CHAPTER I. INTEGRAL EQUATIONS

SECTION 4: SOLVING $y = Ky + f$, where K IS SMALL.

If the kernel is not separable, an alternate hypothesis that will enable one to solve the equation $y = Ky + f$ is to suppose that the kernel for K is small. Of course this does not mean that K is of the form $K(x,t) = .007\, x\, t$. Rather, we ask that K should be small in a sense developed below. The technique for getting a solution in this case is to iterate.

Take $\varphi_0(x)$ to be $f(x)$ and φ_1 to be defined by

$$\varphi_1(x) = \int_0^1 K(x,t)\, \varphi_0(t)\, dt + f(x).$$

In general,

$$\varphi_{n+1}(x) = \int_0^1 K(x,t)\, \varphi_n(t)\, dt + f(x).$$

It is reasonable to ask: does this generated sequence converge to a limit and in what sense does it converge? The answer to both questions can be found under appropriate hypothesis on K.

THEOREM If K satisfies the condition that

$$\max_x \int_0^1 |K(x,t)|\, dt < 1,$$

then $\lim_p \varphi_p(x)$ exists and the convergence is uniform on [0,1] - in the sense that if $u = \lim_p \varphi_p$ then

$$\lim_p \max_x |\, u(x) - \varphi_p(x)\, | = 0.$$

SUGGESTION FOR PROOF: Note that

$$|\, \varphi_1(x) - \varphi_0(x)\, | = \left|\int_0^1 K(s,t)\, f(t)\, dt\right| \le \int_0^1 |K(x,t)|\, dt\ \max_x |f(x)|.$$

Furthermore, if p is a positive integer, the distance between successive iterates can be computed:

$$|\varphi_{p+1}(x) - \varphi_p(x)| = \left| \int_0^1 K(x,t)\,[\varphi_p(t) - \varphi_{p-1}(t)]\,dt \right|$$

$$\leq \int_0^1 |K(x,t)|\,dt \ \max_x |\varphi_p(x) - \varphi_{p-1}(x)|.$$

Inductively, this does not exceed

$$\left[\max_x \int_0^1 |K(x,t)|\,dt\right]^{p+1} \max_x |f(x)|.$$

Thus, if

$$r = \max_x \int_0^1 |K(x,t)|\,dt$$

and n > m then

$$|\varphi_n(x) - \varphi_m(x)| \leq \frac{r^{m+1}}{1-r} \max_x |f(x)|.$$

Hence, the sequence $\{\varphi_p\}_{p=1}^{\infty}$ of functions converges uniformly on [0,1] to a limit function and this limit provides a solution to the equation

$$u(x) = \int_0^1 K(x,t)\,u(t)\,dt + f(x).$$

COROLLARY. If $r = \max_x \int_0^1 |K(x,t)|\,dt$

and

$$u = \lim_p \varphi_p$$

then

$$\max_x |u(x) - \varphi_m(x)| \leq \frac{r^{m+1}}{1-r} \max_x |f(x)|.$$

EXERCISE 1.4:

1. Let $K(x,t) = \begin{cases} x-t & \text{if } t < x \\ 0 & \text{if } x < t \end{cases}.$

(a) Show that if $0 \le x \le 1$, $\int_0^1 |K(x,t)|\, dt < 1$.

In fact,

$$\int_0^1 |K(x,t)|\, dt = x^2/2.$$

(b) Toward solving $y(x) = K[y](x) + x$, **compute** φ_0, φ_1, **and** φ_2.

ans: x, $\frac{x^3}{6} + x$, $\frac{x^5}{5!} + \frac{x^3}{3!} + x$

(c) Give a bound on the error between the solution y and φ_2.

ans: $|y - \varphi_2| \le \frac{1}{4}$

(d) Solve $y(x) = Ky(x) + x$ **in closed form for this K. (Reference Exercise 2.3 IV.)**

2. Repeat 1 (a)-(c) with $K(x,t) = \begin{cases} 1 - t & \text{if } x < t \\ 1 - x & \text{if } t < x \end{cases}.$

ans: $\int_0^1 |k(x,t)|\, dt \le .5$, $\varphi_1(x) = \frac{1-x^3}{6} + x$, $|y - \varphi_2| \le .25$