SECTION 2.2. DIFFERENTIAL OPERATORS AND THEIR ADJOINTS

As suggested in the first chapter the role of the adjoint of a linear function will be critical. If L is a linear function that is defined on some subspace of E, then the task of finding the adjoint L* will involve finding not only how the adjoint is defined, but also what is the subspace composing the domain of L*.

Consider the differential operator L given by

$$L(y) = \sum_{p=0}^{n} a_p(x)\, y^{(p)}(x) = a_n(x)y^{(n)}(x) + \ldots + a_1(x)y'(x) + a_0(x)y(x).$$

One often defines the <u>formal adjoint L* of L</u> by

$$L^*(y) = \sum_{p=0}^{n} (-1)^p\big(a_p(x)\, y(x)\big)^{(p)} = (-1)^n\big(a_n(x)\, y(x)\big)^{(n)} + \ldots + a_0(x)\, y(x).$$

The second order operator $L(y) = a_2(x)\, y''(x) + a_1(x)\, y'(x) + a_0(x)\, y(x)$, according to this formula, will have formal adjoint

$$L^*(y) = \big(a_2(x)\, y(x)\big)'' - \big(a_1(x)\, y(x)\big)' + a_0(x)\, y(x).$$

If L is not defined on all of E, but just some subspace, or manifold M, then one must find where L* is defined. We denote the domain of L* by M*.

DEFINITION. Suppose that L is defined on a manifold, or subspace, M, and that L* is defined on a manifold M*. Then L* is <u>THE</u> adjoint of L if

$$\int_0^1 [v\, L(u) - L^*(v)\, u] = 0$$

for all u in M and v in M*.

EXAMPLE Let $L(u) = u''(x) + 3x\, u'(x) + x^2\, u(x)$ be defined on the manifold consisting of all functions u on [0,1] which satisfy $u(0) = u'(1)$ and $u(1) = u'(0)$,

$$M = \{u : u(0) = u'(1),\ u(1) = u'(0)\}.$$

We indicated how to find L* and M*.

$$L^*(v) = v'' - (3x\, v(x))' + x^2 v(x).$$

The manifold M* is chosen to make the equation

$$\langle L(u), v\rangle = \langle u, L^*(v)\rangle$$

satisfied.

$$\int_0^1 [L(u)(x)\, v(x) - u(x)\, L^*(v)(x)]\, dx$$

$$= \int_0^1 \left([vu'' - uv''] + [3x\, u'\, v + (3x\, v)'\, u] \right) dx$$

$$= \int_0^1 (u'v - uv')'\, dx + \int_0^1 [u(x)\, 3x\, v(x)]'\, dx$$

$$= [u'(1)\, v(1) - u(1)\, v'(1)] - [u'(0)\, v(0) - u(0)\, v'(0)]$$

$$+ u(1)\, 3\, v(1) - u(0)\, 3 \quad 0 \quad v(0)$$

Since u is in M, $u'(1) = u(0)$ and $u(1) = u'(0)$, so that

$< L(u), v > - < u, L^*(v) > = [v(1) + v'(0)]u(0) - v'(1) + v(0) - 3v(1)]\, u'(0)$.

In order for this last line to be zero for all u in M, the manifold M* should consist of functions f such that

$v(1) + v'(0) = 0$ and $v'(1) + v(0) = 3v(1)$.

This previous example has introduced the type problems that we shall consider. We will provide a method to get solutions for ordinary differential equations with boundary conditions. The problem, together with the boundary conditions, defines the operator L and a manifold M.

TYPICAL PROBLEM: The problem $y''+ 3y' + 2y = f$, $y(0) = y'(0) = 0$ leads to an operator L and a manifold M given by

$$L(y) = y'' + 3y'+2y \text{ and } M = \{y: y(0) = y'(0) = 0 \}.$$

The problem is to solve the following equation: given a continuous function f, find y in M such that L(y) = f. The technique is to construct a function G such that u is given by

$$u(x) = \int_0^1 G(x,t)\, f(t)\, dt.$$

For such problems, we will have techniques to construct G. In this case

$$G(x,t) = \begin{cases} 0 & \text{if } 0 < x < t < 1 \\ e^{t-x} - e^{2(t-x)} & \text{if } 0 < t < x < 1. \end{cases}$$

Most frequently, we will consider three types of boundary conditions illustrated below for a second order problem:

Initial conditions

$$\begin{cases} 0 = B_1(u) = u(0) \\ 0 = B_2(u) = u'(0) \end{cases}$$

unmixed, two point boundary conditions

$$\begin{cases} 0 = B_1(u) = au(0)+bu'(0) \\ 0 = B_2(u) = cu(1)+du'(1) \end{cases}$$

mixed, two point boundary conditions

$$\begin{cases} 0 = B_1(u) = a_{11}u(0)+a_{12}u'(0)+b_{11}u(1)+b_{12}u'(1) \\ 0 = B_2(u) = a_{21}u(0)+a_{22}u'(0)+b_{21}u(1)+b_{22}u'(1) \end{cases}$$

EXERCISE 2.2:

(1) Compute the formal adjoint for each of the following:

(a) $L(y) = x^2 y'' + x y' + y$ **(b)** $L(y) = y'' + 9\pi^2 y$

(c) $L(y) = (e^x y'(x))' + 7 y(x)$ **(d)** $L(y) = y'' + 3y' + 2y$

(2) Argue that L is formally self adjoint if it has constant coefficients and derivatives of even order only.

(3) Suppose that $L(y) = y'' + 3y' + 2y$ **and** $y(0)= y'(0) = 0$**.**
Find conditions on v which assure that

$$\int_0^1 [vL(y) - L^*(v)\, y] = 0.$$

(4) Let $L(u) = u'' + u$**. The formal adjoint of L is given by** $L^*(v) = v''+ v$**. For each manifold M given below, find M* such that L* on M* is the adjoint of L on M.**

(a) $M = \{u: u(0)=u(1)=0\}$, **ans:** $M = M^*$

(b) $M = \{u: u(0)=u'(0)=0\}$ **ans:** $M^* = \{z: :z(1) = z'(1) =0\}$

(c) $M = \{u: u(0)+3u'(0)=0, u(1)-5u'(1)=0\}$, **ans:** $M = M^*$

(d) $M = \{u: u(0)=u(1), u'(0)=u'(1) \}$. **ans:** $M = M^*$

(5) Let L and M be as given below; find L* and M*.

(a) $L(u)(x) = u''(x) + b(x)\, u'(x) + c(x)\, u(x)$,
$M = \{u: u(0)=u'(1), u(1) = u'(0) \}$.

(b) $L(u)(x) = -(p(x)\, u'(x))' + q(x)\, u(x)$;
$M = \{u: u(0) = u(1), u'(0) =u'(1) \}$.

(hint: Consider $\int_0^1 [p(z'y-y'z)]'$ **)**

(c) $L(u)(x) = u''(x)$;

M = {u: u(0) + u(1) = 0, u'(0) - u'(1) = 0 }

ans: M* = {z: z(0) = z(1), z′(0) = − z′(1)}

(6) Verify that for L, M, and u as given in the TYPICAL PROBLEM above, u is in M and L(u) = f. (Recall Exercise III in the section AN INTRODUCTION TO THE PROBLEMS OF GREEN'S FUNCTIONS of these notes.)

Suppose G is as in TYPICAL PROBLEM and

$$z(x) = \int_0^1 G^*(x,t)\, h(t)\, dt.$$

Show z solves L* on M*.