Appendix : GREEN'S FUNCTIONS IN R^n.

The setting for this course is in an inner product space. Since the idea of an inner product, or dot product, arises in such a variety of problems, we should recall exactly what are the properties that define an inner product and what are some of the consequences of these properties.

Appendix 1 ARITHMETIC AND GEOMETRY IN R^n

If E is the linear space on which the inner product is defined and x and y are in E, then $< x, y >$ denotes the (perhaps, complex) number which is the inner product of x and y. Moreover

(a) $<x,y> = <y,x>^*$ for all x,y in ExE (Note that * is the complex conjugate),

(b) $< \alpha x+y, z > = \alpha< x, z > + < y, z >$ for all numbers α and all x and y in E,

(c) $< x, x > > 0$ if $x \neq 0$, with $< x, x > = 0$ if $x = 0$.

One consequence of these defining properties is the Cauchy-Schwartz inequality: if x and y are in E then

$$|< x, y >|^2 \leq < x, x > < y, y >.$$

A norm is defined in terms of the inner product:

$$||x||^2 = < x , y >.$$

This norm has those properties which characterize any norm:

$$||x|| > 0 \text{ unless } x = 0,$$

$$||ax|| = |a| \; ||x||, \text{ and}$$

$$||x+y|| \leq ||x|| + ||y||.$$

EXAMPLE: One can think of a *weighted* dot product on R^n by choosing a positive number sequence $\{a_p\}_{p=1}^n$ and defining $< x, y >$ to be given by

$$< x, y > = \sum_{p=1}^{n} a_p x_p y_p.$$

The *usual* inner product on R^n is the one obtained in the above example by choosing $a_p = 1$ for p= 1,2,...n. In this setting, we can think of the structure of R^n in the language of Euclidean geometry: the distance between x and y is $||x-y||$ and the angle between x and y is θ where $0 \le \theta \le \pi$ and

$$\cos(\theta) = \frac{< x, y >}{||x|| \; ||y||}$$

provided neither x nor y is zero. For example, points x and y are perpendicular, or orthogonal, if $< x, y > = 0$. Also, the concept of distance provides a notion of points being "close together". It is natural to say that the sequence $\{v_p\}_{p=1}^{n}$ of points in R^n has limit the point y in R^n provided

$$\lim_p || v_p - y || = 0.$$

It is of value, even at this elementary level, to realize that there are several ways to think of the idea "$\lim_p v_p = y$". In R^n, the following three are equivalent:

The sequence $\{v_p\}_{p=1}^{n}$ in R^n converges to y *component wise* if for each integer i, $1 \le i \le n$,

$$\lim_p (v_p)_i = y_i.$$

The sequence $\{v_p\}_{p=1}^{n}$ in R^n converges to y *uniformly* if

$$\lim_p(\max_i | x_p(i) - y(i) |) = 0.$$

The sequence $\{v_p\}_{p=1}^{n}$ in R^n converges to y in *norm* if

$$\lim_p ||v_p - y|| = 0.$$

It is not difficult to establish that these three notions are equivalent in R^n. The value of thinking of them separately here is that the three methods of convergence have analogues in situations which we will encounter in later sections. In those situations, the three methods of convergence may be not equivalent.

Fundamental in the development of Green's Functions will be the Riesz Representation Theorem: If L is a linear function from R^n to R then there is a vector v in R^n such that $L(u) = < u, v >$ for all u in R^n. Closely

related to the Riesz Representation Theorem is the fact that every linear function from R^n to R^n has a matrix representation. These ideas are familiar.

EXERCISE 1.1:(1) Suppose that L: $R^3 \rightarrow R$ and is defined by L(u) = $2u_1-u_3$. Find v such that L(u) = < u, v > for all u on R^3.

(2) Suppose that L: $R^3 \rightarrow R$ and is defined by L(u)= u_2. Find v such that L(u) = < u, v >.

(3) Give the matrix representation of L: $R^3 \rightarrow R^3$ if L(u) = $\{3u_1+2u_3, -u_1+u_2, u_1-u_2+u_3\}$.

