CHAPTER I. INTEGRAL EQUATIONS

SECTION 2. THE FREDHOLM ALTERNATIVE THEOREMS

A first understanding of the problem of solving an integral equation

$$y = Ky + f$$

can be made by introducing the Fredholm Alternative Theorems in the context of integral equations.

I. Exactly one of the following holds:
(a)(First Alternative) if f is in $L^2\{0,1\}$, then

$$y(x) = \int_0^1 K(x,t)\, y(t)\, dt + f(x)$$

has one and only one solution.

(b)(Second Alternative) $y(x) = \int_0^1 K(x,t)\, y(t)\, dt$

has a nontrivial solution.

II. (a) If the first alternative holds for the equation

$$y(x) = \int_0^1 K(x,t)\, y(t)\, dt + f(x)$$

then it also holds for the equation

$$z(x) = \int_0^1 K(t,x)\, z(t)\, dt + g(x).$$

(b) In either alternative, the equation

$$y(x) = \int_0^1 K(x,t)\, y(t)\, dt$$

and it adjoint equation

$$z(x) = \int_0^1 K(t,x)\, z(t)\, dt$$

have the same number of linearly independent solutions.

III. Suppose the second alternative holds. Then

$$y(x) = \int_0^1 K(x,t)\, y(t)\, dt + f(x)$$

has a solution if and only if

$$\int_0^1 f(t)\, z(t)\, dt = 0$$

for each solution z of the adjoint equation

$$z(x) = \int_0^1 K(t,x)\, z(t)\, dt.$$

Comparing this context for the Fredholm Alternative Theorems with an understanding of similar matrix examples seems irresistible. Since these ideas will re-occur in each section, the student should pause to make these comparisons.

EXAMPLE: Suppose that E is the linear space of continuous functions on the interval [-1,1]. with

$$< f, g > = \int_{-1}^1 f(x)\, g(x)\, dx$$

and that

$$K(x,t) = \frac{1}{2} + x\,(3t^2 - 1).$$

Then $$K^*(x,t) = \frac{1}{2} + t\,(3x^2 - 1)$$

for $$< K(f), g > = \int_{-1}^1 K(f)(t)\, g(t)\, dt$$

$$= \int_{-1}^1 \left(\int_{-1}^1 [\frac{1}{2} + t\,(3s^2 - 1)]\, f(s)\, ds \right) g(t)\, dt$$

$$= \int_{-1}^1 \int_{-1}^1 [\frac{1}{2} + t\,(3s^2 - 1)]\, f(s)\, g(t)\, ds\, dt$$

$$= \int_{-1}^{1}\int_{-1}^{1} [\frac{1}{2} + s\,(3t^2 - 1)]\, f(t)\, g(s)\, dt\, ds$$

$$= \int_{-1}^{1} f(t) \int_{-1}^{1} [\frac{1}{2} + s\,(3t^2 - 1)]\, g(s)\, ds\;\; dt$$

$$= \int_{-1}^{1} f(t)\, K^*(g)(t)\, dt.$$

The equation y = K(y) has a non-trivial solution: the constant function 1. To see this, one computes

$$K[1](x) = \int_{-1}^{1} [\frac{1}{2} + x\,(3t^2 - 1)]\; 1\; dt = 1 + 0\;\; x = 1.$$

One implication of these computations is that the problem y = Ky + f is a second alternative problem. It may be verified that y(x) = 1 is also a nontrivial solution for y = K*y. It follows from the third of the Fredholm alternative theorems that a necessary condition for y = Ky + f to have a solution is that

$$0 = \langle f, 1 \rangle = \int_{-1}^{1} f(x)\; 1\; dx.$$

Note that one such f is $f(x) = x + x^3$.

EXERCISE 1.2

(1) Suppose that E is the linear space of continuous functions on [0,1] with

$$\langle f, g \rangle = \int_{0}^{1} f(x)\, g(x)\, dx$$

and that

$$K(x,t) = 1 + \sin(\pi x)\cos(\pi t).$$

Show that $$K^*(x,t) = 1 + \cos(\pi x)\sin(\pi t).$$

(2) Show that y = Ky has non-trivial solution the constant function 1.

(3) Show that y = K*y has non-trivial solution the function $\pi + 2\cos(\pi x)$.

(4) What conditions must hold on f in order that

$$y = Ky + f$$

should have a solution?