FRAMES AND TIME-FREQUENCY ANALYSIS

LECTURE 4: THE STFT AND THE MODULATION SPACES

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READING

For background on Banach spaces, Hilbert spaces, and operator theory:

Chapters 1–2 in C. Heil, *A Basis Theory Primer*, Birkhäuser, Boston, 2011.

For background on the Fourier transform:

Chapter 9 in C. Heil, *A Basis Theory Primer*, Birkhäuser, Boston, 2011.

Today's lecture is based upon:

C. E. Heil and D. F. Walnut, *Continuous and discrete wavelet transforms*, **SIAM Review, 31 (1989), pp. 628–666.**

K. Gröchenig, *Foundations of Time-Frequency Analysis*, Birkhäuser, Boston, 2001.

For further reading:

C. Heil, *Integral operators, pseudodifferential operators, and Gabor frames*, **in: "Advances in Gabor Analysis," Birkhäuser, Boston, 2003, pp. 153–169.**

C. Heil, *History and evolution of the Density Theorem for Gabor frames*, **J. Fourier Anal. Appl., 13 (2007), pp. 113–166.**

REVIEW

Translation: $(T_a f)(t) = f(t-a), \qquad a \in \mathbb{R}.$

Modulation: $(M_b f)(t) = e^{2\pi ibt} f(t), \qquad b \in \mathbb{R}.$

Time-frequency shifts **are** $M_b T_a$ **and** $T_a M_b$ **(note** $T_a M_b = e^{-2\pi iab} M_b T_a$**)**

(Regular or Lattice) Gabor (Gah-bor) System:

$$\mathcal{G}(g,a,b) = \{M_{bn}T_{ak}g\}_{k,n\in\mathbb{Z}} = \{e^{2\pi ibnt}g(t-ak)\}_{k,n\in\mathbb{Z}}.$$

Analysis map**:**

$$C_g f = \Big\{\langle f, M_{bn}T_{ak}g\rangle\Big\}_{k,n\in\mathbb{Z}}$$

or

$$(C_g f)_{kn} = \langle f, M_{bn}T_{ak}g\rangle = \int f(t)\,\overline{g(t-ak)}\,e^{-2\pi ibnt}\,dt.$$

If $\mathcal{G}(g,a,b)$ **is a frame, then we have the** ***reconstruction formula*****:**

$$f = \sum_{k,n\in\mathbb{Z}} \langle f, M_{bn}T_{ak}\widetilde{g}\,\rangle\, M_{bn}T_{ak}g, \qquad f \in L^2(\mathbb{R}).$$

ASIDE: THE SCHWARTZ SPACE

The *Schwartz space* is

$$\mathcal{S}(\mathbb{R}) \;=\; \bigl\{f \in C^\infty(\mathbb{R}) : t^m f^{(n)}(t) \in L^\infty(\mathbb{R}) \;\; \textbf{for all } m, n \geq 0\bigr\}.$$

An element of $\mathcal{S}(\mathbb{R})$ is called a *Schwartz-class function.* If $f \in \mathcal{S}(\mathbb{R})$ then for each $m,\, n \geq 0$ there exists a constant C_{mn} such that

$$|f^{(n)}(t)| \;\leq\; \frac{C_{mn}}{|t^m|}, \qquad t \in \mathbb{R}.$$

Thus, $f^{(n)}$ decays faster at infinity than the reciprocal of any polynomial. An example is Gaussian $\phi(t) \;=\; 2^{1/4}\, e^{-\pi t^2}$ (normalized so $\|\phi\|_2 = 1$).

$\mathcal{S}(\mathbb{R})$ is not a Banach space, but rather is a *Fréchet space* whose topology is defined by the countable family of seminorms

$$\rho_{mn}(f) \;=\; \|t^m f^{(n)}(t)\|_\infty, \qquad m, n \geq 0.$$

That is, convergence in the Schwartz space is defined by

$$f_k \to f \textbf{ in } \mathcal{S}(\mathbb{R}) \qquad \Longleftrightarrow \qquad \lim_{k\to\infty} \|t^m f^{(n)}(t) - t^m f_k^{(n)}(t)\|_\infty = 0, \quad \textbf{all } m, n \geq 0.$$

In one sense $\mathcal{S}(\mathbb{R})$ is a very "small" space, which makes it a suitable space of *test functions.* On the other hand,

$$C_c^\infty(\mathbb{R}) \;\subsetneq\; \mathcal{S}(\mathbb{R}) \;\subsetneq\; L^p(\mathbb{R}),$$

and $C_c^\infty(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ for every finite p.

The *space of tempered distributions* is the dual space of $\mathcal{S}(\mathbb{R})$. That is, it is the set of all *continuous linear functionals* on $\mathcal{S}(\mathbb{R})$:

$$\mathcal{S}'(\mathbb{R}) \;=\; \bigl\{\mu\colon \mathcal{S}(\mathbb{R}) \to \mathbb{C} : \mu \textbf{ is linear and continuous}\bigr\}.$$

A functional $\mu\colon \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ is continuous if

$$f_k \to f \textbf{ in } \mathcal{S}(\mathbb{R}) \qquad \Longrightarrow \qquad \mu(f_k) \to \mu(f).$$

Example 1. The *delta distribution* is the functional $\delta\colon \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ defined by

$$\delta(f) \;=\; f(0), \qquad f \in \mathcal{S}(\mathbb{R}).$$

Suppose $f_k \to f$ in $\mathcal{S}(\mathbb{R})$. Then

$$\lim_{k\to\infty} \|t^m f^{(n)}(t) - t^m f_k^{(n)}(t)\|_\infty = 0, \quad \text{all } m,n \geq 0.$$

Hence

$$|\delta(f) - \delta(f_k)| \;=\; |f(0) - f_k(0)| \;\leq\; \|f - f_k\|_\infty \;\to\; 0.$$

Therefore δ is continuous. As it is also linear, it is an element of $\mathcal{S}'(\mathbb{R})$. $\diamondsuit$

$\mathcal{S}'(\mathbb{R})$ includes $L^1_{\text{loc}}(\mathbb{R})$ and hence $L^p(\mathbb{R})$ for every p, where we identify a function $g \in L^1_{\text{loc}}(\mathbb{R})$ with the functional

$$\mu_g(f) \;=\; \langle f, g\rangle \;=\; \int f(t)\,\overline{g(t)}\,dt, \qquad f \in \mathcal{S}(\mathbb{R}).$$

$\mathcal{S}'(\mathbb{R})$ also includes the space of bounded Borel measures $M_b(\mathbb{R})$, where we identify a bounded measure ν with the functional

$$\mu_\nu(f) \;=\; \langle f, \nu\rangle \;=\; \int f(t)\,d\overline{\nu(t)}, \qquad f \in \mathcal{S}(\mathbb{R}).$$

In summary, $\mathcal{S}'(\mathbb{R})$ is "very large."