Appendix 2: THE ADJOINT A*

Linear functions and matrices arise in many ways. Here is one that you might not have considered. Choose an nxn matrix A. Consider all points x and y in R^n related by

$$< Au, x > = < u, y >$$

for all u in R^n. To be sure that, given x, there is such a point y, consider the following: pick x; then L(u) = <Au, x > is a linear function of u from R^n to R. By the Riesz Representation Theorem, there is a point y in R^n such that L(u) = < u, y > for all u in R^n. Define this y as B(x). It can be established that B is, itself, a linear function from R^n to R^n. Hence, it has a matrix representation. This linear function B is called the adjoint of A and is denoted as A*. For all x and u in R^n,

$$< Au, x > = < u, A^*x >.$$

The concern of this course, stated in the context of R^n, is the following problem: given a matrix A and a vector f = $\{f_1, f_2, \ldots f_n\}$, can a vector u be found such that Au = f? There are matrices A and vectors f such that the equation Au=f has exactly one solution, or no solution, or an infinity of solutions.

EXERCISE 2: (1) Let A = $\begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$ Find A*.

(2) For the following matrices A and vectors f, determing whether the equation Au = f has exactly one solution, no solution or an infinity of solutions. If there are solutions, find them.

(a) $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ and $f = \begin{pmatrix} 4 \\ 10 \end{pmatrix}$

(b) $A = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$ and $f = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

(c) $A = \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}$ and $f = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

Appendix 3 THE FREDHOLM ALTERNATIVE THEOREMS

The following ideas will persist in each segment of the course. These fundamental results are known as the Fredholm Alternative Theorems. For matrices, the alternatives hinge on whether or not the determinant of A is zero or not.

I. Exactly one of the following two alternatives holds:

(a)(First Alternative) if f is in R^n, then Au = f has one and only one solution.

(b)(Second Alternative) Au= 0 has a nontrivial solution.

II. (a) If the first alternative holds for A, then it also holds for A*.

(b) In either alternative, the equations Au= 0 and A* u = 0 have the same number of linearly independent solutions.

III. Suppose the second alternative holds. Then Au=f has a solution if and only if < f, z > = 0 for each z such that A*z=0.

A matrix problem is *well- posed* if
(a) for each f, the equation Au=f has a solution,
(b) the solution is unique, and
(c) the solution depends continuously on the data-- in the sense that if f is close to g and u and v satisfy Au=f and Av = g, then u is close to v.

EXERCISE 3: (1): For each of the matrices listed below determine what is the dimension of the null space of A and of the null space of A*? Give a basis for each. Find one f such that < f, z > = 0 for each z in the null space of A*. Solve Au = f. Find g such that < g, z > ≠ 0 for each z in the null space of A*. Show that one can not solve Au=g for this g.

(a) $A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$

(b) $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

(2) Let

$A_1 = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$ **and** $A_2 = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$

Show that the problem $A_1u = f$ is well posed and the problem $A_2u = f$ is not.

Appendix 4 SOLVING EQUATIONS

Just as there are many ways to conceive of solving Au=f in case A is a matrix, so there are many ways for finding solutions for problems which are introduced in the next chapters. We concentrate on the method of constructing a Green's function. From what has come before, it may be clear that in some sense, we are finding an inverse for the linear operation A.

THE FIRST ALTERNATIVE

We discuss the solution of the linear equation

$$Au = f,$$

where we suppose we are given a matrix A and know f. Don't be to quick to dismiss the solution as $u = A^{-1} f$. While that is correct, we want perspective here, not just results.

Here is an adaptation of the methods which we will use to find Green's functions in two chapters from here: Let δ_i be the vector which has the property that

$$< \delta_i, u > = u(i)$$

for all u in R^n. One can write the components for δ_i

$$\delta_i = \{0,0,...0. 1, 0, ... 0\}$$

where the 1 is in the i-th component. Note that

$$u = \sum_i \delta_i \, u(i).$$

Let A be an nxn matrix and f be a vector. In order to solve Au=f, we seek G(i,j) such that u defined by

$$< G(j,\ ) , f(\ ) > = u(j) \qquad (*)$$

might provide the solution for Au=f.

Look again at equation (*). Writing the dot product as a sum changes that equation to

$$\sum_{i=1}^{n} G(j,i)\, f(i) = u(j) \qquad (**)$$

or, in the notation of matrix multiplication,

$$Gf = u.$$

In what follows, perhaps you will see that writing the equation as (*) or (**) provides unifying ideas.

