,n\in\mathbb{Z}} \;=\; \{M_{bn}T_{ak}\widetilde{g}\}_{k,n\in\mathbb{Z}}, \;=\; \mathcal{G}(\widetilde{g},a,b), \qquad \textbf{where } \widetilde{g}=S^{-1}g.$$

Assignment 2. Assuming $\mathcal{G}(g,a,b)$ is a frame, prove the following.

(a) $\mathcal{G}(g,a,b)$ is a Riesz basis (i.e., exact) $\iff$ $\langle M_{bn}T_{ak}g, M_{bn'}T_{ak'}\widetilde{g}\rangle = \delta_{nn'}\delta_{kk'}$ $\iff$ $\langle g,\widetilde{g}\rangle = 1$.

(b) The canonical Parseval frame is $\mathcal{G}(g^\sharp,a,b)$ where $g^\sharp = S^{-1/2}g$. $\quad\diamondsuit$

Theorem 1 (Daubechies, Grossmann, Meyer)**.** Fix a, $b > 0$ and $g \in L^2(\mathbb{R})$.

(a) If $0 < ab \leq 1$ and $\mathrm{supp}(g) \subseteq [0, b^{-1}]$, then $\mathcal{G}(g, a, b)$ is a frame for $L^2(\mathbb{R})$ if and only if there exist constants A, $B > 0$ such that

$$Ab \;\leq\; \sum_{k\in\mathbb{Z}} |g(x - ak)|^2 \;\leq\; Bb \text{ a.e.} \tag{1}$$

In this case, A, B are frame bounds for $\mathcal{G}(g, a, b)$.

(b) If $0 < ab < 1$, then there exist g supported in $[0, b^{-1}]$ that satisfy (1) and are as smooth as we like (even infinitely differentiable).

(c) If $ab = 1$, then any g that is supported in $[0, b^{-1}]$ and satisfies (1) must be discontinuous.

(d) If $ab > 1$ and g is supported in $[0, b^{-1}]$, then (1) is not satisfied and $\mathcal{G}(g, a, b)$ is incomplete in $L^2(\mathbb{R})$.

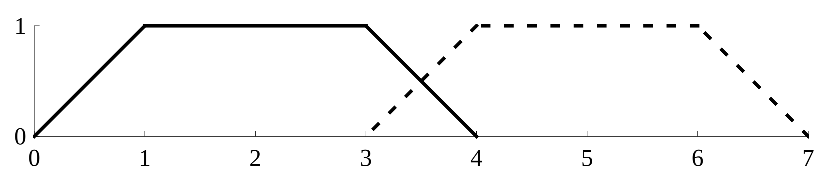


FIGURE 4. Example of $g(x)^2$ and $g(x-a)^2$ using $a = 3$ and $b = 1/4$.

Proof. (a) Set $e_{bn}(x) = e^{2\pi ibnx}$ and $I_k = [ak, ak + b^{-1}]$. Note $\{b^{1/2}e_{bn}\}_{n\in\mathbb{Z}}$ is an ONB for $L^2(I_k)$. Taking f in the dense subspace $C_c(\mathbb{R})$,

$$
\begin{aligned}
b\sum_{n\in\mathbb{Z}} \left|\langle f,\, M_{bn}T_{ak}g\rangle\right|^2 &= b\sum_{n\in\mathbb{Z}} \left|\int_{-\infty}^{\infty} f(x)\, \overline{e^{2\pi ibnx}\, g(x-ak)}\, dx\right|^2 \\
&= b\sum_{n\in\mathbb{Z}} \left|\int_{ak}^{ak+b^{-1}} f(x)\, \overline{g(x-ak)}\, e^{-2\pi ibnx}\, dx\right|^2 \\
&= \sum_{n\in\mathbb{Z}} \left|\langle f\cdot T_{ak}\overline{g},\, b^{1/2}e_{bn}\rangle_{L^2(I_k)}\right|^2 \\
&= \left\|f\cdot T_{ak}\overline{g}\right\|_{L^2(I_k)} \qquad \text{(Plancherel)} \\
&= \int_{ak}^{ak+b^{-1}} \left|f(x)\, T_{ak}\overline{g(x)}\right|^2 dx \;=\; \int_{-\infty}^{\infty} |f(x)\, g(x-ak)|^2\, dx \qquad \text{(only translations!)}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\sum_{k,n\in\mathbb{Z}} \left|\langle f,\, M_{bn}T_{ak}g\rangle\right|^2 &= b^{-1}\sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} |f(x)\, g(x-ak)|^2\, dx \\
&= b^{-1}\int_{-\infty}^{\infty} |f(x)|^2 \sum_{k\in\mathbb{Z}} |g(x-ak)|^2\, dx \\
&\geq \int_{-\infty}^{\infty} |f(x)|^2\, A\, dx \;=\; A\, \|f\|_{L^2(\mathbb{R})}^2.
\end{aligned}
$$

□

Summary:

- **If** $0 < ab < 1$ **then there exist nice Gabor frames for** $L^2(\mathbb{R})$.
- **If** $ab = 1$ **then there exist Gabor frames** $\mathcal{G}(g, a, b)$, **but the ones we constructed are discontinuous.**
- **If** $ab > 1$ **then we cannot construct a Gabor frame using this method.**

Assignment 3. Assume (with same conditions on g) that we have a frame, and set

$$G_0(x) \;=\; \sum_{k\in\mathbb{Z}} |g(x-ak)|^2.$$

Prove the following.

(a) The frame operator is $Sf(x) = b^{-1}G_0(x)\, f(x)$, and $S^{-1}f(x) = bf(x)/G_0(x)$.

(b) If $ab = 1$ then $\mathcal{G}(g, a, b)$ is a Riesz basis (= exact frame).

(c) If $ab < 1$ then $\mathcal{G}(g, a, b)$ is a redundant frame (overcomplete). $\diamondsuit$

Assignment 4. Given $g \in C_c(\mathbb{R})$, $g \neq 0$, prove there exist a, $b > 0$ such that $\mathcal{G}(g, a, b)$ is a frame for $L^2(\mathbb{R})$. $\diamondsuit$

Question (extremely difficult): For which a, $b > 0$ **is** $\mathcal{G}(\chi_{[0,1]}, a, b)$ **a frame?**

Theorem 2. Assume $\mathcal{G}(g,a,b)$ is a frame for $L^2(\mathbb{R})$ with frame bounds A, $B > 0$. Then

$$Ab \;\leq\; \sum_{k\in\mathbb{Z}} |g(x-ak)|^2 \;\leq\; Bb \;\text{ a.e.} \tag{2}$$

In particular, g must be bounded.

