SECTION 2.4 A METHOD FOR CONSTRUCTING A GREEN'S FUNCTION IF THE FIRST ALTERNATIVE HOLDS.

There are, of course, several methods of constructing Green's functions. The one we will present first, and emphasize, is the one students seem to prefer. Perhaps this is because it is easy to remember and has an inherent simplicity. Other methods will be included in these notes for comparison. There are ideas which the other methods use that are important.

As before, we assume a certain form for the differentiable operator L:

$$L(y) = \sum_{p=0}^{n} a_p(x)\, y^{(p)}(x) = a_n(x)y^{(n)}(x) + \dots + a_1(x)y'(x) + a_0(x)y(x).$$

We suppose that $a_n(x)$ is not zero on [0,1] and that each term of the sequence $\{a_p(x)\}_{p=0}^{n}$ has at least n continuous derivatives. We discuss the construction of the Green's function in three cases depending on the nature of the boundary conditions. Until further notice, we assume the first alternative holds and will repeat this warning for emphasis. We continue to denote M and M* as the manifolds associated with {L,B} and {L*,B*}, respectively.

The function G should be constructed on [0,1]x[0,1] to have the following properties: if t is in (0,1) and $0 \le p \le n$ then

$$\frac{\partial^p(G(x,t))}{\partial x^p}$$

exists for $0 < x < t$ and for $t < x < 1$. Further suppose that these derivatives have a continuous extension to the triangles $0 \le x \le t$ and $t \le x \le 1$. The effect of this extension is that

$$\frac{\partial^p G}{\partial x^p}(t^+,t) = \frac{\partial^p G}{\partial x^p}(t,t^-)$$

and

$$\frac{\partial^p G}{\partial x^p}(t^-,t) = \frac{\partial^p G}{\partial x^p}(t,t^+)\,.$$

At this point, all we have asked of G is that it should have n continuous partials on the closed triangles $0 \le x \le t$ and $t \le x \le 1$. The requirement along the boundary will be that for $p \le n-2$, we have continuity. For example at p = 0, the effect is that $G(t^+,t) = G(t^-,t)$. Indeed, $G(t^+,t) = G(t,t^-) = G(t,t^+) = G(t^-,t)$. And, this happens for the p^{th} partials up to p = n-2. For the $(n-1)^{th}$ partial, we allow a jump discontinuity as prescribed in the summary below:

CONSTRUCTION OF G(X,T) FOR NTH ORDER EQUATIONS

Pick t in [0,1].

(a) $L(G(\ ,t))(x) = 0$ for $0 < x < t$ and for $t < x < 1$,

(b) $G(\ ,t)$ is in M,

(c) for $0 \le p \le n-2$, $\partial^p(G(t^+,t))/\partial x^p = \partial^p(G(t^-,t))/\partial x^p$.

(d) $\partial^{n-1}(G(t^+,t))/\partial x^{n-1} - \partial^{n-1}(G(t^-,t))/\partial x^{n-1} = 1/a_n(t)$.

Before showing that the above recipe really does provide solutions to the n^{th} order equation, it would be well do to some examples.

EXAMPLE (First alternative, initial conditions): Here is the problem: given f continuous on [0,1], construct y such that

$$y'' + 3y' + 2y = f \quad \text{with } y(0) = y'(0) = 0.$$

Let's identify the important parts here.

$$L(y) = y'' + 3y' + 2y, \qquad B_1(y) = y(0), \ B_2(y) = y'(0),$$

and $$M = \{y: y(0) = y'(0) = 0\}.$$

We are in the first alternative because for this $\{L,B_1,B_2\}$ the system $L(y) = 0$, $B_1(y) = B_2(y) = 0$ has only one solution and it is zero. We construct G step-by-step from the above directions.

To follow the directions of step (a) we need the general solution of the homogeneous equation $L(y) = 0$, that is, we need the general solution of the homogeneous equation

$$y'' + 3y' + 2y = 0.$$

It's not so hard to see that linearly independent solutions for this equation are e^{-2x} and e^{-x}. Thus G satisfies step (a) if

$$G(x,t) = \begin{cases} Ae^{-2x} + Be^{-x} & \text{for } x < t \\ Ce^{-2x} + De^{-x} & \text{for } t < x \end{cases}$$

Note that A, B, C, and D are constant in x, but may change with t.

