CHAPTER I. INTEGRAL EQUATIONS

SECTION 5: ALTERNATE "K IS SMALL"

There is an alternate, and independent, concept of K being small which leads to convergence of the iteration process in the norm of $L^2[0,1]$ This alternate hypothesis asks that

$$\int_0^1 \int_0^1 |K(x,t)|^2 \, dt \, dx < 1.$$

THEOREM If K satisfies the condition that

$$\int_0^1 \int_0^1 |K(x,t)|^2 \, dt \, dx < 1,$$

then $\lim_p \varphi_p(x)$ exists and the convergence is in norm- meaning that if $u = \lim_p \varphi_p$ then

$$\lim_p \; || u(x) - \varphi_p(x) || = 0.$$

INDICATION OF PROOF. The analysis of the nature of the convergence will go like this:

$$|| \varphi_1 - \varphi_0 ||^2$$

is defined to be

$$\int_0^1 |\varphi_1(x) - \varphi_0(x)|^2 \, dx$$

$$= \int_0^1 \left| \int_0^1 K(x,t) f(t) \, dt \right|^2 dx$$

$$\leq \int_0^1 \int_0^1 |K(x,t)|^2 \, dt \, dx \int_0^1 |f(t)|^2 \, dt.$$

As before,

$$|\varphi_n - \varphi_m|^2 \leq \frac{r^{m+1}}{1-r} \, || f ||^2$$

where

$$r = \int_0^1 \int_0^1 |K(x,t)|^2 \, dt \, dx.$$

COROLLARY. If $r = \int_0^1 \int_0^1 |K(x,t)|^2 \, dt \, dx$

and

$$u = \lim_p \varphi_p$$

then

$$|u - \varphi_m|^2 \leq \frac{r^{m+1}}{1-r} |f|^2.$$

THE RESOLVENT.

Before addressing the final case - where

K does not have a separable kernel,

nor is $\int_0^1 |K(x,t)| \, dt < 1$,

nor is $\int_0^1 \int_0^1 |K(x,t)|^2 \, dt \, dx < 1$,

we generate "resolvents" for the integral equations.

Re-examining the iteration process:

$\varphi_0(x) = f(x)$,

$\varphi_1(x) = K\varphi_0(x) + f(x)$

$\varphi_2(x) = K(K(\varphi_0))x + K(f)(x) + f(x)$

.

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One writes $\varphi_0 = f$, $\varphi_1 = Kf + f$, $\varphi_2 = K[Kf+f] + f = K^2 f + Kf + f$, ...

In fact, with

$$Kf(x) = \int_0^1 K(x,t) f(t) \, dt$$

$$K^2 f(x) = \int_0^1 K(x,t)[K(f)](t) \, dt$$

$$= \int_0^1 K(x,t) \left[\int_0^1 K(t,s) f(s)\, ds \right] dt$$

$$= \int_0^1 \left[\int_0^1 K(x,t)\, K(t,s)\, dt \right] f(s)\, ds.$$

Hence, the kernel K_2 associated with K^2 is

$$K_2(x,t) = \int_0^1 K(x,s)\, K(s,t)\, ds.$$

Inductively, $K^n f(x) = \int_0^1 K(x,t)\, [K^{n-1} f](t)\, dt$, $K_n(x,t)$

$$= \int_0^1 K(x,s)\, K_{n-1}(s,t)\, ds$$

and

$$\varphi_n(x) = f(x) + \sum_{p=1}^{\infty} K^p f\, (x).$$

We have, in this section, conditions which imply that

$$\sum_{p=1}^{\infty} K^p f$$

converges and that its limit y satisfies $y = Ky + f$. Many authors call this series of operators the "resolvent" and denote

$$R = \sum_{p=1}^{\infty} K^p.$$

Note that R is a function which operates on elements of $L^2[0,1]$. One writes that

$$y = Ky + f$$

has solution

$$y(x) = [(1 + R) f](x) = f(x) + \int_0^1 R(x,t)\, f(t)\, dt.$$

Suggestive algebra can be made by identifying (1 + R) as

$$(1 - K)^{-1} = 1 + K(1 - K)^{-1}, \text{ so that } R = K(1 - K)^{-1}.$$

EXERCISE 1.5.

1. Suppose that $K(x,y) = \begin{cases} f(x)\,g(y) & \text{if } x \le y \\ h(x)\,J(y) & \text{if } y \le x \end{cases}$**. Give a formula for**

$$\int_0^1 K(x,s)\,K(s,y)\,ds.$$

2. Compute $\int_0^1 \int_0^1 |K(x,t)|^2\,dt\,dx$ **for each K in the previous exercise set.**

ans: 1/12 and 1/6.

3. Let $K(x,t) = \begin{cases} 2 & \text{if } t < 1/4 \\ 0 & \text{if } t > 1/4 \end{cases}$**.**

For this K, find y such that y(x) = K[y](x) + x. Note that

$$\int_0^1 K(x,t)\,dt < 1 \text{ and } \int_0^1 \int_0^1 K(x,t)^2\,dt\,dx = 1.$$

What is the significance of this observation?

ans: x +1/8

4. Let $K(x,t) = \begin{cases} 1 & \text{if } t < x \\ 0 & \text{if } x < t \end{cases}$**.**

For this K, find y such that y(x) = K[y](x) + x. Note that

$$\max_x \int_0^1 K(x,t)\,dt = 1 \text{ and } \int_0^1 \int_0^1 K(x,t)^2\,dt\,dx < 1.$$

What is the significance of this observation?

ans: exp(x) - 1

5. Suppose that

$$K[y](x) = \int_0^1 \cos(x+t)y(t)dt \text{ and } H[y](x) = \int_0^1 \sin(x+t)y(t)dt,$$

so that the kernel of K is cos(x+t) and the kernel of H is sin(x+t). What is the kernel of K[H]?

ans: $\int_0^1 \cos(x+s)\sin(s+t)ds.$

6. Find the kernel for the resolvent of the K whose kernel is K(x,t) = x t.

Ans: $R(x,t) = K(x,t) + K_2(x,t) + K_3(x,t) + . = 3xt/2.$