Proof. Note that we are not assuming that g has compact support, but *if f is supported in an interval I of length b^{-1}*, then

$$\begin{aligned}\int_{-\infty}^{\infty} |f(x)|^2 \sum_{k\in\mathbb{Z}} g(x-ak)|^2\,dx \;&=\; b\sum_{k,n\in\mathbb{Z}} \bigl|\bigl\langle f,\, M_{bn}T_{ak}g\bigr\rangle\bigr|^2 \qquad \text{(just as before)}\\ &\geq\; bA\,\|f\|_2^2 \qquad \text{(since it's a frame)}\\ &=\; bA\int_{-\infty}^{\infty} |f(x)|^2\,dx.\end{aligned}$$

Hence

$$\int_{-\infty}^{\infty} |f(x)|^2 \left(\sum_{k\in\mathbb{Z}} g(x-ak)|^2 - bA\right) dx \;\geq\; 0, \qquad \text{whenever } \operatorname{supp}(f) \subseteq I.$$

Therefore the lower inequality in (2) holds a.e. any interval I of length b^{-1}. □

Corollary 3 (Density and Frame Bounds)**.** If $\mathcal{G}(g,a,b)$ is a frame for $L^2(\mathbb{R})$ with frame bounds A, B, then the following statements hold.

(a) $Aab \le \|g\|_2^2 \le Bab$.

(b) If $\mathcal{G}(g,a,b)$ is Parseval $(A = B = 1)$, then $\|g\|_2^2 = ab$.

(c) $0 < ab \le 1$.

(d) $\langle g, \widetilde{g}\,\rangle = ab$, where $\widetilde{g} = S^{-1}g$.

(e) $\mathcal{G}(g,a,b)$ is a Riesz basis if and only if $ab = \langle g, \widetilde{g}\,\rangle = 1$.

Proof. (a) Recall that $\quad Ab \;\le\; \sum_{k\in\mathbb{Z}} |g(x-ak)|^2 \;\le\; Bb$ a.e.

Integrating over $[0,a]$:

$$Aab \;=\; \int_0^a Ab\,dx \;\le\; \int_0^a \sum_{k\in\mathbb{Z}} |g(x-ak)|^2\,dx \;=\; \int_{-\infty}^{\infty} |g(x)|^2\,dx \;=\; \|g\|_2^2.$$

(c) The canonical Parseval frame is $\mathcal{G}(g^\sharp,a,b)$ where $g^\sharp = S^{-1/2}g$. The elements of a Parseval frame have at most unit norm, so $ab \;=\; \|g^\sharp\|_2^2 \;\le\; 1$.

(d) Since $S^{-1/2}$ is self-adjoint,

$$\langle g, \widetilde{g}\,\rangle \;=\; \langle g, S^{-1/2}S^{-1/2}g\rangle \;=\; \langle S^{-1/2}g, S^{-1/2}g\rangle \;=\; \|g^\sharp\|_2^2 \;=\; ab.$$

(e) $\mathcal{G}(g,a,b)$ is a Riesz basis if and only if $\langle g, \widetilde{g}\,\rangle = 1$. □

Summary:

- **If** $ab > 1$ **then** $\mathcal{G}(g, a, b)$ **is not a frame.**
- **If** $\mathcal{G}(g, a, b)$ **is a frame and** $ab = 1$ **then it is a Riesz basis.**
- **If** $\mathcal{G}(g, a, b)$ **is a frame and** $0 < ab < 1$ **then it is a redundant frame.**

The value $1/(ab)$ **is called the *density* of the Gabor system** $\mathcal{G}(g, a, b)$, **because the number of points of** $a\mathbb{Z} \times b\mathbb{Z}$ **that lie in a given ball in** $\mathbb{R}^2$ **is asymptotically** $1/(ab)$ **times the volume of the ball as the radius increases to infinity. We refer to the density** $1/(ab) = 1$ **as the *critical density* or the *Nyquist density.***

Remark: $0 < ab \leq 1$ **is not *sufficient* for** $\mathcal{G}(g, a, b)$ **to be a frame. For example,**

$$\mathcal{G}(\chi_{[0,\frac{1}{2}]}, 1, 1)$$

is not a frame because its elements are supported on intervals $[k, k + \frac{1}{2}]$ **with** k **integer.**

The following example shows some of the limitations of L^p-norms.

Example 4. If $\chi = \chi_{[0,1)}$, then $\mathcal{G}(\chi, 1, 1)$ is an ONB for $L^2(\mathbb{R})$. Note that χ is discontinuous, but at least it is well-localized in time.

Create a new function ("Walnut's function") by dividing $[0, 1)$ into infinitely many pieces and then "sending those pieces off to infinity":

$$\begin{aligned} g &= \chi_{[0,\frac{1}{2})} + T_1\chi_{[\frac{1}{2},\frac{3}{4})} + T_2\chi_{[\frac{3}{4},\frac{7}{8})} + \cdots \\ &= \chi_{[0,\frac{1}{2})} + \chi_{[1+\frac{1}{2},1+\frac{3}{4})} + \chi_{[2+\frac{3}{4},2+\frac{7}{8})} + \cdots . \end{aligned}$$

Then g is not continuous and does not decay at infinity, but $\|g\|_2 = \|\chi\|_2$. Further, because we translated the "pieces" by integers, $\mathcal{G}(g, 1, 1)$ is an ONB for $L^2(\mathbb{R})$. $\diamondsuit$

We cannot distinguish a well-localized function from a poorly-localized function only by considering their L^p-norms.

We create a space of functions which are "locally L^p" and "globally ℓ^q."

Definition 5. Given $1 \le p \le \infty$ and $1 \le q < \infty$, the *Wiener amalgam space* $W(L^p, \ell^q)$ consists of those functions $f \in L^p_{\mathrm{loc}}(\mathbb{R})$ for which the norm

$$\|f\|_{W(L^p,\ell^q)} \;=\; \biggl(\sum_{k\in\mathbb{Z}} \|f \cdot \chi_{[k,k+1]}\|_p^q\biggr)^{1/q}$$

is finite. For $q = \infty$ we substitute the ℓ^∞-norm for the ℓ^q-norm above, i.e.,

$$\|f\|_{W(L^p,\ell^\infty)} \;=\; \sup_{k\in\mathbb{Z}} \|f \cdot \chi_{[k,k+1]}\|_p.$$

We also define

$$W(C,\ell^q) \;=\; \bigl\{f \in W(L^\infty,\ell^q) : f \text{ is continuous}\bigr\},$$

and we impose the norm $\|\cdot\|_{W(L^\infty,\ell^q)}$ on $W(C,\ell^q)$. $\diamondsuit$

Example 6. We will be most interested in $W(L^\infty, \ell^1)$.