Here is a proposal for how humans find G. Find G such that

$$A(G(\ ,i)) = \delta_i.$$

That is, $A(G(\ ,i))(m) = \delta_i(m).$

Having such a G, define u by

$$u(j) = \sum_i G(j,i)f(i).$$

Then

$$[A(u)](m) = \sum_j A(m,j)\, u(j) = \sum_j A(m,j) \sum_i G(j,i)f(i)$$

$$= \sum_i \sum_j A(m,j)G(j,i)f(i)$$

$$= \sum_i \delta_i(m)\, f(i) = f(m).$$

This is what was desired: u solves the equation

$$Au = f.$$

Here is an alternate approach to the problem: Consider these relations:

$$\langle \delta_i,u\rangle = u(i) = \langle G(i,.),f(.)\rangle = \langle G(i,.),Au\rangle = \langle A^*G(i,.),u\rangle.$$

Thus, we might seek G such that $\delta_i = A^*G(i,.)$. Go back and re-do the above exercise this way to see if you get the same answer. Here's a proof that you should.

THEOREM. Suppose that A and G are matrices. These are equivalent:
(a) $A(G(\ ,i)) = \delta_i$ and (b) $A^*(G(i,\ )) = \delta_i$.

Suggestion for a proof. Let G_1 be defined by the first equation:

$$A(G_1(\ ,i)) = \delta_i$$

and G_2 be defined by the second equation:

$$A^*(G_2(i,\ )) = \delta_i.$$

Then $G_1(j,i) = < \delta_j , G_1(\ ,i) > \ = < A^*(G_2(j,\ )) , G_1(\ ,i)) >$

$$= < G_2(j,\ ) , A(G_1(\ ,i)) > = < G_2(j,\ ) , \delta_i > = G_2(j,i).$$

THE SECOND ALTERNATIVE

In case the determinant of A is zero and we are in the second alternative, we can still conceive of the possibility of constructing G such that if f is perpendicular to the null space of the adjoint of A, then

$$u(j) = < G(j,\ ), f >$$

provides a solution to the equation Au = f. The methods developed above will not work for, given i, we cannot find a vector G(i,) that solves $A(G(i,\ )) = \delta_i.$ To see this, recall the third part of the Fredholm Alternative Theorem and then note that

$$< \delta_i,v > \neq 0$$

for all vectors v in the nullspace of A*. Thus, in this second alternative, we must modify the method.

Let $\{v_p\}_{p=1}^{m}$, m < n, be a maximal orthonormal sequence in the nullspace of A*. We know there is G(,p) in R^n such that

$$A(G(\ ,p)) = \delta_p - \sum_{p=1}^{n} v_i(\ ) v_i(p)$$

for $$< \delta_p - \Sigma_{i=1} v_i(\ )\ v_i(p) , w > = 0$$

for all w in the nullspace of A*. We will show that u(i) defined by

$$< G(i,\ ), f >$$

satisfies the equation Au = f. In fact,

$$(Au)_p = \sum_{i=1}^{n} A_{pi}\, u_i = \sum_{i=1}^{n} A_{pi} < G(i,\ ),f >$$

$$= \sum_{i=1}^{n} A_{pi} \sum_{j=1}^{n} G(i,j) f_j$$

$$= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} A_{pi} G_{ij}\right) f_i = \sum_{j=1}^{n} (AG(\ ,j))_p\, f_j$$

$$= \sum_{j=1}^{n} \left(\delta_j(p) - \sum_{i=1}^{n} v_i(p)\, v_i(j)\right) f_j$$

$$= \sum_{j=1}^{n} \delta_j(p)\, f_j - \sum_{i=1}^{n} v_i(p) < v_i,f > = f_p - 0.$$

EXERCISE 4:

(1) For the following matrices A, show that det(A) = 0, Find an orthonormal basis for the null space of A*. Make up G. For the given f show that is it is perpendicular to the null space of A*. Show that u as defined by equation ( ) in the discussion of the First Alternative solves Au=f.

(a) $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$ **and** $f = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$**.**

(b) $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ **and** $f = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$**.**

(2) Take

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}.$$

Find G such that the equation

$$< G(i\ .), f(.) > = u(i)$$

provides a solution for Au = f for any vector f. (Be aware that high school students know how to do this without ever thinking of the vector δ_i!)