The Fourier transform $\mathcal{F}$ is a continuous bijective mapping of $\mathcal{S}(\mathbb{R})$ onto itself. "By duality" we can extend $\mathcal{F}$ to a continuous bijective mapping $\mathcal{F}\colon \mathcal{S}'(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$. Thus "almost everything" has a Fourier transform.

Definition 2 (STFT)**.** The *Short-Time Fourier Transform* of $f \in L^2(\mathbb{R})$ with respect to the *window* (or *atom*) $g \in L^2(\mathbb{R})$ is

$$V_g f(x,\xi) \;=\; \langle f, M_\xi T_x g\rangle \;=\; \int f(t)\,\overline{g(t-x)}\,e^{-2\pi i\xi t}\,dt, \qquad (x,\xi)\in\mathbb{R}^2. \qquad \diamondsuit$$

The STFT is also known as the *windowed Fourier transform* or *continuous Gabor transform*. The plane $\mathbb{R}^2$ is called the *time-frequency plane* (math, engineering) or *phase space* (physics).

Note that the analysis map is

$$(C_g f)_{kn} \;=\; \langle f, M_{bn} T_{ak} g\rangle \;=\; V_g f(ak, bn), \qquad k,n\in\mathbb{Z},$$

i.e., it is obtained by *sampling* the STFT on the lattice $a\mathbb{Z}\times b\mathbb{Z}$. The STFT is defined at all points in the time-frequency plane, while the analysis map is defined on a grid determined by the values of a and b. In one sense, the STFT is the limit of taking $a\to 0$ and $b\to 0$.

The STFT is an approximation to the ideal of defining the "instantaneous frequency content of f at time x and frequency ξ" (which, because of the Uncertainty Principle, cannot be done perfectly).

STFT:

$$V_g f(x,\xi) \;=\; \langle f, M_\xi T_x g\rangle \;=\; \int f(t)\,\overline{g(t-x)}\,e^{-2\pi i\xi t}\,dt, \qquad (x,\xi)\in\mathbb{R}^2.$$

Assignment 1. If $f,\, g \in L^2(\mathbb{R})$, then $V_g f$ is bounded and uniformly continuous on $\mathbb{R}^2$, with $\|V_g f\|_\infty \le \|f\|_2 \|g\|_2$. $\diamondsuit$

Covariance property of the STFT:

$$\begin{aligned} V_g(T_u M_\eta f)(x,\xi) &= \langle T_u M_\eta f, M_\xi T_x g\rangle \\ &= \langle f, M_{-\eta} T_{-u} M_\xi T_x g\rangle \\ &= e^{-2\pi i u\xi}\,\langle f, M_{\xi-\eta} T_{x-u} g\rangle \\ &= e^{-2\pi i u\xi}\,V_g f(x-u,\xi-\eta). \end{aligned}$$

The STFT converts a time-frequency shift of f into a translation of $V_g f$ in the time-frequency plane.

$$V_g f(x,\xi) \;=\; \langle f, M_\xi T_x g\rangle \;=\; \int f(t)\,\overline{g(t-x)}\,e^{-2\pi i\xi t}\,dt, \qquad (x,\xi)\in\mathbb{R}^2.$$

Equivalent forms of the STFT for $f,\, g \in L^2(\mathbb{R})$:

$$\begin{aligned}
V_g f(x,\xi) &= (f\cdot T_x\bar{g})^\wedge(\xi) \\
&= \langle f, M_\xi T_x g\rangle \\
&= \langle \widehat{f}, T_\xi M_{-x}\widehat{g}\,\rangle \\
&= e^{-2\pi i x\xi}\,\langle \widehat{f}, M_{-x}T_\xi\widehat{g}\,\rangle \\
&= e^{-2\pi i x\xi}\, V_{\widehat{g}}\widehat{f}(\xi,-x) && \textbf{(Fundamental Identity of TFA)} \\
&= e^{-2\pi i x\xi}\,(f * M_{-\xi}\widetilde{g}\,)(x) && (\textbf{Involution: } \widetilde{g}(t) = \overline{g(-t)}\ ) \\
&= (\widehat{f} * M_{-x}\widehat{\widetilde{g}}\,)(\xi) \\
&= e^{\pi i x\xi}\int f(t+\tfrac{x}{2})\,\overline{g(t-\tfrac{x}{2})}\,e^{-2\pi i t\xi}\,dt && \textbf{(cross ambiguity function)}
\end{aligned}$$

We can extend the definition of the STFT to pairs that lie in dual spaces. For example,

$$V_g f(x,\xi) = \langle f, M_\xi T_x g\rangle$$

is defined at every point (x,ξ) **whenever:**

- $f \in L^p(\mathbb{R})$ **and** $g \in L^{p'}(\mathbb{R})$ $(1 \le p \le \infty)$**, or**
- $f \in M_b(\mathbb{R})$ **(bounded Radon measures) and** $g \in C_0(\mathbb{R})$, **or**
- $f \in \mathcal{S}'(\mathbb{R})$ **(tempered distributions) and** $g \in \mathcal{S}(\mathbb{R})$ **(Schwartz space).**

We can analyze a wide range of objects f **by restricting the class of windows.**

Technical issue: Where do you keep your conjugates?