- $\chi = \chi_{[0,1]} \in W(L^\infty,\ell^1) \subsetneq L^p(\mathbb{R})$ for every p.
- Walnut's function belongs to $L^p(\mathbb{R})$ for every p, but does not belong to $W(L^\infty,\ell^1)$. $\diamondsuit$

We will be most interested in $W(L^\infty, \ell^1)$:

$$\|f\|_{W(L^\infty,\ell^1)} \;=\; \sum_{k\in\mathbb{Z}} \|f \cdot \chi_{[k,k+1]}\|_\infty.$$

Theorem 7. Let $C_a \;=\; \max\bigl\{1+a,\, 2\bigr\}$.

(a) $W(L^\infty, \ell^1) \subseteq L^p(\mathbb{R})$ for $1 \le p \le \infty$, and is dense for $1 \le p < \infty$.

(b) $W(L^\infty, \ell^1)$ is closed under translations, and $\|T_b f\|_{W(L^\infty,\ell^1)} \;\le\; 2\, \|f\|_{W(L^\infty,\ell^1)}$.

(c) $W(L^\infty, \ell^1)$ is an ideal in $L^\infty(\mathbb{R})$, i.e., $f \in L^\infty(\mathbb{R}),\; g \in W(L^\infty, \ell^1) \implies fg \in W(L^\infty, \ell^1)$.

(d) For each $a > 0$,

$$|||f|||_a \;=\; \sum_{k\in\mathbb{Z}} \|f \cdot \chi_{[ak,a(k+1)]}\|_\infty$$

is an equivalent norm for $W(L^\infty, \ell^1)$, with

$$\frac{1}{C_{1/a}}\, |||f|||_a \;\le\; \|f\|_{W(L^\infty,\ell^1)} \;\le\; C_a\, |||f|||_a. \qquad \diamondsuit$$

Corollary 8. If $f \in W(L^\infty, \ell^1)$, then

$$\sum_{k\in\mathbb{Z}} \|T_{ak}f \cdot \chi_{[0,a]}\|_\infty \;=\; \sum_{k\in\mathbb{Z}} \|f \cdot T_{ak}\chi_{[0,a]}\|_\infty \;=\; \sum_{k\in\mathbb{Z}} \|f \cdot \chi_{[ak,a(k+1)]}\|_\infty \;\le\; C\,\|f\|_{W(L^\infty,\ell^1)}. \qquad \diamondsuit$$

Corollary 9. If $f \in W(L^\infty, \ell^1)$, then its a-periodization

$$\varphi(x) \;=\; \sum_{n\in\mathbb{Z}} T_{ak}f(x) \;=\; \sum_{n\in\mathbb{Z}} f(x-ak)$$

is bounded, and

$$\left\|\sum_{k\in\mathbb{Z}} T_{ak}f\right\|_\infty \;=\; \left\|\sum_{k\in\mathbb{Z}} T_{ak}f \cdot \chi_{[0,a]}\right\|_\infty \;\le\; C\,\|f\|_{W(L^\infty,\ell^1)}. \qquad \diamondsuit$$

The Painless Nonorthogonal Expansions require supp$(g) \subseteq [0, b^{-1}]$**. For a generic** $g \in L^2(\mathbb{R})$ **we can break** g **into pieces supported on intervals of length** b^{-1}**, analyze each piece, and past the pieces back together (to do this we will need to assume that** g **belongs to** $g \in W(L^\infty, \ell^1)$**).**

Definition 10. Given $g \in W(L^\infty, \ell^1)$ and a, $b > 0$, we define associated *correlation functions*

$$G_n(x) \;=\; \sum_{k \in \mathbb{Z}} g(x - ak)\, \overline{g(x - ak - \tfrac{n}{b})}, \qquad n \in \mathbb{Z}.$$

In particular, $G_0(x) = \sum_{k \in \mathbb{Z}} |g(x - ak)|^2$. $\diamondsuit$

The usual lattice $a\mathbb{Z} \times b\mathbb{Z}$ **and the *adjoint lattice*** $\frac{1}{b}\mathbb{Z} \times \frac{1}{a}\mathbb{Z}$ **implicitly appear here. Equivalent forms:**

$$G_n \;=\; \sum_{k \in \mathbb{Z}} T_{ak} g \cdot T_{ak + \frac{n}{b}} \bar{g} \;=\; \sum_{k \in \mathbb{Z}} T_{ak}\big(g \cdot T_{\frac{n}{b}} \bar{g}\big).$$

Thus G_n **is the** a**-periodization of** $g \cdot T_{\frac{n}{b}} \bar{g}$**. Since** $g \in W(L^\infty, \ell^1)$**, we have** $G_n \in L^\infty(\mathbb{R})$**.**

Lemma 11. If $g \in W(L^\infty, \ell^1)$ then

$$\sum_{n \in \mathbb{Z}} \|G_n\|_\infty \;\leq\; C \|g\|^2_{W(L^\infty, \ell^1)}.$$

Proof. We compute that

$$\|G_n\|_\infty \;=\; \Big\|\sum_{k\in\mathbb{Z}} T_{ak}\big(g\cdot T_{\frac{n}{b}}\bar{g}\big)\Big\|_\infty \;\le\; C\,\big\|g\cdot T_{\frac{n}{b}}\bar{g}\big\|_{W(L^\infty,\ell^1)}.$$

Therefore

$$\begin{aligned}\sum_{n\in\mathbb{Z}}\|G_n\|_\infty \;&\le\; C\sum_{n\in\mathbb{Z}}\big\|g\cdot T_{\frac{n}{b}}g\big\|_{W(L^\infty,\ell^1)}\\ &=\; C\sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\big\|g\cdot\chi_{[k,k+1]}\cdot T_{\frac{n}{b}}g\cdot\chi_{[k,k+1]}\big\|_\infty\\ &\le\; C\sum_{k\in\mathbb{Z}}\|g\cdot\chi_{[k,k+1]}\|_\infty\bigg(\sum_{n\in\mathbb{Z}}\|T_{\frac{n}{b}}g\cdot\chi_{[k,k+1]}\|_\infty\bigg).\end{aligned}$$

