CHAPTER II. BOUNDARY VALUE PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS.

This section will continue to use the basic idea that a collection of functions defined on an interval might form a vector space and that the vector space of functions might have an inner product defined on it.

SECTION 2.1 MORE ABOUT $L^2[0,1]$

Recall $L^2[0,1]$ with the "usual" inner product as an inner product space of functions on [0,1]. One should be aware that there are many choices that can be made for an inner product for the functions on [0,1]. One might have a weighted inner product. That is, if w(x) is a continuous, non-negative function, we might have

$$\langle f, g \rangle = \int_0^1 f(x)\, g(x)\, w(x)\, dx.$$

The choices for w(x) are usually suggested by the context in which the space arises.

Some inner product spaces are called HILBERT SPACES. A Hilbert Space is simply a vector space on which there is an inner product and on which there is one more bit of structure. A Hilbert space is an inner product space that is *complete*. This means that if $\{f_p\}_{p=0}^{\infty}$ is an infinite sequence of vectors in the space which is Cauchy convergent - - meaning that

$$\lim_{n,m} \langle f_n - f_m , f_n - f_m \rangle = \lim_{n,m} | f_n - f_m |^2 = 0$$

-- then there is a vector g, also in the space, such that $\lim_n f_n = g$.

To illustrate these ideas, two examples follow. In the first, there is a sequence $\{f_p\}_{p=0}^{\infty}$ and a function g in the space with $\lim_n f_n = g$. In the second, there is no such g.

EXAMPLE: Let E be the vector space of continuous functions on [0,1] with the usual inner product. Let

$$f_n(x) = \begin{cases} 1/n & \text{if } 0 \le x \le 1/n \\ x & \text{if } 1/n \le x \le 1 \end{cases}$$

and let g(x) = x on [0,1]. Then $\lim_n |f_n - g|^2$

$$= \lim_n \int_0^1 [\, f_n(x) - g(x)\,]^2 \, dx$$

$$= \lim_n \int_0^{1/n} (1/n - x)^2 \, dx = 0.$$

EXAMPLE. This space E of continuous functions on [0,1] with the "usual" inner product is not complete. To establish this, we provide a sequence $\{f_p\}_{p=0}^{\infty}$ for which there is no <u>continuous</u> function g such that $\lim_n f_n = g$.

$$f_n(x) = \begin{cases} 0 & \text{if } 0 \le x \le (n-1)/2n \\ n(x - 1/2) + 1/2 & \text{if } (n-1)/2n \le x \le (n+1)/2n \\ 1 & \text{if } (n+1)/2n \le x \le 1 \end{cases}$$

Sketch the graphs of f_1, f_2, and f_3 to see that the limit of this sequence of functions is not continuous.

The Riesz Representation Theorem is an important idea in a Hilbert Space. You will recall that in R^n, this result declared that if L is a linear function from R^n to R then there is a vector v in R^n such that $L(x) = \langle x, v \rangle$ for each x in R^n. In general, where the vector space is not R^n, more is required.

THEOREM If $\{E, \langle\ ,\ \rangle\}$ is a Hilbert space, then these are equivalent:
(a) L is a continuous, linear function from E to R, and
(b) there is a member v of E such that $L(x) = \langle x, v \rangle$ for each x in E.

It is not hard to show that (b) implies (a). Recall only the Cauchy-Schwartz inequality:

$$|\langle x, y \rangle| \le |x|\ |y|.$$

To show that (a) implies (b) is more interesting. The argument would use the fact that E is complete.

EXAMPLE. Since the space of continuous functions on [0,1], denoted C[0,1], with the usual inner product is <u>not</u> a Hilbert space, one should expect that there might be a linear function L from C[0,1] to R for which there is no v in C[0,1] such that

$$L(f) = \int_0^1 f(x)\, v(x)\, dx$$

for each f in C[0,1]. In fact, here is such an L:

Let $L(f) = \int_{3/4}^{1} f(x)\,dx$. **The candidate for v is v(x) =1 on [3/4, 1] and 0 on [0, 3/4). But this v is not continuous! It is only piecewise continuous.**

DEFINITION. The Heavyside function H is defined as follows:

$$H(x) = \begin{cases} 1 \text{ if } x > 0 \\ 0 \text{ if } x < 0 \end{cases}.$$

Note that $\int_{0}^{1} f(x)\,H(x- 3/4)\,dx = \int_{3/4}^{1} f(x)\,dx.$

Thus, the Heavyside function provides an element v for which the linear function

$$L(f) = \int_{3/4}^{1} f(x)\,dx$$

has a Riesz representation. As noted, v is not in C[0,1].

It is not always possible to have a piecewise continuous v which will rectify the situation. Consider the following linear function: L(f) = f(1/2). It is not so hard to see that there is no piecewise continuous function v on [0,1] having the property that for every continuous f,

$$\int_{0}^{1} f(x)\,v(x)\,dx = f(1/2).$$

DEFINITION. The symbol δ is used to denote the "generalized" function which has the property that

$$\int_{0}^{1} f(x)\,\delta(x,a)\,dx = f(a)$$

for some suitably large class of functions f. It is no surprise that some effort has been made to develop a theory of generalized functions in which the delta function can be found. These functions are called "distributions".

The theory of distributions is attractive and establishes a precise basis for the ideas which these notes will use. It is the choice of this course and these notes to use the delta function without exploring the mathematical framework in which it should be studied.

We will return to this δ function when its properties are needed to understand how to construct Green's functions.

Exercise 2.1

1. Let $L(f) = \int_0^1 f(x)\, H(x-a)\, dx$ **for all f in** $L^2([0,1])$**. Give a number b such that**

$$|L(f)| \le b\, |f|$$

for all f in $L^2([0,1])$**.**

2. Let $L(f) = f(1/2)$ **for all f in** $L^2([0,1])$**. Give a sequence** f_n **so that** $|f_n| = 1$ **and** $|L(f_n)| = n$**.**

3. Show that

$$\lim_{n \to \infty} \int_0^1 f(x) \frac{H(x+h-a)-H(x-a)}{h}\, dx = \int_0^1 f(x)\, \delta(x,a)\, dx.$$