- $f \in L^p(\mathbb{R}),\ g \in L^{p'}(\mathbb{R}), \qquad V_g f(x,\xi) = \int f(t)\,\overline{M_\xi T_x g(t)}\,dt.$
  This is a *sesquilinear form* (linear in f**, antilinear in** g**).**
- $f \in M_b(\mathbb{R}),\ g \in C_0(\mathbb{R}), \qquad V_g f(x,\xi) = \int \overline{M_\xi T_x g(t)}\,df(t).$ **Sesquilinear again.**
- $f \in \mathcal{S}'(\mathbb{R}),\ g \in \mathcal{S}(\mathbb{R}), \qquad V_g f(x,\xi) = \langle f, M_\xi T_x g\rangle = \overline{\langle M_\xi T_x g, f\rangle}.$
  Sesquilinear if we think of the distribution f **acting antilinearly on** $M_\xi T_x g$ **(backwards from the usual linear action).**

Assignment 2. Show that $V_g f(x,\xi)$ is continuous in any of the above cases, is bounded in the first two cases, and has at most polynomial growth in the third case. *Even if $f \in \mathcal{S}'(\mathbb{R})$ is a tempered distribution, its STFT $V_g f$ is a continuous function if we choose $g \in \mathcal{S}(\mathbb{R})$.* $\diamondsuit$

Example 3. What is the STFT of a delta? Ideally it should be entirely concentrated along $x = 0$ in the time-frequency plane ... is it? Fix a window $g \in C_0(\mathbb{R})$:

$$\begin{aligned} V_g\delta(x,\xi) \;=\; \langle \delta, M_\xi T_x g\rangle \;&=\; \int \overline{M_\xi T_x g(t)}\, d\delta(t) \\ &=\; \overline{M_\xi T_x g(0)} \\ &=\; e^{-\pi i \xi \cdot 0}\overline{g(0-x)} \;=\; \overline{g(-x)}. \end{aligned}$$

This is constant in frequency (expected), but does have some dependency on x. In practice we usually choose g to be a smooth function reasonably well-concentrated about the origin (e.g., the Gaussian function), but there is no way to make g supported *only* along $x = 0$. $\diamondsuit$

It is possible to extend the definition of $V_g f$ further, by duality. For example, if f and g are both tempered distributions (i.e., f, $g \in \mathcal{S}'(\mathbb{R})$), then $V_g f$ is a tempered distribution on the time-frequency plane (i.e., $V_g f \in \mathcal{S}'(\mathbb{R}^2)$). But $V_g f$ is no longer a *function* in general. So, we can take $g = \delta$, but only at the cost of having a very "weak" object $V_\delta f$ as our time-frequency representation of $f \in \mathcal{S}'(\mathbb{R})$.

Assignment 3. (a) If $g \in L^1(\mathbb{R})$ then the STFT of the constant function 1 is defined. What is $V_g 1$? Ideally, it should be supported entirely along $\xi = 0$ in the time-frequency plane. Is it?

(b) If we restrict g so that $V_g \delta$ and $V_{\widehat{g}} \widehat{\delta}$ are both defined, is the Fundamental Identity of Time-Frequency Analysis satisfied for this pair? Is it true that

$$V_g \delta(x, \xi) \;=\; e^{-2\pi i \xi x}\, V_{\widehat{g}} \widehat{\delta}(\xi, -x)?$$

Remark: Viewed either distributionally or as the Fourier transform of a measure, the Fourier transform of delta is $\widehat{\delta} = 1$. $\diamondsuit$

ORTHOGONALITY RELATIONS FOR THE STFT

Theorem 4. If $f_1,\ f_2,\ g_1,\ g_2 \in L^2(\mathbb{R})$ then $V_{g_1}f_1,\ V_{g_2}f_2 \in L^2(\mathbb{R}^2)$, and

$$\big\langle V_{g_1}f_1,\ V_{g_2}f_2\big\rangle \;=\; \langle f_1, f_2\rangle\,\overline{\langle g_1, g_2\rangle}.$$

Consequently, if $g \in L^2(\mathbb{R})$ is fixed, then $f \mapsto V_g f$ is a multiple of an isometry:

$$\|V_g f\|_2 \;=\; \|f\|_2\,\|g\|_2.$$

Proof. Let $\mathcal{T}$ denote an asymmetric change of coordinates: $\quad \mathcal{T}F(x,t) \;=\; F(t, t-x).$
This is a unitary mapping of $L^2(\mathbb{R}^2)$ onto itself.

Let $\mathcal{F}_2$ denote the partial Fourier transform (Fourier transform in the 2nd variable)

$$\mathcal{F}_2F(x,\xi) \;=\; \int F(x,t)\,e^{-2\pi i\xi dt}.$$

This is defined for integrable functions, but extends to a unitary mapping of $L^2(\mathbb{R}^2)$ onto itself.

Let $(f \otimes g)(x,t) = f(x)g(t)$. Then $\|f \otimes g\|_2 = \|f\|_2\,\|g\|_2$. We have

$$\begin{aligned} V_g f(x,\xi) \;=\; \langle f, M_\xi T_x g\rangle \;&=\; \int f(t)\,\overline{g(t-x)}\,e^{-2\pi i\xi t}\,dt \\ &=\; \int \mathcal{T}(f \otimes \bar{g})(x,t)\,e^{-2\pi i\xi t}\,dt \;=\; \mathcal{F}_2\mathcal{T}(f \otimes \bar{g})(x,t). \end{aligned}$$

That is,

$$V_g f \;=\; \mathcal{F}_2\mathcal{T}(f \otimes \bar{g}). \tag{1}$$

Therefore

$$\begin{aligned}
\Big\langle V_{g_1} f_1,\, V_{g_2} f_2 \Big\rangle \;&=\; \Big\langle \mathcal{F}_2\mathcal{T}(f_1 \otimes \bar{g}_1),\; \mathcal{F}_2\mathcal{T}(f_2 \otimes \bar{g}_2) \Big\rangle \\
&=\; \big\langle f_1 \otimes \bar{g}_1,\; f_2 \otimes \bar{g}_2 \big\rangle \qquad\qquad (\mathcal{T} \text{ and } \mathcal{F}_2 \text{ are unitary}) \\
&=\; \iint (f_1 \otimes \bar{g}_1)\, \overline{(f_2 \otimes \bar{g}_2)} \\
&=\; \iint f_1(x)\, \overline{g_1(t)}\;\; \overline{f_2(x)}\; g_2(t)\, dx\, dt \\
&=\; \langle f_1, f_2 \rangle\, \overline{\langle g_1, g_2 \rangle}. \qquad \square
\end{aligned}$$