The series in parentheses on the last line resembles the $W(L^\infty,\ell^1)$ norm of $T_{\frac{n}{b}}g$, but it is not since the summation is over n instead of k. But for an appropriate constant C' we have

$$\begin{aligned}\sum_{n\in\mathbb{Z}}\|T_{\frac{n}{b}}g\cdot\chi_{[k,k+1]}\|_\infty \;&=\; \sum_{n\in\mathbb{Z}}\|g\cdot\chi_{[-\frac{n}{b}+k,-\frac{n}{b}+k+1]}\|_\infty\\ &\le\; C'\sum_{m\in\mathbb{Z}}\|g\cdot\chi_{[m,m+1]}\|_\infty \;=\; C'\,\|g\|_{W(L^\infty,\ell^1)}.\end{aligned}$$

Hence

$$\sum_{n\in\mathbb{Z}}\|G_n\|_\infty \;\le\; C\sum_{k\in\mathbb{Z}}\|g\cdot\chi_{[k,k+1]}\|_\infty\,C'\,\|g\|_{W(L^\infty,\ell^1)} \;\le\; C\,C'\,\|g\|^2_{W(L^\infty,\ell^1)}.$$ □

Theorem 12 (Walnut Representation)**.** If $g \in W(L^\infty, \ell^1)$ and $a,\ b > 0$, then $\mathcal{G}(g,a,b)$ is a Bessel sequence, with Bessel bound $B = C\,\|g\|^2_{W(L^\infty,\ell^1)}$ and frame operator

$$Sf \;=\; \sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \langle f, M_{bn}T_{ak}g\rangle\, M_{bn}T_{ak}g \;=\; b^{-1}\sum_{n\in\mathbb{Z}} T_{\frac{n}{b}}f\cdot G_n, \qquad f \in L^2(\mathbb{R}). \tag{3}$$

Proof. Because $g \in W(L^\infty, \ell^1)$, the series in (3) converges absolutely in $L^2(\mathbb{R})$, and

$$\|Sf\|_2 \;\le\; b^{-1}\sum_{n\in\mathbb{Z}} \|T_{\frac{n}{b}}f\|_2\,\|G_n\|_\infty \;\le\; B\,\|f\|_2, \qquad \text{where } B \;=\; C\,\|g\|^2_{W(L^\infty,\ell^1)}.$$

Fix $f \in C_c(\mathbb{R})$ and $k \in \mathbb{Z}$. Then $f\cdot T_{ak}\bar{g}$ is bounded and compactly supported, so its b^{-1}-periodization

$$F_k(x) \;=\; \sum_{j\in\mathbb{Z}} f\bigl(x-\tfrac{j}{b}\bigr)\,\overline{g\bigl(x-ak-\tfrac{j}{b}\bigr)}$$

belongs to $L^\infty[0,b^{-1}]$. We compute that

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} |\langle f, M_{bn}T_{ak}g\rangle|^2 \;&=\; \sum_{n\in\mathbb{Z}} \left|\int_{-\infty}^{\infty} f(x)\,e^{-2\pi i bnx}\,\overline{g(x-ak)}\,dx\right|^2 \\
&=\; \sum_{n\in\mathbb{Z}} \left|\int_0^{b^{-1}} \sum_{j\in\mathbb{Z}} f\bigl(x-\tfrac{j}{b}\bigr)\,e^{-2\pi i bn\left(x-\frac{j}{b}\right)}\,\overline{g\bigl(x-ak-\tfrac{j}{b}\bigr)}\,dx\right|^2 \\
&=\; \sum_{n\in\mathbb{Z}} \left|\int_0^{b^{-1}} \sum_{j\in\mathbb{Z}} f\bigl(x-\tfrac{j}{b}\bigr)\,\overline{g\bigl(x-ak-\tfrac{j}{b}\bigr)}\,e^{-2\pi i bnx}\,dx\right|^2 \\
&=\; \sum_{n\in\mathbb{Z}} \bigl|\bigl\langle F_k,\, e_{bn}\bigr\rangle_{L^2[0,b^{-1}]}\bigr|^2
\end{aligned}$$

$$= \left\|F_k\right\|^2_{L^2[0,b^{-1}]} = b^{-1} \int_0^{b^{-1}} |F_k(x)|^2 \, dx.$$

Using Fubini,

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} |\langle f, M_{bn}T_{ak}g\rangle|^2 &= b^{-1} \sum_{k\in\mathbb{Z}} \int_0^{b^{-1}} F_k(x)\, \overline{F_k(x)}\, dx \\
&= b^{-1} \sum_{k\in\mathbb{Z}} \int_0^{b^{-1}} \sum_{j\in\mathbb{Z}} f\big(x - \tfrac{j}{b}\big)\, \overline{g\big(x - ak - \tfrac{j}{b}\big)\, F_k\big(x - \tfrac{j}{b}\big)}\, dx \\
&= b^{-1} \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} f(x)\, \overline{g(x-ak)\, F_k(x)}\, dx \\
&= b^{-1} \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} f(x)\, \overline{g(x-ak)} \sum_{j\in\mathbb{Z}} \overline{f\big(x - \tfrac{j}{b}\big)}\, g\big(x - ak - \tfrac{j}{b}\big)\, dx \\
&= b^{-1} \sum_{j\in\mathbb{Z}} \int_{-\infty}^{\infty} f(x)\, \overline{f\big(x - \tfrac{j}{b}\big)} \sum_{k\in\mathbb{Z}} \overline{g(x-ak)}\, g\big(x - ak - \tfrac{j}{b}\big)\, dx \\
&= b^{-1} \sum_{j\in\mathbb{Z}} \int_{-\infty}^{\infty} f(x)\, \overline{f\big(x - \tfrac{j}{b}\big)\, G_j(x)}\, dx \\
&= \Big\langle f,\, b^{-1} \sum_{j\in\mathbb{Z}} T_{\frac{j}{b}} f \cdot G_j(x) \Big\rangle_{L^2(\mathbb{R})} \\
&= \big\langle f,\, Sf \big\rangle \;\le\; \|f\|_2 \, \|Sf\|_2 \;\le\; B\, \|f\|^2. \qquad \square
\end{aligned}$$

Remark: The Bessel bound can be given explicitly as

$$B = \frac{2}{b}\, C_{1/a}\, C_b\, \|g\|^2_{W(L^\infty,\ell^1)}, \qquad \textbf{where } C_a = \max\{1+a,\, 2\}. \qquad \diamondsuit$$

Assignment 5 (Frame Perturbations)**.** Let $\{x_n\}_{n\in\mathbb{N}}$ be a frame for a Hilbert space H with frame bounds A, B. Given $\{y_n\}_{n\in\mathbb{N}} \subseteq H$, prove that if $\{x_n - y_n\}_{n\in\mathbb{N}}$ is Bessel with Bessel bound $K < A$, then $\{y_n\}_{n\in\mathbb{N}}$ is a frame. (Typos in printouts.) $\diamondsuit$

Assignment 6 (Perturbation in amalgam norm)**.**

(a) Assume $\mathcal{G}(g, a, b)$ is a frame for $L^2(\mathbb{R})$. Show that there exists a $\delta > 0$ such that if $h \in L^2(\mathbb{R})$ and $\|g - h\|_{W(L^\infty,\ell^1)} < \delta$, then $\mathcal{G}(h, a, b)$ is a frame for $L^2(\mathbb{R})$.