To follow the directions of step (b) which requires that $G(\ ,t)$ be in M, we need

$$G(0,t) = 0 \text{ and } d(G(x,t))/dx \,|_{x=0} = 0$$

The implications of this are that

$$A + B = 0 \text{ and } -2A - B = 0.$$

This implies that A = 0 and B = 0.

To follow the directions of step (c) which requires that $G(t^+,t) = G(t^-,t)$, we need

$$(C - A)e^{-2t} + (D - B)e^{-t} = 0.$$

Or, knowing that A = B = 0,

$$Ce^{-2t} + De^{-t} = 0.$$

To follow the directions of step (d) which requires that

$$\partial(G(x,t))/\partial x\,|_{x=t+} - \partial(G(x,t))/\partial x\,|_{x=t-} = 1,$$

we need

$$-2(C - A)e^{-2t} - (D - B)e^{-t} = 1.$$

Knowing that $A = B = 0$,

$$-2Ce^{-2t} - De^{-t} = 1.$$

This gives two equations, two unknowns in C and D. The solution is

$$C = -e^{2t} \quad \text{and } D = e^{t}.$$

Try to get an over view of this example: after finding two linearly independent solutions of the second order equation L(y) = 0, we would know G provided we solved for A, B, C, and D. Steps b, c, and d gave four equations in these four unknowns. Written in matrix form,

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ -2 & -1 & 0 & 0 \\ 0 & 0 & e^{-2t} & e^{-t} \\ 0 & 0 & -2e^{-2t} & -e^{-t} \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The problem has been reduced to a matrix equation! We have solved the equations to find A = 0, B = 0, $c = -e^{2t}$, and $d = e^{t}$. This gives that

$$G(x,t) = \begin{cases} 0 & \text{if } x < t \\ -e^{2(t-x)} + e^{t-x} & \text{if } t < x. \end{cases}$$

We are confident that if f is continuous then the equation y = Gf provides a solution for L(y) = f because of EXERCISE III of the INTRODUCTION.

VERIFICATION THAT THIS METHOD GIVES SOLUTIONS FOR SECOND ORDER EQUATIONS

Suppose that $L(y)(x) = a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)\, y(x)$. Let

$$u(x) = \int_0^1 G(x,t)\, f(t)\, dt.$$

Since G(, t) is in M then u is in M. It remains to see that L(u) = f. Note that

$$u(x) = \int_0^x G(x,t)\, f(t)\, dt + \int_x^1 G(x,t)\, f(t)\, dt,$$

$$u'(x) = G(x,x-)f(x) + \int_0^x \partial(G(x,t))/\partial x\, f(t)\, dt$$

$$- G(x,x^+) f(x) + \int_x^1 \partial(G(x,t))/\partial x \, f(t)\, dt$$

$$= \int_0^x \partial(G(x,t))/\partial x \, f(t)\, dt + \int_x^1 \partial(G(x,t))/\partial x \, f(t)\, dt.$$

This last equality holds because of the assumption that $G(x,x^-) = G(x,x^+)$.

Also $u''(x) = \partial(G(x,x^-))/\partial x \; f(x) + \int_0^x \partial^2(G(x,t))/\partial x^2 \, f(t)\, dt$

$$- \partial(G(x,x^+))/\partial x \; f(x) + \int_x^1 \partial^2(G(x,t))/\partial x^2 \, f(t)\, dt$$

$$= f(x)/a_2(x) + \int_0^x \partial^2(G(x,t))/\partial x^2 \, f(t)\, dt + \int_x^1 \partial^2(G(x,t))/\partial x^2 \, f(t)\, dt$$

This last equality uses the condition $\partial(G(x,x^-))/\partial x - \partial(G(x,x^+))/\partial x = 1/a_2(x)$. Finally, we use the fact that $G(x, t)$, as a function of x, satisfies $L(y) = 0$ on $[0,x]$ and on $[x,1]$ to get that $L(u) = f$.

EXERCISE 2.4:

I. Find a Green's function such that if f is continuous, then the equation $y = Gf$ provides a solution for $L(y) = f$, $y(0) = y'(0) = 0$, where L is as defined below. In each case, first give L* and M* and verify that the first alternative holds.