Equation (1) **also shows how** $V_g f$ **can be extended to larger domains:** $f \otimes \bar{g}$, **the translation** $\mathcal{T}$, **and the partial Fourier transform** $\mathcal{F}_2$ **all have extensions to tempered distributions, and therefore** $V_g f \;=\; \mathcal{F}_2\mathcal{T}(f \otimes \bar{g})$ **makes sense for** $f,\, g \in \mathcal{S}'(\mathbb{R})$. **However, in this case** $V_g f$ **is itself only a tempered distribution, not a function.**

Corollary 5. If $f,\, g \in \mathcal{S}(\mathbb{R})$, then $V_g f \in \mathcal{S}(\mathbb{R}^2)$.

Proof. Write

$$V_g f \;=\; \mathcal{F}_2 \mathcal{T}(f \otimes \bar{g}),$$

and observe that $f \otimes \bar{g} \in \mathcal{S}(\mathbb{R}^2)$. Since $\mathcal{S}(\mathbb{R}^2)$ is invariant under the change of coordinates $\mathcal{T}$ and under the partial Fourier transform $\mathcal{F}_2$, we obtain $V_g f \in \mathcal{S}(\mathbb{R}^2)$. □

Corollary 6. Choose $g \in L^2(\mathbb{R})$, $g \neq 0$. If $f \in L^2(\mathbb{R})$, then

$$V_g f \;=\; 0 \text{ a.e.} \qquad \Longleftrightarrow \qquad f \;=\; 0 \text{ a.e.}$$

Proof. This follows from the isometric nature of the STFT: $\|V_g f\|_2 \;=\; \|f\|_2 \, \|g\|_2$. □

Corollary 7. If $g \in L^2(\mathbb{R})$ with $g \neq 0$, then $\{M_\xi T_x g\}_{(x,\xi)\in\mathbb{R}^2}$ is complete in $L^2(\mathbb{R})$.

Proof. If $f \perp M_\xi T_x g$ for every x and ξ then $V_g f(x,\xi) = \langle f, M_\xi T_x g\rangle = 0$, so $f = 0$ a.e. □

Remark: Completeness by itself is a rather weak property and does not by itself ensure a reconstruction algorithm. But based on the isometric properties and our experience with frames, we might hope that such a reconstruction formula exists.

We will prove the following theorem.

Theorem 8 (Inversion Formula for the STFT)**.** If g, $\gamma \in L^2(\mathbb{R})$ and $\langle g, \gamma \rangle \neq 0$, then for each $f \in L^2(\mathbb{R})$ then following equality holds in a *weak sense*:

$$f \;=\; \frac{1}{\langle \gamma, g \rangle} \iint V_g f(x,\xi)\, M_\xi T_x \gamma \, d\xi \, dx. \qquad \diamondsuit \tag{2}$$

"Weak sense" roughly means that this makes sense if we think of these objects as bounded antilinear functionals on $L^2(\mathbb{R})$ rather than actual functions. For example, $f \in L^2(\mathbb{R})$ determines a bounded antilinear functional via the rule

$$h \;\mapsto\; \langle f, h \rangle \;=\; \int f(t)\, \overline{h(t)}\, dt, \qquad h \in L^2(\mathbb{R}).$$

The following result shows we can likewise interpret the "integral" in (2) as a bounded antilinear functional. We will also see some "strong" versions later.

Theorem 9. If $g \in L^2(\mathbb{R})$ and $F \in L^2(\mathbb{R}^2)$ are given, then

$$h \mapsto \iint F(x,\xi)\, \langle M_\xi T_x \gamma, h \rangle \, d\xi \, dx$$

is a bounded antilinear functional on $L^2(\mathbb{R})$.

Theorem 10. If $\gamma \in L^2(\mathbb{R})$ and $F \in L^2(\mathbb{R}^2)$ are given, then

$$h \mapsto \iint F(x,\xi)\, \langle M_\xi T_x \gamma, h\rangle \, d\xi\, dx \tag{3}$$

is a bounded antilinear functional on $L^2(\mathbb{R})$.

Proof. If $h \in L^2(\mathbb{R})$, then $V_\gamma h(x,\xi) = \langle h, M_\xi T_x \gamma\rangle$ is measurable and belongs to $L^2(\mathbb{R}^2)$. Therefore $\langle F, V_\gamma h\rangle$ exists. Define a functional $\mu\colon L^2(\mathbb{R}) \to \mathbb{C}$ by

$$\mu(h) \;=\; \langle F, V_\gamma h\rangle, \qquad h \in L^2(\mathbb{R}).$$

This is an antilinear functional, and it is bounded because

$$|\mu(h)| \;=\; |\langle F, V_\gamma h\rangle| \;\le\; \|F\|_2\, \|V_\gamma h\|_2 \;\le\; \|F\|_2\, \|\gamma\|_2\, \|h\|_2 \;=\; C\, \|h\|_2.$$

Therefore, by the Riesz Representation Theorem, there is a unique $f \in L^2(\mathbb{R})$ such that

$$\langle f, h\rangle \;=\; \mu(h) \;=\; \langle F, V_\gamma h\rangle, \qquad h \in L^2(\mathbb{R}).$$

It only remains to observe that μ actually is the functional that we wanted:

$$\mu(h) \;=\; \langle F, V_\gamma h\rangle \;=\; \iint F(x,\xi)\, \overline{V_\gamma h(x,\xi)}\, d\xi\, dx \;=\; \iint F(x,\xi)\, \langle M_\xi T_x \gamma, h\rangle\, d\xi\, dx. \qquad \square$$

In essence, there is a linear functional that is naturally associated with the symbols $\iint F(x,\xi)\, M_\xi T_x \gamma\, d\xi\, dx,$ **and it is the one given by** (3)**.**

Definition 11. If $g \in L^2(\mathbb{R})$ and $F \in L^2(\mathbb{R}^2)$ are given, then the statement

$$f \;=\; \iint F(x,\xi)\, M_\xi T_x \gamma \, d\xi \, dx$$

holds weakly if

$$\forall\, h \in L^2(\mathbb{R}), \quad \langle f, h\rangle \;=\; \iint F(x,\xi)\, \langle M_\xi T_x \gamma, h\rangle \, d\xi \, dx. \qquad \diamondsuit \tag{4}$$

That is, the equality in (4) holds weakly if the functional naturally associated with f is the same as the functional naturally associated with the symbols on the right-side of the equation.

