CHAPTER I. INTEGRAL EQUATIONS

SECTION 6: K DOES NOT HAVE A SEPARABLE KERNEL AND IS NOT "SMALL".

In case K neither has a separable kernel nor is small, then the next best choice is to approximate K with an operator which has a separable kernel.

Theorem. If

$$\int_0^1 \int_0^1 |K(x,t)|^2 \, dx\, dt < \infty$$

then there are kernels K_n and G such that

(1) $K = K_n + G$,

(2) K_n has a separable kernel,

and (3) $\int_0^1 \int_0^1 |G(x,t)|^2 \, dx\, dt < 1.$

In the succeeding pages, we show how to compute K_n and G. However, we first illustrate that the problem is - in theory - solved if we have such a resolution of K into K_n and G. We seek y such that

$$y = Ky + f = K_n y + Gy + f$$

or

$$y - Gy = K_n y + f.$$

Use the resolvent for G:

$$(1-G)^{-1} = 1 + R_G,$$

to get that

$$y = K_n y + R_G(K_n y + f) + f$$

$$=[K_n + R_G K_n]y + (R_G f + f).$$

Define z to be $R_G f + f$, or, what is the same, solve the equation

$$z = Gz + f.$$

We can solve this equation because Gis small. Now, we seek y such that

$$y = (K_n + R_G K_n)y + z.$$

Re-writing this as an integral equation, we seek y such that

$$y(x) = \int_0^1 H(x,t)\, y(t)\, dt + z(x)$$

where

$$H(x,t) = K_n(x,t) + \int_0^1 R_G(x,s)\, K_n(s,t)\, ds.$$

What is astonishing is that this last integral equation is separable! To see this, suppose

$$K_n(x,t) = \sum_{p=1}^{n} a_p(x)\, b_p(t).$$

Then

$$\int_0^1 R_G(x,s) K_n(s,t)\, ds$$

$$= \int_0^1 R_G(x,s) \sum_{p=1}^{n} a_p(s) b_p(t)\, ds$$

$$= \sum_{p=1}^{n} \int_0^1 R_G(x,s) a_p(s)\, ds\ \ b_p(t)$$

So, here is the conclusion. If K is K_n + G as in the above Theorem, in order to solve y = Ky + f, use the fact that

$$\int_0^1 \int_0^1 |G(x,t)|^2\, dx\, dt < 1$$

to form the resolvent for R_G; then find z such that z = $(1+R_G)$f. Finally, solve the separable equation y = $(K_n + R_G K_n)$y + z °

We now must address the question of how to achieve the decomposition of K into K_n + G. The ideas are familiar to persons knowledgeable about the techniques of Fourier Series. In summary of those ideas, recall that if p and q are integers, then

$$\int_0^1 \sin(p\pi x) \sin(q\pi x)\, dx = 0 \text{ if } p \neq q$$

$$= \frac{1}{2} \text{ if } p = q.$$

We seek Apq such that

$$K(x,y) = \sum_{p=1}^{\infty}\sum_{q=1}^{\infty} A_{pq} \sin(p\pi x)\sin(q\pi y).$$

In fact, by integrating both sides of this last equation after multiplying by sin(mπx) sin(nπy), we have

$$\int_0^1\int_0^1 K(x,y)\sin(m\pi x)\sin(n\pi y)\,dx\,dy = \frac{A_{mn}}{4}$$

From the theory of Fourier series,

$$\lim_n \sum_{p=1}^{n}\sum_{q=1}^{n} A_{pq}\sin(p\pi x)\sin(q\pi y) = K(x,y)$$

in the sense that

$$\left|\int_0^1\int_0^1 [K(x,y) - \sum_{p=1}^{n}\sum_{q=1}^{n} A_{pq}\sin(p\pi x)\sin(q\pi y)]^2\,dx\,dy\right| \to 0$$

as n $\to \infty$. Let n be an integer such that

$$\left|\int_0^1\int_0^1 [K(s,y) - \sum_{p=1}^{n}\sum_{p=1}^{n} A_{pq}\sin(p\pi x)\sin(q\pi y)]^2\,dx\,dy\right| < 1.$$

Define K_n and G by

$$K_n(x,y) = \sum_{p=1}^{n}\sum_{q=1}^{n} A_{pq}\sin(p\pi x)\sin(q\pi y)$$

and $$G = K - K_n.$$

Then these three requirements are met:

(1) $K = K_n + G$,
(2) K_n is separable,

and (3) $$\int_0^1\int_0^1 |G(x,y)|^2\,dx\,dy < 1.$$

Thus, we have an analysis of an integral equation y = Ky + f where

$$\int_0^1 \int_0^1 |K(x,y)|^2 \, dx \, dy < \infty .$$

The engineer will want to know about approximations. Here are two appropriate questions:

(a) Suppose one hopes to solve y = Ky + f and that K_n is separable and approximates K. How well does the solution u for u = K_nu + f approximate y?

(b) Suppose K = K_n + G and S = $\sum_{p=1}^{n} G^p$ approximates R_G. How well does the solution u for

$$u = [K_n + SK_n]\, u + [1+S]f$$

approximate y?

Exercise 1.6.

Solve the integral equation y = K[y] + f in case f(t) = t and

$$K(x,t) = \begin{cases} x\,t + x - t & \text{if } t < x \\ x\,t & \text{if } t > x \end{cases}$$