(a) $L(y) = y''$ **(b)** $L(y) = y'' + 4\pi^2 y$ **(c)** $L(y) = 2y'' + y' - y$

(d) $L(y)(x) = (e^x y'(x))'$.

II. (Ad hoc method) Suppose that u is a function on [0,1] which satisfies $L(u) = 0$, $u(0) = 0$, and $u'(0) = 1/a_2(t)$. Let $H(x,t) = 0$ if $x < t$ and $= u(x-t)$ if $t < x$. Show that H is a Green's function for the problem $\{L, B_1(y) = y(0), B_2(y) = y'(0)\}$.

III. (Equivalent integral equation) Let G(x,t) be the Green's function for problem I(a) above. Suppose that b and f are continuous functions. Let h be the function given by

$$h(x) = \int_0^1 G(x,t) f(t)\, dt$$

and H(x,t) be the function given by H(x,t) = G(x,t) b(t). Show these are equivalent:

(a) $y''(x) - b(x)y(x) = f(x),\ \ y(0) = y'(0) = 0$, **and**

(b) $y(x) = \int_0^1 H(x,t)\, y(t)\, dt + h(x).$

IV. Let $L(y) = y'' + 3y' + 2y$

and

$$M = \{y: y(0) = 0,\ y'(0) = 0\}.$$

Find the Green's function for L on M and for L* on M*. Compare the graphs of the two.

ANSWERS FOR I.

A. $G(x,t) = s-t$ **if** $0 \le t \le x$.

B. $G(x,t) = \frac{1}{2\Pi} \sin(2\Pi(x-t))$ **if** $0 \le t \le x$

C. $G(x,t) = \frac{1}{3} [\ e^{(x-t)/2} - e^{(t-x)}$ **if** $0 \le t \le x$

D. $G(x,t) = (1-e^{t-x})\ e^{-t}$ **if** $0 \le t \le x$.

SECTION 2.4 (cont) Constructing the Green's function in the first alternative

EXAMPLE(first alternative; unmixed, two point boundary conditions):
We will construct the Green's function for the problem

$$y'' + 3y' + 2y = f \text{ with } y(0) = 0 \text{ and } y(1) = 0.$$

Here are the important parts:

$$L(y) = y'' + 3y' + 2y, \qquad B_1(y) = y(0), \qquad B_2(y) = y(1),$$

and $$M = \{y: \ y(0) = y(1) = 0\}.$$

A little bit of work needs to be done to verify that we are in the first alternative. If $L(y) = 0$, then there are numbers a and b such that

$$y(x) = ae^{-2x} + be^{-x}.$$

To require that $B_1(y) = 0$ and $B_2(y) = 0$ requires that

$$0 = a + b$$

and $$0 = ae^{-2} + be^{-1}.$$

The only solution to this pair of equations is $a = 0 = b$, which verifies that we are in the first alternative.

The construction for G is as before:

$$G(x,t) = \begin{cases} Ae^{-2x} + Be^{-x} & \text{for } x < t \\ Ce^{-2x} + De^{-x} & \text{for } t < x \end{cases}$$

The two boundary conditions and the continuity conditions lead to the equations

$$0 = A + B$$
$$0 = Ce^{-2} + De^{-1}$$
$$0 = (C-A)e^{-2t} + (D-B)e^{-t}$$
$$1 = -2(C-A)e^{-2t} - (D-B)e^{-t}.$$

Certainly, these equations can be solved, although the details are tedious. Here is a better idea. Instead of choosing e^{-2t} and e^{-t} as linearly independent solutions of the equation $L(y) = 0$, choose another pair having these properties:

$$u_0(0) = 0 \text{ and } u_0(1) \neq 0$$
$$u_1(0) \neq 0 \text{ and } u_1(1) = 0.$$

(For this example, $u_0(t) = e^{-2t} - e^{-t}$ and $u_2(t) = e^{-2(t-1)} - e^{-(t-1)}$.)
Now, make up G this way,

$$G(x,t) = \begin{cases} Au_0(x) + Bu_1(x) & \text{for } x < t \\ Cu_0(x) + Du_1(x) & \text{for } x > t \end{cases}$$