Theorem 12 (Inversion Formula for the STFT)**.** If g, $\gamma \in L^2(\mathbb{R})$ and $\langle g, \gamma\rangle \neq 0$, then for each $f \in L^2(\mathbb{R})$ then following equality holds weakly:

$$f \;=\; \frac{1}{\langle \gamma, g\rangle} \iint V_g f(x,\xi)\, M_\xi T_x \gamma \, d\xi \, dx. \tag{5}$$

Proof. If $h \in L^2(\mathbb{R})$ then (using the orthogonality relations):

$$\begin{aligned} \langle f, h\rangle \;&=\; \frac{1}{\langle \gamma, g\rangle} \big\langle V_g f,\, V_\gamma h \big\rangle \;=\; \frac{1}{\langle \gamma, g\rangle} \iint V_g f(x,\xi)\, \overline{V_\gamma h(x,\xi)}\, dx\, d\xi \\ &=\; \frac{1}{\langle \gamma, g\rangle} \iint V_g f(x,\xi)\, \overline{\langle h, M_\xi T_x \gamma\rangle}\, dx\, d\xi \;=\; \big\langle (5),\, h \big\rangle. \qquad \square \end{aligned}$$

Here is one "strong" version of the Inversion Formula (see Gröchenig, p. 98, inspired by Benedetto's formulas).

Theorem 13 (Inversion for the STFT)**.** Fix $g, \gamma \in L^2(\mathbb{R})$ with $\langle g, \gamma \rangle \neq 0$. Given $f \in L^2(\mathbb{R})$, set

$$f_n \;=\; \frac{1}{\langle \gamma, g \rangle} \int_{-n}^{n} \int_{-n}^{n} V_g f(x, \xi)\, M_\xi T_x \gamma \, d\xi \, dx.$$

Then $f_n \in L^2(\mathbb{R})$, and $f_n \to f$ in L^2-norm, i.e., $\lim_{n\to\infty} \|f - f_n\|_2 \;=\; 0.$ $\diamondsuit$

The mapping $f \mapsto V_g f$ is *analysis*, analogous to taking a musical signal f and turning it into a musical score $V_g f$:

The mapping $V_g f \to f$ is *synthesis*, analogous to an orchestra turning a musical score into music by combining (superimposing) notes.

Theorem 14 (Uncertainty Principle for the STFT)**.** Assume $\|f\|_2 = \|g\|_2 = 1$ (so $\|V_g f\|_2 = 1$). If $U \subseteq \mathbb{R}^2$ and

$$\iint_U |V_g f| \;\geq\; 1 - \varepsilon,$$

then

$$|U| \geq 2\,(1-\varepsilon)^2. \qquad \diamondsuit$$

THE MODULATION SPACES: MOTIVATION

A frame for a Hilbert space is defined in terms of a certain ***norm equivalence***:

$$\forall\, x \in H, \quad A\,\|x\|^2 \;\leq\; \sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2 \;\leq\; B\,\|x\|^2.$$

This automatically implies the existence of a reconstruction formula:

$$\forall\, x \in H, \quad x \;=\; \sum_{n=1}^{\infty} \langle x, \widetilde{x}_n\rangle\, x_n.$$

These facts do not have any immediate generalization to Banach spaces that are not Hilbert—norm equivalence does not imply reconstruction, and reconstruction does not imply norm equivalence.

Once approach is define a frame for a Banach space to be a system that has both the norm equivalence and reconstruction properties. However, this tends to restrict us to considering particular Banach spaces, whereas what we really want is to understand the entire ***family of spaces*** in which (for example) Gabor expansions hold. Rather than starting from scratch, the question is ***how to extend what we know from*** $L^2(\mathbb{R})$ ***to a broader family of spaces***? Or:

What are the right spaces in which to do time-frequency analysis?

First guess: The L^p spaces.

Problem: Although the trigonometric system is an ONB for $L^2[0,1]$, *frequency analysis is very complicated in* L^p *when* $p \neq 2$.

Theorem 15. If $1 < p < \infty$ and $p \neq 2$, then $\{e^{2\pi inx}\}_{n\in\mathbb{Z}}$ is a *conditional Schauder basis* for $L^p[0,1]$. Specifically, if we order the index set as $\mathbb{Z} = \{0, 1, -1, 2, -2, \dots\}$, then

$$\forall\, f \in L^p[0,1], \quad f(x) \;=\; \sum_{n\in\mathbb{Z}} \widehat{f}(n)\, e^{2\pi inx}, \tag{6}$$

where the series converges in L^p-norm, but there exist other orderings of $\mathbb{Z}$ for which this series does not converge. ◇

The underlying problem is that the Dirichlet kernel is not an *approximate identity* (or *summability kernel*) for convolution in $L^1[0,1]$. (A proof of Theorem 15 can be found in Chapter 14 of the *Basis Primer*.) A consequence of conditionality is that the ℓ^q norm of $\widehat{f}$ is not equivalent to the L^p norm of f when $p \neq 2$.

A related result (much deeper and more difficult than Theorem 15) is the *Carleson–Hunt Theorem*, which states that the Fourier series in (6) *converge pointwise a.e.* when $1 < p < \infty$ (again with a restriction on the ordering of $\mathbb{Z}$—for $p \neq 2$ we are relying on "miraculous cancellations," not absolute or unconditional convergence).

