SECTION 2.3 THE FREDHOLM ALTERNATIVE THEOREMS

Before developing a technique for solving these ordinary differential equations with boundary conditions, attention should be paid to the statement of the Fredholm Alternative Theorems in this setting.

Suppose that L is an n^{th} order differentiable operator with n boundary conditions. $B_1, B_2, ...B_n$. The problem is posed as follows: Given f, find u such that $L(u) = f$ with $B_p(u) = 0$, $p = 1, 2, ...n$.

I. Exactly one of the following two alternatives holds:
(a)(First Alternative) if f is continuous then $L(u) = f$, $B_p(u) = 0$, $p = 1, 2, ..., n$, has one and only one solution..
(b)(Second Alternative) $L(u) = 0$, $B_p(u) = 0$, $p = 1, 2, ...n$, has a nontrivial solution.

II. (a) If $L(u) = f$, $B_p(u) = 0$, $p = 1, 2, ...n$, has exactly one solution then so does $L^*(u) = f$, $B_p^*(u) = 0$, $p = 1, 2, ...n$ have exactly one solution.
(b) $L(u) = 0$, $B_p(u) = 0$, $p = 1, 2, ... n$, has the same number of linearly independent solutions as $L^*(u) = 0$, $B_p^*(u) = 0$, $p = 1, 2, ...n$.

III. Suppose the second alternative holds. Then $L(u) = f$, $B_p(u) = 0$, $p = 1, 2, ...n$ has a solution if and only if $\langle f, w \rangle = 0$ for each w that is a solution for

$$L^*(w) = 0,\ B_p^*(w) = 0,\ p = 1, 2, ...n.$$

EXERCISE 2.3.

1. Decide if the following operators L are formally self adjoint and if they are self adjoint on M. Decide if the equation $L(y) = f$ on M is in the First Alternative.
(a) $L(y) = y''$, $M = \{y: y(0) = y'(0) = 0\}$. ans: yes, no, yes.
(b) $L(y) = y''$, $M= \{y: y(0) = y(1) = 0\}$. ans: yes, yes, yes.
(c) $L(y) = y'' + 4\pi^2 y$, $M=\{y: y(0) = y(1), y'(0) = y'(1)\}$.
ans: yes, yes, no.
(d) $L(y) = y'' + 3y' + 2y$, $M = \{y: y(0) = y(1) = 0\}$. ans: no, no, yes.

2. Suppose that $L(y)(x) = y''(x) + 4\pi^2 y(x)$. $B_1(y) = y(0)$ and $B_2(y) = y(1)$.
(a) Show that the problem $L(y) = 0$, $B_1(y)= B_2(y) = 0$ has $\sin(2\pi x)$ as a non-trivial solution.
(b) What is the adjoint problem for $\{L, B_1, B_2\}$?
(c) What conditions must be satisfied by f in order that $L(y) = f$, $y(0) = 0 = y(1)$ has a solution?
(d) Show that $y''(x) + 4\pi^2 y(x) = 1$, $y(0) = 0 = y(1)$ has $[1 - \cos(2\pi x)]/4\pi^2$ as a solution.

3. Show that $y'' = x$, $y'(0) = 0 = y'(1)$ has no solution.

4. Show that $y'' = \sin(2\pi x)$, $y'(0) = 0 = y'(1)$ has a solution.