CHAPTER I. INTEGRAL EQUATIONS

SECTION 1.1. GEOMETRY AND ONE TYPE OF LINEAR FUNCTION

Most often, freshman and sophomore mathematics is a study of the calculus of R^n, likely with n = 1, 2, or 3. Instead of working in the space R^n, we now work in a space of functions on a finite interval. Most often, we will take that interval to be [0,1]. Of course, we will not work in the class of *all* functions on [0,1]; rather we ask that the linear space should consist of functions f for which

$$\int_0^1 |f(x)|^2 dx < \infty.$$

Then, we have an inner product space as we did in the finite dimensional calculus. This space is called $L^2([0,1])$. The dot product of two functions is given by

$$< f, g > = \int_0^1 f(x)\, g(x)\, dx$$

and the norm of f is defined in terms of the dot product:

$$|\ f\ |^2 = \int_0^1 |f(x)|^2 dx.$$

It does not seem appropriate to study in detail the nature of $L^2[0,1]$ at this time. Rather, suffice it to say that the space is large enough to contain all continuous functions - even functions which are continuous except at a finite number of places. The interested student can find what $L^2[0,1]$ is by looking in standard books in Real Analysis.

Having an inner product space, we can now decide if f and g in the space are perpendicular. The distance and the angle between f and g are given by the same formulas as we understood from the finite dimensional calculus: the distance from f to g is $|\ f - g\ |$ and the angle α between f and g satisfies

$$\text{cosine}(\alpha) = < f, g >/|f|\ |g|$$

provided neither f nor g is zero.

Suppose $\{ f_p \}_{p=1}^{\infty}$ is a sequence of functions in $L^2([0,1])$. It is

valuable to consider the possible meanings for $\lim_p f_p(x) = g(x)$. There are at least three meanings.

The sequence $\{f_p\}_{p=1}^{\infty}$ converges *point-wise* to g at each x in [0,1] provided that for each x in [0,1],

$$\lim_p f_p(x) = g(x).$$

The sequence converges to g *uniformly* on [0,1] provided that

$$\lim_p \sup_x |f_p(x) - g(x)| = 0.$$

And, the sequence converges to g *in norm* if

$$\lim_p \| f_p - g \| = 0.$$

An understanding of these three modes of convergence should be sought. These are ideas that re-occur in mathematics. In class, we will give examples to contrast these methods of convergence.

A type of integral equation will be studied in this section. For example, given a function called the kernel

$$K: [0,1]x[0,1] \rightarrow R$$

and a function $f: [0,1] \rightarrow R$, we seek a function y such that for each x in [0,1],

$$y(x) = \int_0^1 K(x,t)\, y(t)\, dt + f(x).$$

Such equations are called Fredholm equations of the second kind. An equation of the form

$$0 = \int_0^1 K(x,t)\, y(t)\, dt + f(x)$$

is a Fredholm equation of the first kind.

The requirements in this section on K and f will be that

$$\int_0^1 \int_0^1 |K(x,t)|^2\, dx\, dt < \infty \quad \text{and} \quad \int_0^1 |f(x)|^2\, dx < \infty.$$

These requirements are met if K and f are continuous.

For simplicity, we denote by **K** the linear function given by

$$\mathbf{K}(y)(x) = \int_0^1 K(x,t)\, y(t)\, dt.$$

Note that **K** has a domain large enough to contain all functions y which are continuous on [0,1]. Also, if y is continuous then **K**(y) is a function and its value at x is denoted **K**(y)(x). In spoken conversation, it is not so easy to distinguish the number valued function K and the function valued **K**. The bold character will be used in these notes to denoted the latter.

It is well to note the resemblence of this function **K** to the multiplication of a matrix A by a vector u:

$$A(u)\,(p) = \sum_{q=1}^{\infty} A(p,q)\, u(q).$$

This formula has the same form as that for **K** given above. The analogy should be instructive.

In order to understand **K***, one must consider < **K**(f), g > and seek **K*** such that < **K**f, g > = < f, **K***g >.

$$< \mathbf{K}(f), g > = \int_0^1 \mathbf{K}(f)(x)\, g(x)\, dx$$

$$= \int_0^1 \int_0^1 K(x,t)\, f(t)\, g(x)\, dt\, dx.$$

An examination of these last equations leads one to guess that **K*** is given by

$$\mathbf{K}^*(g)\,(t) = \int_0^1 K(x,t)\, g(x)\, dx,$$

or, keeping t as the variable of integration,

$$\mathbf{K}^*(g)\,(x) = \int_0^1 K(t,x)\, g(t)\, dt,$$

Those last equations verified that

$$< \mathbf{K}(f), g > = < f, \mathbf{K}^*(g) >.$$

Care had to be taken to watch whether the "variable of integration" is t or x in the integrals involved.