We will introduce a class of spaces which tie together:

- **norm equivalence in terms of the STFT,**
- **norm equivalence in terms of Gabor frames,**
- **reconstruction in terms of Gabor frames,**

among other properties. We will define these *modulation spaces* using the STFT, and see the connection to Gabor frames later. To begin, we fix a specific window:

$$\phi(t) \;=\; 2^{1/4}\, e^{-\pi t^2} \qquad \textbf{(normalized so } \|\phi\|_2 = 1\textbf{)}.$$

Because the Gaussian function ϕ belongs to the Schwartz class, $V_g f$ is a continuous function for any tempered distribution f.

Definition 16 (The Modulation Space M^p)**.** The (unweighted) modulation space $M^p(\mathbb{R})$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R})$ such that

$$\|f\|_{M^p} \;=\; \|V_\phi f\|_p \;=\; \left(\iint |V_\phi f|^p\right)^{1/p} \;<\; \infty,$$

with the usual adjustment if $p = \infty$ (use the essential supremum). $\diamondsuit$

We will not prove it, but $M^p(\mathbb{R})$ is a Banach space, and replacing ϕ with a different window $g \in \mathcal{S}(\mathbb{R})$ gives exactly the same space $M^p(\mathbb{R})$ under an equivalent norm.

The modulation spaces were introduced by Feichtinger, and extensively studied by Feichtinger and Gröchenig. We can replace time-frequency, the Heisenberg group, and the Schrödinger representation, with other groups and representations. Some facts work in general, some depend strongly on the group. The Heisenberg group is "simpler" in many ways than the affine group (which gives rise to wavelets), and consequently we sometimes obtain *stronger* results for time-frequency than for wavelets or general systems. (Even so, there are many unanswered questions!)

In contrast to the Schwartz space $\mathcal{S}(\mathbb{R})$, which is only a Fréchet space (a complete space whose topology is induced from a countable family of seminorms), and the space of tempered distributions $\mathcal{S}'(\mathbb{R})$, which is the dual of the Schwartz space, the modulation spaces $M^p(\mathbb{R})$ are *Banach spaces*, even for $p = \infty$.

Example 17. Because the STFT is isometric, we have $M^2(\mathbb{R}) = L^2(\mathbb{R})$:

$$f \in M^2(\mathbb{R}) \quad \Longleftrightarrow \quad \|V_\phi f\|_2 < \infty \quad \Longleftrightarrow \quad \|f\|_2\, \|\phi\|_2 < \infty \quad \Longleftrightarrow \quad f \in L^2(\mathbb{R}).$$

(Well, to be precise, there is a technical issue of showing that if $f \in \mathcal{S}'(\mathbb{R})$ is such that $\|V_\phi f\|_2 < \infty$, then $f \in L^2(\mathbb{R})$. This does take some work.) $\diamondsuit$

But $M^p(\mathbb{R}) \neq L^p(\mathbb{R})$ for $p \neq 2$. (Not a surprise: *frequency analysis* isn't suited to L^p.)

Lemma 18. $\mathcal{S}(\mathbb{R}) \subseteq M^1(\mathbb{R}) \;=\; \Big\{ f \in \mathcal{S}'(\mathbb{R}) : \|f\|_{M^1} = \|V_\phi f\|_1 = \iint |V_\phi f| < \infty \Big\}.$

Proof. If $f \in \mathcal{S}(\mathbb{R})$, then $V_\phi f \in \mathcal{S}(\mathbb{R}^2) \subseteq L^1(\mathbb{R}^2)$. □

Lemma 19. $M^1(\mathbb{R})^* = M^\infty(\mathbb{R}) \subseteq \mathcal{S}'(\mathbb{R})$.

Proof. (Sketch) Given $h \in M^\infty(\mathbb{R})$, define a functional $\mu_h \colon M^1(\mathbb{R}) \to \mathbb{C}$ by

$$\mu_h(f) \;=\; \big\langle V_\phi f,\, V_\phi h \big\rangle \;=\; \iint V_\phi f(x,\xi)\, \overline{V_\phi h(x,\xi)}\, dx\, d\xi, \qquad f \in M^1(\mathbb{R}).$$

This is well-defined because $V_\phi f$ is integrable and $V_\phi h$ is bounded. It is a linear function of f, and by Hölder,

$$|\mu_h(f)| \;=\; \big|\big\langle V_\phi f,\, V_\phi h\big\rangle\big| \;\le\; \big\|V_\phi f\big\|_1 \, \big\|V_\phi h\big\|_\infty \;=\; \|f\|_{M^1}\, \|h\|_{M^\infty}.$$

Taking the supremum over the unit vectors, we see that μ_h is bounded, with operator norm

$$\|\mu_h\| \;=\; \sup_{\|f\|_{M^1}=1} |\mu_h(f)| \;\le\; \|h\|_{M^\infty}.$$

Thus every element of $M^\infty(\mathbb{R})$ determines a bounded linear functional on $M^1(\mathbb{R})$, so $M^\infty(\mathbb{R})$ is a subset of the dual space $M^1(\mathbb{R})^*$. □

$M^1(\mathbb{R})$ **and** $M^\infty(\mathbb{R})$ **are Banach spaces, and** $M^\infty(\mathbb{R})$ **is the dual of** $M^1(\mathbb{R})$. **In this sense,** $M^1(\mathbb{R})$ **is a good substitute for** $\mathcal{S}(\mathbb{R})$ **and** $M^\infty(\mathbb{R})$ **is a good substitute for** $\mathcal{S}'(\mathbb{R})$.