(b) Does this remain true if we perturb in L^2-norm instead? $\diamondsuit$

Here is another consequence of the Walnut representation (but this takes some work to prove, e.g., Theorem 4.1.8 in H/Walnut survey "Continuous and discrete wavelet transforms").

Corollary 13. If $g \in W(L^\infty, \ell^1)$, then $\mathcal{G}(g, a, b)$ is a frame for all small enough a, b. $\diamondsuit$

We have studied Gabor systems $\mathcal{G}(g,a,b)$ that are complete, incomplete, a frame, exact, inexact, a Riesz basis, an ONB, etc. But what about the most basic linear algebra property of all?

Question 14 (HRT, 1996)**.** *Are Gabor systems finitely independent?*
(Reference: C. Heil, J. Ramanathan, and P. Topiwala, Linear independence of time-frequency translates, Proc. Amer. Math. Soc., 124 (1996), pp. 2787–2795.) $\diamondsuit$

***Independence* here is linear independence in the most ordinary linear algebra sense: If there is some *finite linear combination* of elements of $\mathcal{G}(g,a,b)$ that equals zero, then the system is dependent, otherwise it is independent.**

If $\mathcal{G}(g,a,b)$ is minimal (has a biorthogonal system), then certainly it is independent. In particular, every exact system, every Riesz basis, and every ONB is independent. But what about redundant Gabor frames? Are they redundant because they are linearly dependent? For *lattice* Gabor systems the answer is known (but the proof is far from trivial!).

Theorem 15 (Linnell, 1999)**.** If $g \in L^2(\mathbb{R})\backslash\{0\}$ and $a, b > 0$, then $\mathcal{G}(g,a,b)$ is independent. In particular, if $\Lambda = \{(a_k, b_k)\}_{k=1}^N \subseteq a\mathbb{Z} \times b\mathbb{Z}$ then $\mathcal{G}(g,\Lambda) = \{M_{b_k} T_{a_k} g\}_{k=1}^N$ is independent. $\diamondsuit$

For general Gabor systems the answer is unknown.

Conjecture 16 (HRT Conjecture)**.** If $g \in L^2(\mathbb{R})$ is not the zero function and $\Lambda = \{(a_k, b_k)\}_{k=1}^N$ is any set of finitely many distinct points in $\mathbb{R}^2$, then

$$\mathcal{G}(g, \Lambda) \;=\; \bigl\{M_{b_k} T_{a_k} g\bigr\}_{k=1}^N$$

is a linearly independent set of functions in $L^2(\mathbb{R})$. That is,

$$\sum_{k=1}^N c_k\, e^{2\pi i b_k x}\, g(x - a_k) \;=\; 0 \qquad \Longrightarrow \qquad c_1 = \cdots = c_N = 0. \qquad \diamondsuit$$

Assignment 7. Either prove that the HRT Conjecture is valid, or find a counterexample. Or, "for simplicity," just prove the special case given in the next conjecture. $\diamondsuit$

Conjecture 17 (HRT Subconjecture)**.** If $g \in L^2(\mathbb{R}) \backslash \{0\}$ then

$$\bigl\{g(x), \;\; g(x-1), \;\; e^{2\pi i x} g(x), \;\; e^{2\pi i \sqrt{2} x} g(x - \sqrt{2})\bigr\}$$

is linearly independent $\bigl($here, $\Lambda = \{(0,0), \; (1,0), \; (0,1), \; (\sqrt{2}, \sqrt{2})\}\bigr)$. $\diamondsuit$

Assignment 8. Let $\Lambda = \{(0,0), \; (1,0), \; (0,b)\}$. Linnell's theorem implies that HRT holds for $\mathcal{G}(g, \Lambda)$ for any nonzero $g \in L^2(\mathbb{R})$. Find a "simple" proof that HRT holds for this case! $\diamondsuit$

Evidence *for* HRT

If we use only time-shifts $g(x-a_k)$ or frequency shifts $e^{2\pi i b_k x} g(x)$ then it is true:

$$\sum_{k=1}^{N} c_k\, g(x-a_k) \;=\; 0 \textbf{ a.e.} \qquad \Longrightarrow \qquad \left(\sum_{k=1}^{N} c_k\, g(x-a_k)\right)^{\wedge}(\xi) \;=\; 0 \textbf{ a.e.}$$

$$\Longrightarrow \qquad \sum_{k=1}^{N} c_k\, e^{-2\pi i a_k \xi}\, \widehat{g}(\xi) \;=\; 0 \textbf{ a.e.}$$

$$\Longrightarrow \qquad \widehat{g}(\xi)\left(\sum_{k=1}^{N} c_k\, e^{-2\pi i a_k \xi}\right) \;=\; 0 \textbf{ a.e.}$$

$$\Longrightarrow \qquad \widehat{g} \;=\; 0 \textbf{ a.e.} \qquad \Longrightarrow \qquad g \;=\; 0 \textbf{ a.e.}$$

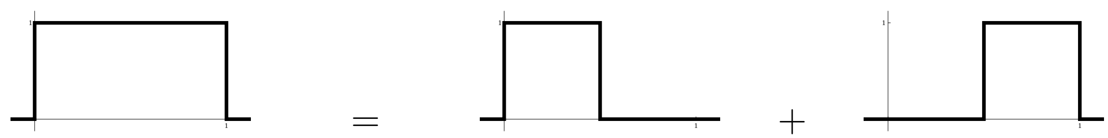
(Interesting side question: What if $g \in L^p$ with $p > 2$? $\widehat{g}$ is a distribution!)

Assignment 9. Use the fact that a nontrivial trigonometric polynomial $\sum_{k=1}^{N} c_k\, e^{-2\pi i a_k \xi}$ cannot vanish on any set of positive measure to prove that the HRT holds for all *compactly supported* atoms g (using any finite set Λ). $\diamondsuit$

<u>Evidence *against* HRT</u>

It's false if we use *time-scale* instead of *time-frequency*. Let $\chi = \chi_{[0,1]}$. Then

$$\chi(x) \;=\; \chi(2x) + \chi(2x-1).$$



The *Haar wavelet*, which generates a wavelet ONB for $L^2(\mathbb{R})$, is

$$\psi(x) \;=\; \chi(2x) - \chi(2x-1).$$

The box function χ is the *scaling function* for the Haar ONB.