Apply the boundary conditions:

$$0 = Bu_1(0) \qquad \text{which implies that } B = 0,$$

and $$0 = Cu_0(1) \qquad \text{which implies that } C = 0.$$

The continuity conditions give the two equations

$$0 = G(t^+,t) - G(t^-,t) = Du_1(t) - Au_0(t)$$
$$1/a_2(t) = \partial(G(t^+,t))/\partial x - \partial(G(t^-,t))/\partial x = D\,u'_1(t) - A\,u'_0(t).$$

From these equations we get

$$A = u_2(t) / a_2(t)w(t)$$

and $$D = u_0(t) / a_2(t)w(t)$$

where $w(t) = \det \begin{pmatrix} u_0(t) & u_1(t) \\ u_0'(t) & u_1'(t) \end{pmatrix}$ and is called the <u>Wronskian</u> of u_0 and u_1.

Here is the final result:

$$G(x,t) = \begin{cases} u_1(t)\, u_0(x)/a_1(t)w(t) & \text{for } x < t \\ u_1(t)\, u_1(x)/a_2(t)w(t) & \text{for } t < x. \end{cases}$$

There is one more important piece of information that you will learn, or be reminded of, if we work out the formulas for both parts of the formula for G. Recall that

$$u_0(t) = e^{-2t} - e^{-t} \quad \text{and } u_1(t) = e^{-2(t-1)} - e^{-(t-1)}.$$

And now, to compute w(t). The chore of that computation seems too tedious to be fun. Not to worry! Look up "Wronskian" is some <u>good</u> sophomore differential equations book and you will find a convenient formula :

$$W(t) = W(0) \exp\left(\int_0^t -[a_{n-1}(s)/a_n(s)]\, ds \right).$$

Now, the computation is easy:

$$W(0) = \det \begin{pmatrix} 0 & e^2 - e^1 \\ -1 & -2e^2 + e^1 \end{pmatrix} = e^2 - e^1.$$

and $$w(t) = (e^2 - e^1)\, e^{-3t}.$$

Hence, $$\frac{u_0(t)\, u_1(x)}{w(t)} = (e^{2+t} - e^{1+2t})\,(e^{-2x} - e^{-x}) / (e^2 - e^1)$$

$$= e^{2(t-x)}\,(e^{1-t} - 1)\,(1 - e^x)/(e-1)$$

and $$\frac{u_0(t)\, u_1(x)}{w(t)} = (e^t - e^{2t})\,(e^{2-2x} - e^{1-x}) /(e^2 - e)$$

$$= e^{t-x}\,(1-e^t)\,(e^{1-x} - 1)/ (e-1).$$

We got more from this example than the answer: we also got the following quick method that works for this type problem.

AD HOC METHOD TO CONSTRUCT GREEN'S FUNCTIONS FOR SECOND ORDER, FIRST ALTERNATIVE,UNMIXED, TWO POINT BOUNDARY CONDITIONS

Pick u_1 and u_2 such that $B_1(u_1) = 0$, $B_2(u_1) \neq 0$, $B_2(u_2) = 0$, and $B_1(u_2) \neq 0$. Then

$$G(x,t) = \begin{cases} u_2(t)\, u_1(x)/a_2(t)w(t) & \text{for } x < t \\ u_1(t)\, u_2(x)/a_2(t)w(t) & \text{for } t < x. \end{cases}$$

where w is the Wronskian of u_1 and u_2.