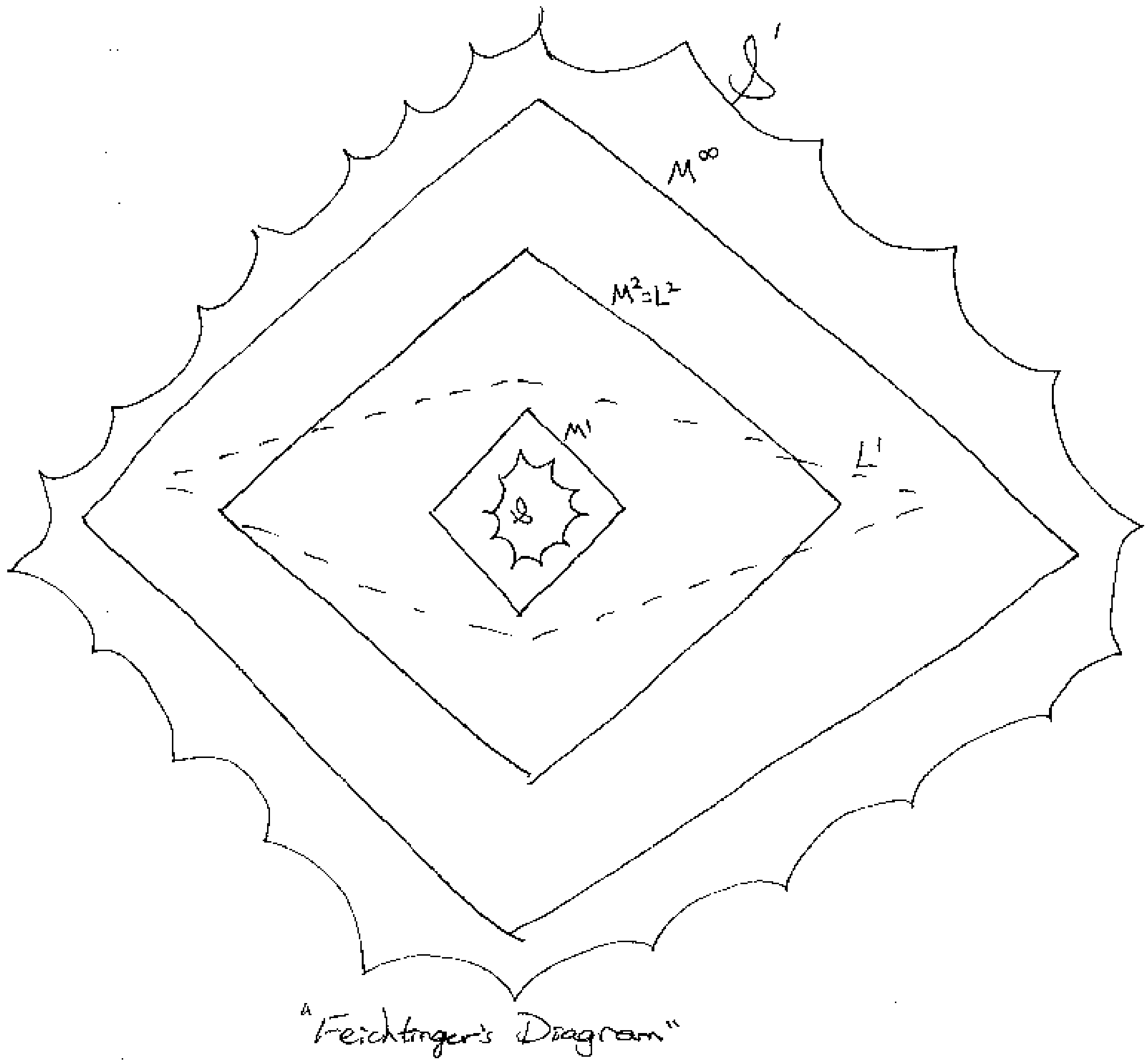

FIGURE 1. The Feichtinger diagram.

Theorem 20. $M^p(\mathbb{R})$ is invariant under the Fourier transform.

Proof. The Fundamental Identity of Time-Frequency Analysis is

$$V_g f(x,\xi) \;=\; e^{-2\pi i x\xi}\, V_{\widehat{g}} \widehat{f}(\xi,-x).$$

Therefore $\|f\|_{M^p} \;=\; \|V_\phi f\|_p \;=\; \|V_{\widehat{\phi}} \widehat{f}\,\| \;=\; \|V_\phi \widehat{f}\,\| \;=\; \|\widehat{f}\,\|_{M^p}.$ $\square$

The Lebesgue space $L^1(\mathbb{R})$ is closed under convolution (in fact, it is a Banach algebra because $\|f*g\|_1 \le \|f\|_1\, \|g\|_1$), but it is not closed under pointwise products.

The *Wiener algebra* (or *Fourier algebra*) is

$$A(\mathbb{R}) \;=\; \bigl\{\widehat{f} : f \in L^1(\mathbb{R})\bigr\} \;\subsetneq\; C_0(\mathbb{R}).$$

It is closed (in fact, it is a Banach algebra) under pointwise multiplication, but it is not closed under convolution.

Theorem 21. $M^1(\mathbb{R}) \;\subsetneq\; L^1(\mathbb{R}) \cap A(\mathbb{R}) \;=\; \bigl\{f \in L^1(\mathbb{R}) : \widehat{f} \in L^1(\mathbb{R})\bigr\}$,
and $M^1(\mathbb{R})$ is a Banach algebra under both convolution and pointwise products. $\diamondsuit$

$M^1(\mathbb{R})$ **(also known as $S_0(\mathbb{R})$) is the *Feichtinger algebra.* Another useful inclusion (think of Gabor frames) is $M^1(\mathbb{R}) \subsetneq W(C,\ell^1)$.**

Example 22. The STFT of the delta distribution is

$$V_\phi\delta(x,\xi) \;=\; \phi(-x) \;=\; 2^{1/4}\, e^{-\pi x^2}.$$

For $1 \le p < \infty$ we have

$$\|\delta\|_{M^p} \;=\; \|V_\phi\delta\|_p \;=\; \left(\iint \phi(x)^p\, dx\, d\xi\right)^{1/p} \;=\; \infty,$$

so $\delta \notin M^p(\mathbb{R})$. On the other hand, $V_\phi\delta$ is bounded, so we do have $\delta \in M^\infty(\mathbb{R})$.

