CHAPTER I. INTEGRAL EQUATIONS

SECTION 3. SOLVING y = K(y) + f WHERE K HAS A SEPARABLE KERNEL.

We now analyze the equation y = Ky + f in three cases:

K has a separable kernel,

K has norm less than one,

and K is approximated by K's with separable kernels.

This section develops the calculus in case K has a separable kernel.

Definition: The integral operator

$$K[y](x) = \int_0^1 K(x,t)\, y(t)\, dt$$

has a *separable kernel* if its kernel is given by

$$K(x,t) = \sum_{p=1}^{n} a_p(x)\, b_p(t),$$

where the functions

$$\{a_p(x)\}_{p=1}^{n} \text{ and } \{b_p(t)\}_{p=1}^{n}$$

are in $L^2[0,1]$.

With the supposition that K is separable, it is not hard to find y such that y = Ky + f, for this equation can be re-written as

$$y(x) = \sum_{p=1}^{n} a_p(x) \int_0^1 b_p(t)\, y(t)\, dt + f(x)$$

or, using the notation of inner products,

$$y(x) = \sum_{p=1}^{n} a_p(x) \langle b_p, y\rangle + f(x).$$

One can guess that, if the sequence

$$\{a_p(x)\}_{p=1}^{n}$$

of functions on [0,1] is a linearly independent sequence, then y will have this special form:

there is a sequence $\{c_p\}_{p=1}^{n}$ of numbers such that

$$y(x) = \sum_{p=1}^{n} c_p\, a_p(x) + f(x).$$

In fact, supposing there is such a sequence, we determine what it should be.

Suppose

$$y(x) = \sum_{p=1}^{n} c_p\, a_p(x) + f(x).$$

Substitute this in the equation to be solved:

$$\sum_{p=1}^{n} c_p a_p(x) + f(x)$$

$$= \sum_{p=1}^{n} a_p(x) \;\langle b_p, \sum_{q=1}^{n} c_q a_q + f \rangle + f(x)$$

and we see that

$$c_p = \sum_{q=1}^{n} \langle b_p, a_q \rangle c_q + \langle b_p, f \rangle.$$

This now reduces to a matrx problem:

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} \langle b_1,a_1\rangle & \cdots & \langle b_1,a_n\rangle \\ \langle b_2,a_1\rangle & \cdots & \langle b_2,a_n\rangle \\ \vdots & & \vdots \\ \langle b_n,a_1\rangle & \cdots & \langle b_n,a_n\rangle \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} + \begin{pmatrix} \langle b_1,f\rangle \\ \langle b_2,f\rangle \\ \vdots \\ \langle b_n,f\rangle \end{pmatrix}$$

Define K and f to be the matrix and vector so defined that the last equation is rewritten as

$$c = K\,c + f.$$

We now employ ideas from linear algebra. The equation $c = K\,c + f$ has exactly one solution provided

$$\det(1 - K) \neq 0.$$

The Fredholm Alternative Theorems for matrices address these ideas. If the sequence

$$\{c_p\}_{p=1}^{n}$$

is found then we have a formula for y(x).

EXAMPLE: In Exercises 2.2, it should have been established that if

$$K(x,t) = 1 + \sin(\pi x)\cos(\pi t),$$

then

$$K^*(x,t) = 1 + \sin(\pi t)\cos(\pi x).$$

Also,

$y = Ky$ **has solution** $y(x) = 1$

and

$y = K^*y$ **has solution** $y(x) = \pi + 2\cos(\pi x)$.

It is the promise of the Fredholm Alternative theorems that

$$y = Ky + f$$

has a solution provided that

$$\int_0^1 [\pi + 2\cos(\pi t)]\, f(t)\, dt = 0.$$

Let us try to solve $y = Ky + f$ and watch to see where the requirement that f should be perpendicular to the function $\pi + 2\cos(\pi t)$ appears.

To solve $y = Ky + f$ is to solve

$$y(x) = \int_0^1 y(t)\, dt + \sin(\pi x)\int_0^1 \cos(\pi t)\, y(t)\, dt + f(x).$$

We guess that the solution is of the form $y(x) = a + b\sin(\pi x) + f(x)$ and substitute this for y:

$$a + b\sin(\pi x) + f(x) = \int_0^1 [a + b\sin(\pi t) + f(t)]\, dt +$$

$$\sin(\pi x)\Big[\, a\int_0^1 \cos(\pi t)dt + b\int_0^1 \cos(\pi t)\sin(\pi t)dt +$$

$$\int_0^1 \cos(\pi t) f(t) dt\Big] + f(x).$$

From this, we get the algebraic equations

$$a = a + \frac{2}{\pi} b + \int_0^1 f(t)\, dt$$

$$b = 0\ \ a + 0\ \ b + \int_0^1 \cos(\pi t)\, f(t)\, dt.$$

Hence, in our guess for y, we find that a can be anything and that b must be

$$-\frac{\pi}{2}\int_0^1 f(t)\,dt$$

and also must be

$$\int_0^1 \cos(\pi t)\, f(t)\, dt.$$

The naive pupil might think this means there are two (possibily contradictory) requirements on b. The third of the Fredholm Alternative theorems assures the student that there is only one requirement!

EXERCISE 2.3:

I. With K, f, and an interval as given, solve the integral equation y = Ky + f.

(a) $K(x,t) = 2x-t$, $f(x) = x^2$ on [1,2]. ans: $y(x) = x^2 - (75x - 61)/6$.

(b) $K(x,t) = x + 2xt$, $f(x) = x$ on [0,1]. ans: $y(x) = -6x$.

(c) $K(x,t) = 2x^2 -3t$, $f(x) = x$ on [0,1]. ans: $y(x) = x +(6x^2-13)/28$.

(d) $K(x,t) = t(t+x)$, $f(x) = x$ on [0,1] ans: $y(x) =(18+48x)/23$.

(e) $K(x,t) = xt^2+1$, $f(x) = x$ on [0,1] ans: $y(x) = -3$.

(f) $K(x,t) = 1/2 + x t$, $f(x) = 3x^2-1$ on [-1,1]. ans: $y(x) =3x^2 + c$

(g) $K(x,t) = x t$, $f(x) = \exp(x)$ on [0, ln(7)]. ans: $y(x) = e^x+ax$ where a is $3(7\ln(7)-6)/(3-(\ln(7)^3)$

(h) $K(x,t) = x - t$, $f(x) = x$ on [0,1]. ans: $y(x) = (18x-4)/13$

(i) $K(x,t) = \sin(\pi x) \cos(\pi t)$, $f(x) = \sinh(x)$ on [0,1].

II. Show that if f is continuous and $1 + \lambda/2 - \lambda^2/240 \neq 0$, then

$$y(x) = -\lambda \int_0^1 (x^2 t + x t^2)\, y(t)\, dt + f(x)$$

has a solution.

III. (a) For what functions f does the equation

$$y(x) = \int_{-1}^1 [1/2 + \sin(\pi x) \sin(\pi t)]\, y(t)\, dt + f(x)$$

have a solution? **Ans:** $\int_{-1}^{1} f(t)dt$ **and** $\int_{-1}^{1} \sin(\pi t)f(t)dt$ **are 0.**

(b) Repeat (a) for $y(x) = \int_{-1}^{1} \sin(\pi(x+t))\, y(t)\, dt + f(x)$.

IV. Solve the integral equation y = Ky + f where

$$K(y)(x) = \int_{0}^{x} (x - t)\, y(t)\, dt$$

and f(x) = x. (Hint: take the derivative of both sides.)

ans: sinh(x)