In summary, if K is the kernel associated with the linear operator K, then the kernel associated with K* is given by $K^*(x,y) \equiv K(y,x)$. It is of value to compare how to get K* from K with the process of how to get A* from A:

$$A^*_{p,q} = A_{q,p}.$$

Consistent with the rather standard notation we have adopted above, it is clear that a briefer representation of the equation

$$y(x) = \int_0^1 K(x,t)\, y(t)\, dt + f(x)$$

is the concise equation $y = K(y) + f$, or $(1 - K)\, y = f$.

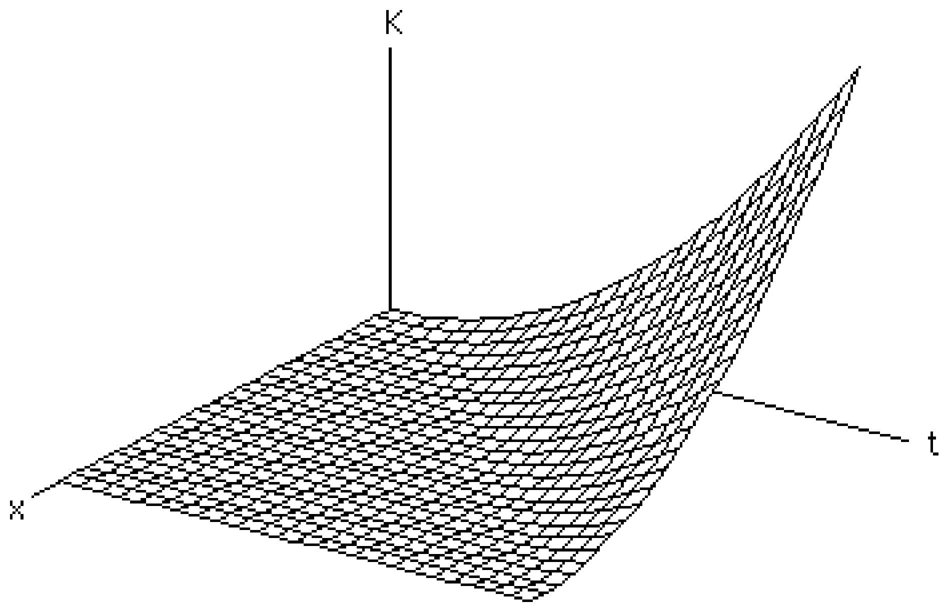
EXAMPLE: Suppose that

$$K(x,t) = \begin{cases} (x-t)^2 & \text{if } 0 < x < t < 1 \\ 0 & \text{if } 0 < t < x < 1 \end{cases} .$$

To get K*, let's use other letters for the argument of K* and K to avoid confusion. Suppose that $0 < u < v < 1$. Then, $K^*(u,v) = K(v,u) = 0$. In a similar manner, $K^*(u,v) = (u-v)^2$ if $0 < v < u < 1$. Note that K* is not K.

$$K^*(x,t) = \begin{cases} 0 & \text{if } 0 < x < t < 1 \\ (x-t)^2 & \text{if } 0 < t < x < 1 \end{cases} .$$

The discussion of this example has been algebraic to this point. Consider this geometric notion that is suggested by the alternate name for "self-adjoint", namely, some call K "symmetric" if $K(x,t) = K(t,x)$. The geometric name suggests a picture and the picture is the graph of K. The K of this example is not symmetric in x and t. Its graph is not symmetric about the line $x = t$. The function K is different from the function K*.



EXERCISE 2.1:

1. (a) Find the distance from $\sin(\pi x)$ to $\cos(\pi x)$ in $L^2[0,1]$ and $L^2[-1,1]$.

   Ans: 1., $\sqrt{2}$

   (b) Find the angle between $\sin(\pi x)$ and $\cos(\pi x)$ in $L^2[0,1]$ and $L^2[-1,1]$.

   Ans: $\pi/2, \pi/2$.

2. Repeat 1. (a) and (b) for x and x^2.

   Ans: $1/\sqrt{30}$, $4/\sqrt{15}$, $\text{Arccos}(\sqrt{15}/4)$, $\pi/2$

3. Suppose $K(x,t) = 1 + 2\,x\,t^2$ on $[0,1]\times[0,1]$ and $y(x) = 3 - x$. Compute K(y) and K*(y).

   Ans: $(5+3x)/2$, $(15+14x^2)/6$

4. Suppose $K(x,t) = \begin{cases} x\,t & \text{if } 0 < x < t < 1 \\ x\,t^2 & \text{if } 0 < t < x < 1 \end{cases}$. For y(x) = 3 - x, compute K(y) and K*(y).

   Ans: $K[y](x) = -\frac{x^5}{4} + \frac{4x^4}{3} - \frac{3x^3}{2} + \frac{7x}{6}$