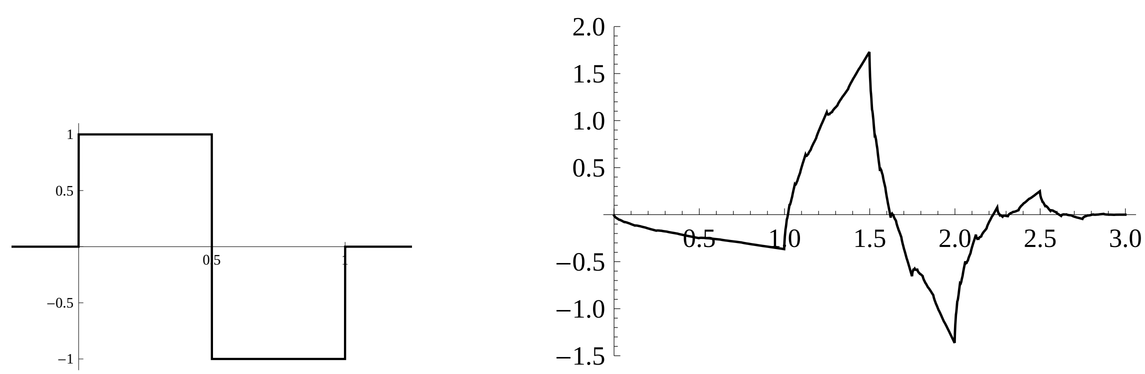
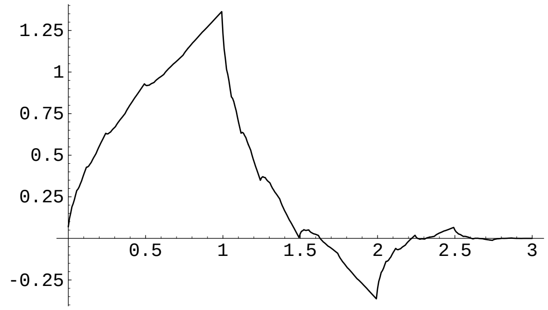


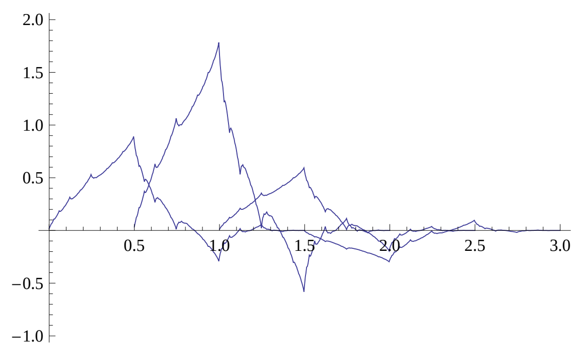
FIGURE 5. Left: Haar wavelet ψ. Right: Daubechies wavelet W_4.

The Daubechies D_4 function:



Time-scale shifts of D_4 are dependent:

$$D_4(x) \;=\; \tfrac{1+\sqrt{3}}{4}D_4(2x) + \tfrac{3+\sqrt{3}}{4}D_4(2x-1) + \tfrac{3-\sqrt{3}}{4}D_4(2x-2) + \tfrac{1-\sqrt{3}}{4}D_4(2x-3).$$



The function D_4 is a *scaling function* that generates a *multiresolution analysis.* This MRA gives rise to the *Daubechies wavelet* W_4, which generates a wavelet ONB for $L^2(\mathbb{R})$. *Dependency* is critical for the construction of wavelet ONBs!

Recall the unitary dilation operator $D_r f(x) = r^{1/2} f(rx)$. **We have**

$$\begin{aligned} D_r(M_{bn}T_{ak}g)(x) &= r^{1/2} M_{bn}T_{ak}g(rx) \\ &= r^{1/2} e^{2\pi i bnrx} g(rx - ak) \\ &= e^{2\pi i (bnrx)} r^{1/2} g(r(x - ak/r)) = M_{brn}T_{ak/r}D_r g(x). \end{aligned}$$

Therefore

$$D_r\big(\mathcal{G}(g,a,b)\big) = \big\{D_r(M_{bn}T_{ak}g)\big\}_{k,n\in\mathbb{Z}} = \big\{M_{brn}T_{ak/r}D_r g\big\}_{k,n\in\mathbb{Z}} = \mathcal{G}(D_r g, a/r, br).$$

Since D_r **is unitary,**

$$\mathcal{G}(g,a,b) \text{ is independent} \iff \mathcal{G}(D_r g, a/r, br) \text{ is independent}.$$

The dilation D_r **is one example of a** ***metaplectic transform.*** **We can replace the lattice** $a\mathbb{Z} \times b\mathbb{Z}$ **with the rescaled lattice** $(a/r)\mathbb{Z} \times (br)\mathbb{Z}$, **at the cost of replacing** g **with its image** $D_r g$ **under a unitary map. More generally, if** Λ **is an arbitrary set of points in** $\mathbb{R}^2$, **then the image of** $\mathcal{G}(g,\Lambda)$ **under the unitary map** D_r **is** $\mathcal{G}(D_r g, \Delta_r(\Lambda))$, **where**

$$\Delta_r = \begin{bmatrix} 1/r & 0 \\ 0 & r \end{bmatrix}.$$

Note that Δ_r **is** ***area preserving*** **(has determinant** 1**).**

The Fourier transform is another metaplectic transform. We have

$$(M_b T_a g)^\wedge \;=\; T_b M_{-a} \widehat{g} \;=\; e^{2\pi i a b} M_{-a} T_b.$$

Scalars of unit modulus do not affect properties like completeness, independence, or being a frame:

$$\begin{aligned}
\mathcal{G}(g,\Lambda) \text{ is a frame} \quad &\iff \quad \bigl\{M_b T_a g\bigr\}_{(a,b)\in\Lambda} \text{ is a frame} \\
&\iff \quad \bigl\{e^{2\pi i a b} M_{-a} T_b \widehat{g}\bigr\}_{k,n\in\mathbb{Z}} \text{ is a frame} \\
&\iff \quad \mathcal{G}(\widehat{g}, R(\Lambda)) \text{ is a frame},
\end{aligned}$$

where $R = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ **is rotation by** $-\pi/2$ **(an area-preserving transformation!).**

Now, *every* 2×2 **matrix with** $\det(A) = 1$ **can be written as a product of rotations** R, **dilations** Δ_r, **and shears** $S_r \;=\; \left[\begin{smallmatrix} 1 & 0 \\ r & 1 \end{smallmatrix}\right]$.