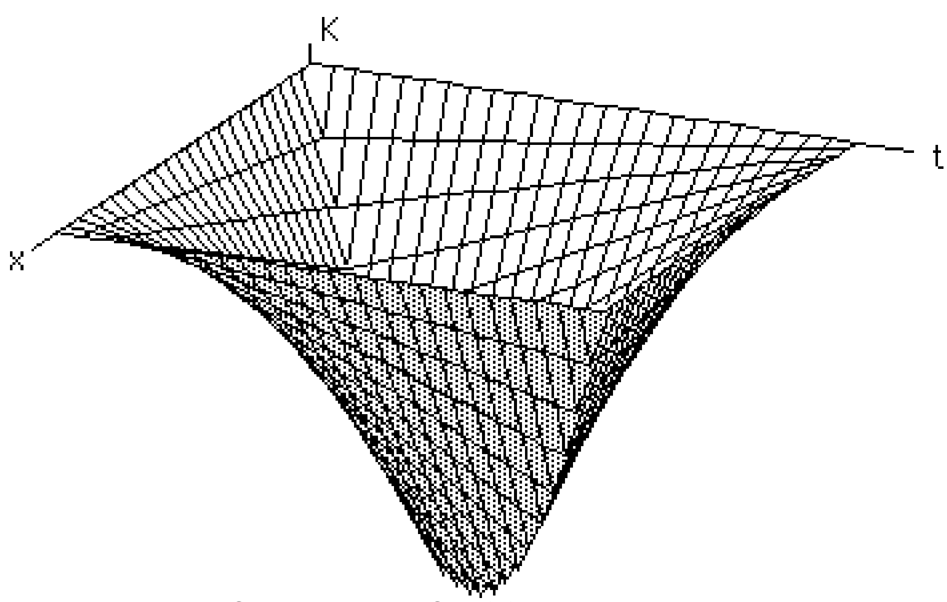
EXERCISE 2.4b. Construct L*, B*, and G for the following :
(a) $L(y) = y''$, $B_1(y) = y(0)$, $B_2(y) = y(1)$,
(b) $L(y) = y''$, $B_1(y) = y(0) + y'(0)$, $B_2(y) = y(1) + y'(1)$.
(c) $L(y) = y''$, $B_1(y) = y(0) + y'(0)$, $B_2(y) = y(1) - y'(1)$,
(d) $L(y) = y'' + 4\pi^2 y$, $B_1(y) = y(0) + y'(0)$, $B_2(y) = y(1) - y'(1)$.
(e) $L(y) = 2y'' + y' - y$, $B_1(y) = y(0) + y'(0)$, $B_2(y) = y(1) - y'(1)$.
(f) $L(y)(x) = (e^x y'(x))'$, $B_1(y) = y(0) + y'(0)$, $B_2(y) = y(1) - y'(1)$.

Ans (a)

$$G(x,t) = \begin{cases} x\,(t-1) & \text{if } 0 \le x \le t \\ t\,(x-1) & \text{if } 0 \le t \le x \end{cases}.$$

(c)

$$G(x,t) = \begin{cases} (1-x)\,t & \text{if } 0 \le x \le t \\ (1-t)\,x & \text{if } 0 \le t \le x \end{cases}.$$



Graph of the answer for (a)

SECTION 2.4(CONT) Constructing the Green's function

EXAMPLE (first alternative; mixed, two point boundary conditions): Suppose

$$L(y) = y'',\ B_1(y) = y(0) + y(1), \text{ and } B_2(y) = y'(0) + y'(1).$$

First, we verify that we have the first alternative by supposing that

$$L(y) = 0 \text{ and } B_1(y) = B_2(y) = 0.$$

Then $y(x) = a + bx$, for constants a and b. Since

$$y(0) + y(1) = 0$$

then

$$2a + b = 0.$$

Since

$$y'(0) + y'(1) = 0,$$

then

$$2a = 0.$$

These two equations imply that $a = b = 0$. We now begin the construction of the Green's function.
Pick $0 < t < 1$.

$$G(x,t) = \begin{cases} A + Bx & \text{for } x < t \\ C + Dx & \text{for } t < x \end{cases}$$

We have four constants to determine; here are four equations:

$$0 = G(0,t) + G(1,t) = A + C + D,$$

$$0 = \partial(G(x,t))/\partial x \ \big|_{x=0} + \partial(G(x,t))/\partial x \ \big|_{x=1} = B + D,$$

$$0 = G(t^+,t) - G(t^-,t) = (C - A) + (D - B)\, t,$$

$$1/a_2(t) = \partial(G(x,t))/\partial x \ \big|_{x=t+} - \partial(G(x,t))/\partial x \ \big|_{x=t-} = D - B.$$

The solution for these four equations is $A = (2t - 1)/4$, $B = -1/2$, $C = -(2t + 1)/4$, and $D = 1/2$.

EXERCISE 2.3c. So that you will remember why we are constructing Green's functions, use the above result to provide a solution for the equation $y''(x) = x^2$, $y(0) + y(1) = 0$, and $y'(0) + y'(1) = 0$.