We can refine the modulation spaces by distinguishing between time and frequency. In particular, if we let $M^{1,\infty}$ denote a mixed L^1-L^∞ norm on the STFT, then

$$\|\delta\|_{M^{1,\infty}} \;=\; \int \sup_\xi |V_\phi\delta(x,\xi)|\, dx \;=\; \int \sup_\xi \phi(x)\, dx \;=\; \|\phi\|_1 \;<\; \infty,$$

so $\delta \in M^{1,\infty}(\mathbb{R})$. A similar computation shows that $1 \in M^{\infty,1}(\mathbb{R})$. $\diamondsuit$

If $p \neq q$ then $M^{p,q}(\mathbb{R})$ is not invariant under the Fourier transform, but it does behave "correctly" under $\mathcal{F}$.

We can also allow weights in the definition of the modulation spaces. Assume that $w > 0$ **is even (for convenience) and *submultiplicative*, i.e.,**

$$w(x+y) \;\le\; w(x)\,w(y).$$

Typical example: $w(x) = (1+|x|)^s$ **or** $w(x) = (1+x^2)^{s/2}$ **with** $s \ge 0$. **Then**

$$\begin{aligned}
\|f\|_{M^2_w}^2 \;=\; \|V_\phi f\|_{L^2_w}^2 \;&=\; \iint |V_\phi f(x,\xi)|^2\, w(x)^2\, dx\, d\xi \\
&=\; \int \left(\int |(f\cdot T_x\phi)^\wedge(\xi)|^2\, d\xi \right) w(x)^2\, dx \\
&=\; \int \left(\int |(f\cdot T_x\phi)(t)|^2\, dt \right) w(x)^2\, dx \\
&=\; \iint |f(t)|^2\, |\phi(t-x)|^2\, w(x)^2\, dt\, dx \\
&=\; \int |f(t)|^2 \int |\phi(t-x)|^2\, w(x)^2\, dx\, dt \\
&=\; \int |f(t)|^2 \int |\phi(x)|^2\, w(t-x)^2\, dx\, dt \\
&\le\; \int |f(t)|^2 \int |\phi(x)|^2\, w(x)^2\, dx\; w(t)^2\, dt \\
&=\; \|f\|_{L^2_w}\, \|\phi\|_{L^2_w}.
\end{aligned}$$

Hence, when w **depends only on the time variable, we have**

$$M^2_w(\mathbb{R}) \;=\; L^2_w(\mathbb{R}).$$

Assignment 4. Let w depend solely on frequency. Specifically, take

$$w(\xi) \;=\; (1+|\xi|)^s \qquad \text{where } s \geq 0.$$

Prove that $M^2_w(\mathbb{R})$ equals the *Sobolev space* $H^s(\mathbb{R})$, which is defined by the norm

$$\|f\|^2_{H^s} \;=\; \int |\widehat{f}(\xi)|^2\,(1+|\xi|)^{2s}\,d\xi. \qquad \diamondsuit$$

If $w(x,\xi)$ depends on both time and frequency, then $M^{p,q}_w(\mathbb{R})$ is defined by the norm

$$\|f\|_{M^{p,q}_w} \;=\; \left(\int\left(\int |V_\phi f(x,\xi)|^p\,w(x,\xi)^p\,dx\right)^{q/p} d\xi\right)^{1/q},$$

with the usual adjustments if $p=\infty$ or $q=\infty$.

In this way we obtain a broad class of Banach spaces $M^{p,q}_w(\mathbb{R})$ that includes weighted L^2 spaces and Sobolev spaces, but not $L^p(\mathbb{R})$ or other classical families such as Besov spaces (at least when $p \neq 2$).

Theorem 23. Set $v_s(x,\xi) = (1+|x|+|\xi|)^s$. Then

$$\|V_\phi f\|_{L^\infty_{v_s}} \;=\; \sup_{x,\xi}\, |(1+|x|+|\xi|)^s\,|V_\phi f(x,\xi)|, \qquad s \geq 0,$$

forms an equivalent family of seminorms for $\mathcal{S}(\mathbb{R})$. $\diamondsuit$

We can formulate the BLT in terms of the modulation spaces. Let

$$w(x, \xi) \;=\; (1 + |x| + |\xi|).$$

Given $f \in L^2(\mathbb{R})$ **we have**

$$\begin{aligned}
&\left(\int |xf(x)|^2\,dx\right)\left(\int |\xi\widehat{f}(\xi)|^2\,d\xi\right) \;<\; \infty \\
&\iff \int |f(x)|^2\,(1+|x|)^2 dx \;<\; \infty \quad \textbf{and} \quad \int |\widehat{f}(\xi)|^2\,(1+|\xi|)^2\,d\xi \;<\; \infty \\
&\iff \iint |V_\phi f(x,\xi)|^2\,(1+|x|+|\xi|)^2\,dx\,d\xi \;<\; \infty \\
&\iff f \in M^2_w(\mathbb{R}).
\end{aligned}$$

Theorem 24. (Classical BLT) Given $g \in L^2(\mathbb{R})$,

$$\mathcal{G}(g,1,1) \text{ is is a Riesz basis for } L^2(\mathbb{R}) \quad \implies \quad g \notin M^2_w(\mathbb{R}). \qquad \diamondsuit$$

PROBLEMS ON STFTs

1. If $f,\ g \in L^2(\mathbb{R})$, then given $x \in \mathbb{R}$ we have $f \cdot T_x\bar{g} \in L^1(\mathbb{R})$ by Hölder. Prove that

$$f \cdot T_x\bar{g} \in L^2(\mathbb{R}) \quad \textbf{for a.e. } x,$$

and consequently for each $1 \le p \le 2$ we have $f \cdot T_x\bar{g} \in L^p(\mathbb{R})$ for a.e. x.

Hint: $V_g f \in L^2(\mathbb{R})$. How does $f \cdot T_x\bar{g}$ relate to $V_g f$?

2. Use the fact that $V_g f(x,\xi) = (f \cdot T_x\bar{g})^\wedge(\xi)$ to show directly that $\|V_g f\|_2 = \|f\|_2\, \|g\|_2$.

3. Prove Theorem 13 (the strong version of the Inversion Formula for the STFT).