Assignment 10. Given $g \in L^2(\mathbb{R})$ and $\Lambda \subseteq \mathbb{R}^2$, show that

$$\mathcal{G}(g,\Lambda) \text{ is independent} \quad \iff \quad \mathcal{G}(h, S_r(\Lambda)) \text{ is independent},$$

where $h(x) = e^{\pi i r x^2} g(x)$. $\quad \diamondsuit$

Remark: If $d > 1$, **then it is not true that every** $2d \times 2d$ **matrix with** $\det(A) = 1$ **can be factored this way. Only the *symplectic* matrices can be so factored.**

Combining the previous results gives us the following corollary.

Corollary 18. Let A be a 2×2 matrix with $\det(A) = 1$. Then there exists a unitary transform U on $L^2(\mathbb{R})$ such that

$$\mathcal{G}(g, \Lambda) \text{ is independent} \quad \Longleftrightarrow \quad \mathcal{G}(Ug, A(\Lambda)) \text{ is independent.} \qquad \diamondsuit$$

We can also translate Λ **by** $z = (c, d)$, **at the cost of replacing** g **by** $M_d T_c g$. **Combining these results with Linnell's Theorem gives the following corollary.**

Corollary 19. If $g \in L^2(\mathbb{R})$ is nonzero and Λ is any finite subset of $A(a\mathbb{Z} \times b\mathbb{Z}) + z$ where $\det(A) = 1$ and $z \in \mathbb{R}^2$, then $\mathcal{G}(g, \Lambda)$ is linearly independent.

In other words, HRT holds whenever g **is nonzero and** Λ **is a finite subset of a translate of a full-rank lattice. Any 3 points in** $\mathbb{R}^2$ **lie on a translate of a full-rank lattice, so HRT holds when** $|\Lambda| \leq 3$.

Corollary 20. If $g \in L^2(\mathbb{R})$ is nonzero and Λ contains 1, 2, or 3 distinct points of $\mathbb{R}^2$, then $\mathcal{G}(g, \Lambda)$ is linearly independent. $\diamondsuit$

The 4-point set $\quad \Lambda = \big\{(0,0),\ (1,0),\ (0,1),\ (\sqrt{2}, \sqrt{2})\big\}$
from the HRT Subconjecture is *not* a subset of a translate of a full-rank lattice! HRT is open for 4-point sets (though some special cases are known).

Let's sketch the proof that $\left\{g(x),\ g(x-a),\ g(x)e^{2\pi i x}\right\}$ **is independent. Suppose**

$$c_1\,g(x) \;+\; c_2\,g(x-a) \;+\; c_3\,g(x)e^{2\pi i x} \;=\; 0, \qquad \textbf{all } x \in \mathbb{R}.$$

Rewriting,

$$\boxed{g(x-a) \;=\; m(x)\,g(x),}$$

where $m(x) \;=\; -\dfrac{c_1}{c_2} - \dfrac{c_3}{c_2}\,e^{2\pi i x}$ **is** 1**-periodic. Iterating:**

$$g(x-2a) \;=\; g((x-a)-a) \;=\; m(x-a)\,g(x-a) \;=\; m(x-a)\,m(x)\,g(x).$$

Hence

$$\boxed{g(x-ka) \;=\; \left(\prod_{j=0}^{k-1} m(x-ja)\right) g(x).}$$

Taking absolute values and logarithms:

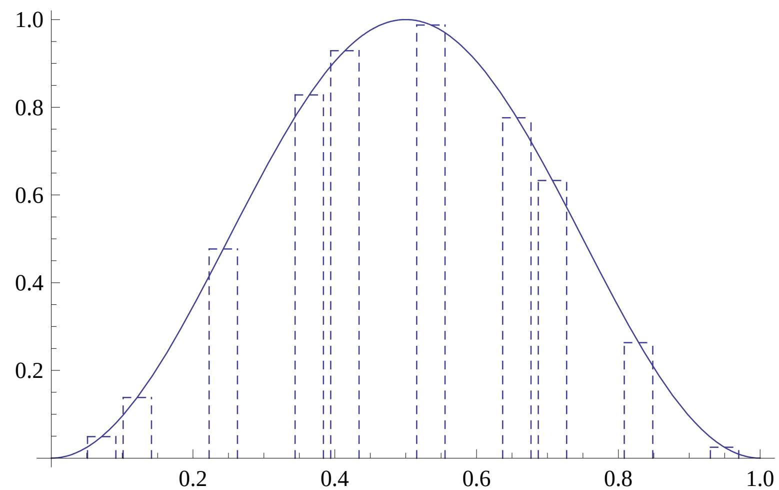
$$\ln|g(x-ka)| \;=\; \sum_{j=0}^{k-1} \ln|m(x-ja)| \;+\; \ln|g(x)| \;=\; \sum_{j=0}^{k-1} p(x-ja) \;+\; \ln|g(x)|,$$

where p **is** 1**-periodic. Therefore**

$$\boxed{\frac{1}{k}\ln|g(x-ka)| \;=\; \frac{1}{k}\sum_{j=0}^{k-1} p(x-ja) \;+\; \frac{1}{k}\ln|g(x)|.}$$

$$\frac{1}{k}\ln|g(x-ka)| \;=\; \frac{1}{k}\sum_{j=0}^{k-1} p(x-ja) \;+\; \frac{1}{k}\ln|g(x)|.$$

The term $\frac{1}{k}p(x-ja)$ is the area of the box with base centered at $x-ja$ (mod 1) with height $p(x-ja)$ and width $1/k$. The sum is the sum of the areas of the boxes:



It's like a Riemann sum approximation to $\int_0^1 p(x)\,dx$, except the boxes are "randomly distributed" — at least, if a is irrational.

Ergodic Theorem: $\frac{1}{k}\sum_{j=0}^{k-1} p(x-ja) \;\to\; \int_0^1 p(x)\,dx$ **as** $k\to\infty$.

