CHAPTER I. INTEGRAL EQUATIONS

A COMPENDIUM OF PROBLEMS

I. Consider the problem

$$y(x) = \int_{-1}^{1} \left(1/2 + x\,t\right) y(t)\,dt + f(x).$$

(a) Explain how you know this problem is in the second alternative.

ans: y(x) = c is a non-trivial solution to the non-homogeneous problem.

(b) Find linearly independent solutions for the equation y=K*(y).

(c) Let $f_1(x) = 3x - 1$ and $f_2(x) = 3x^2 - 1$. For one of these there is a solution to the equation y = K(y) + f, for the other there is not. Which has a solution?

ans: $3x^2-1$.

(d) For the f for which there is a solution, find two.

ans: $3x^2 -1 + 7$ and $3x^2- 1 + 11$.

II Consider the problem

$$y(x) = \int_0^1 x^2\,t^2\,y(t)\,dt + f(x).$$

(a) Show that the associated K is small in both senses of this section.

(b) Compute φ_2 where f(x) = 1. ans: $\frac{2}{5}x^2 + 1$

(c) Give an estimate for how much φ_2 differs from the solution y of y=K(y)+f.

ans: error $\le \frac{1}{24\,25^2}$

(d) Using the kernel k for K, compute the kernel k_2 for K^2 and k_3 for K^3.

ans: $k_2(x,t) = x^2t^2/5$.

(e) Compute the kernel for the resolvent of this problem.

ans: $r(x,t) = \frac{5x^2t^2}{4}$

(f) What is the solution for y=Ky+f in case f(x) = 1.

ans: $y(x) = 1 + \frac{5}{12}x^2$

III. Consider the problem

$$y(x) = \int_0^1 x\,t^3\,y(t)\,dt + x^2.$$

(a) Compute the associated approximations φ_0, φ_1, φ_2 and φ_3.

ans: $\varphi_1(x) = x^2 + x/6$

(b) Give an estimate for how much φ_3 differs from the solution.
(c) Give the kernel for the resolvent of this problem.

$$\text{ans: } r(x,t) = \frac{5xt^3}{4}$$

(d) Using the resolvent, give the solution to this problem.

$$\text{ans: } y(x) = x^2 + \frac{5x}{24}$$

(e) Using the fact that the kernel of the problem separates, solve the equation.

IV. Suppose that $K(x,t) = \begin{cases} 1-t \text{ if } x < t \\ 1-x \text{ if } t < x. \end{cases}$

(a) Show that $\int_0^1 |K(x,t)| \, dt < 1$ **for all x in [0,1].**

(b) Solve the problem y = K[y] + 1. ans: $y(x) = \cos(x)/\cos(1)$.

V. a. Find a nontrivial solution for y = K[y] in $L^2[0,1]$ where

$$K(x,t) = 1 + \cos(\pi x) \cos(\pi t).$$

b. Find a nontrivial solution for z = K*[z].
c. What condition must hold on f in order that

$$y = K[y] + f$$

shall have a solution. Does $f(x) = 3 x^2$ meet this condition.

ans: Constant functions are nontrivial solutions for both equations and the equation of IV(c) has a solution provided

$$\int_0^1 f(t) \, dt = 0.$$

The function $3 x^2$ does not meet this condition.