Consequently:

$$\frac{1}{k}\sum_{j=0}^{k-1} p(x-ja) \;\approx\; \int_0^1 p(x)\,dx \;=\; C$$

$$\sum_{j=0}^{k-1} \ln|m(x-ja)| \;\approx\; Ck$$

$$\prod_{j=0}^{k-1} |m(x-ja)| \;\approx\; e^{Ck}$$

$$|g(x-ka)| \;=\; \left(\prod_{j=0}^{k-1} |m(x-ja)|\right)|g(x)| \;\approx\; e^{Ck}\,|g(x)|$$

If $C > 0$ then g is growing as $x \to -\infty$, contradicting $\int_{-\infty}^{\infty} |g(x)|^2\,dx < \infty$.

Symmetric argument if $C < 0$. And $C = 0$ takes more work.

Why care?

(a) It's an amazingly simple-to-state question.

(b) The HRT Conjecture can be reformulated in terms of the the Heisenberg group. In this form, it is somewhat related to the following deep open conjecture.

Conjecture 21 (Zero Divisor Conjecture; Higgins, 1940)**.** The group algebra FG of a torsion-free group G over a field F is a domain. $\diamondsuit$

(c) The nature of redundancy in infinite-dimensional settings is surprisingly subtle. Here is an example.

Conjecture 22 (Feichtinger Conjecture, 1990)**.** Every frame can be written as a finite union of nonredundant (= Riesz) subsequences. $\diamondsuit$

The Feichtinger conjecture has been shown to be equivalent to another famous problem.

Conjecture 23 (Kadison–Singer (Paving) Conjecture, 1959)**.**
$\forall\, \varepsilon > 0,\ \exists\, M$ such that $\forall\, n,\ \forall\, n \times n$ matrices S having zero diagonal,
$\exists$ partition $\{\sigma_j\}_{j=1}^{M}$ of $\{1, \dots, n\}$ such that

$$\|P_{\sigma_j} S P_{\sigma_j}\| \le \varepsilon\, \|S\|, \quad j = 1, \dots, M,$$

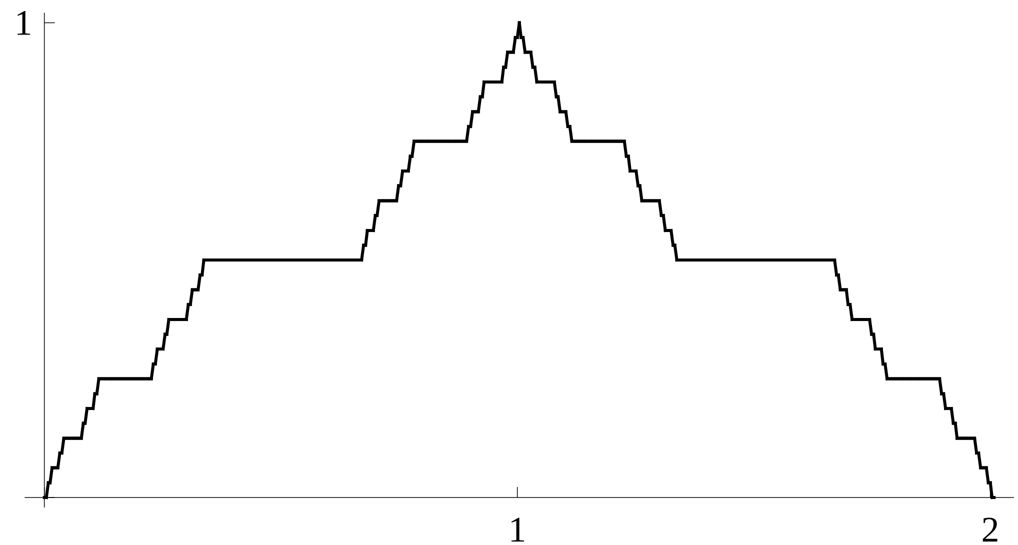
where P_I is the orthogonal projection onto $\mathrm{span}\{e_i\}_{i \in I}$. $\diamondsuit$

KADISON–SINGER HAS RECENTLY BEEN SETTLED!!

Adam Marcus, Dan Spielman, and Nikhil Srivastava, 2013
"Interlacing Families II: Mixed Characteristic Polynomials and the Kadison-Singer Problem"

K-S is true; so Feichtinger is true as well—but *how many* subsequences is still unknown.

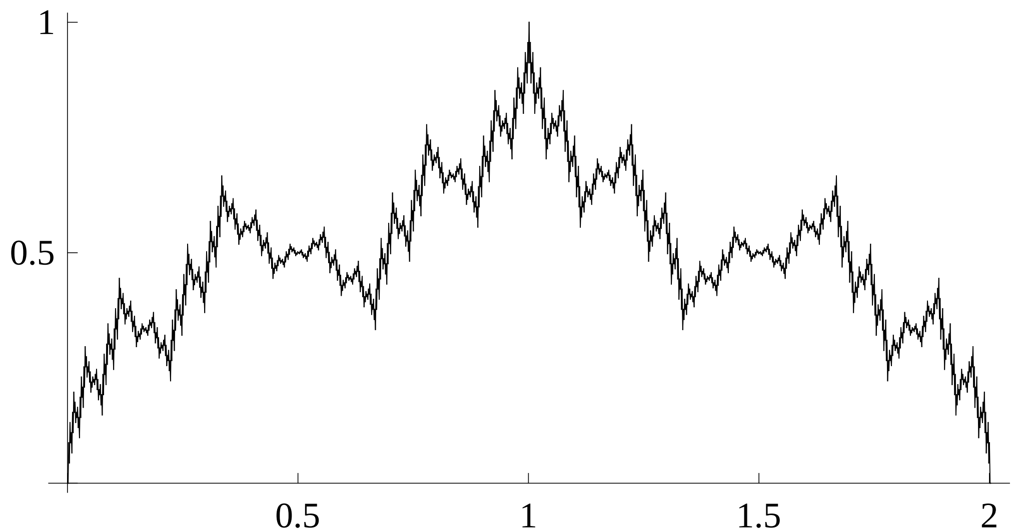
The Devil's Staircase (Cantor–Lebesgue Function)



$$\varphi(x) \;=\; \tfrac{1}{2}\varphi(3x) + \tfrac{1}{2}\varphi(3x-1) + \varphi(3x-2) + \tfrac{1}{2}\varphi(3x-3) + \tfrac{1}{2}\varphi(3x-4).$$

Refinable functions **like these play important roles in wavelet theory and in subdivision schemes in computer-aided graphics.**

de Rham's Nowhere Differentiable Function



$$\varphi(x) \ = \ \tfrac{2}{3}\varphi(3x) + \tfrac{1}{3}\varphi(3x-1) + \varphi(3x-2) + \tfrac{1}{3}\varphi(3x-3) + \tfrac{2}{3}\varphi(3x-4).$$