SECTION 2.5 AN UNDERSTANDING OF THE EQUATION $L(y) = \delta(x,t)$.

By now you should believe that except for arithmetic details, you can work any of these problems. We have come to the place where we need to get this problem into perspective.

We know that the requirements of Section 2.3 give the Green's function. That was established in the verification given in that section. What we would like to do is to connect the inversion of the differential operator L with the inversion of matrices, and other linear problems. One unifying idea is contained in the Appendix of these notes where the inverse of the matrix A is obtained by finding G such that $A(G(\ ,j)) = \delta_j$. In this situation, we might hope to find $G(\ ,t)$ as a solution to the equation $L(G(\ ,t)) = \delta(\ ,t)$. Some understanding of this equation is in order for the right side is not a function in the ordinary sense. As has already been pointed out, it is a "distribution".

We present here, not a proof, but an understanding that

$$L(G(\ ,t))(x) = \delta(x,t). \qquad (*)$$

The ideas should be examined, and re-examined in later courses as a theory for distributions is developed:
Suppose that

$$y(x) = \int_0^1 G(x,t)\, f(t)\, dt$$

and that equation (*) holds. Intuition is a guide:

$$L(y)(x)$$

$$= L\left(\int_0^1 G(x,t)\, f(t)\, dt\right)$$

$$= \int_0^1 L(G(\ ,t))(x)\, f(t)\, dt$$

$$= \int_0^1 \delta(x,t)\, f(t)\, dt = f(x).$$

If one were asked to solve the equation $L(y) = f$, where L is a reasonable second order operator, in the context of a sophomore differential equations course, one would think of the variations of parameter formula.

In that setting, and for second order problems with u_0 and u_1 linearly independent solutions of the homogeneous equation,

$$y(x) = C_0(x)\, u_0(x) + C_1(x)\, u_1(x).$$

Here,

$$\frac{\partial C_0}{\partial x}(x)\, u_0(x) + \frac{\partial C_1}{\partial x}(x)\, u_1(x) = 0$$

and

$$\frac{\partial C_0}{\partial x}(x)\, u'_0(x) + \frac{\partial C_1}{\partial x}(x)\, u'_1(x) = f(x)/a_2(x).$$

From this, one derives

$$\frac{\partial C_0}{\partial x}(x) = \frac{-\, u_1(x)\, f(x)}{a_2(x)\, w(x)}$$

and

$$\frac{\partial C_1}{\partial x}(x) = \frac{u_0(x)\, f(x)}{a_2(x)\, w(x)}$$

This suggests an interpretation for "solution" of the second order distribution equation:

$$L(G(\ ,t))(x) = \delta(x,t).$$

Namely, $G(\ ,t)$ is the continuous function given by

(a) $$G(x,t) = C_0(x,t)u_0(x) + C_1(x,t)u_1(x)$$

where

(b) $$\frac{\partial C_0}{\partial x}(x)\, u_0(x) + \frac{\partial C_1}{\partial x}(x)\, u_1(x) = 0$$

and

(c) $$\frac{\partial C_0}{\partial x}(x)\, u'_0(x) + \frac{\partial C_1}{\partial x}(x)\, u'_1(x) = \delta(x,t)/a_2(x).$$

As above, the distribution equations should have solution

(d) $$\frac{\partial C_0}{\partial x}(x) = \frac{-\,u_1(x)\ \delta(x,t)}{a_2(x)\ w(x)}$$

and

(e) $$\frac{\partial C_1}{\partial x}(x) = \frac{u_0(x)\ \delta(x,t)}{a_2(x)\ w(x)}.$$

<u>THEOREM</u>. If, for each t, G(,t) is in M and L(G(,t))(x) = δ(x,t) then G satisfies the four equations of Section 2.3.

Proof. We hope to recognize the four equations which we used to define G for second order problems as arising from the above requirements for G. Two of those equations come from asking that G(,t) should satisfy the two boundary equations. One other, $G(t^+,t) - G(t^-,t) = 0$, comes from the requirement that G(,t) should be continuous. To derive the equation

$$\partial G(x,t)/\partial x\,|_{x=t+} - \partial G(x,t)/\partial x\,|_{x=t-} = 1/a_2(t),$$

we first compute $\partial G(x,t)/\partial x$.

$$\partial G(x,t)/\partial x =$$
$$\partial C_0(x,t)/\partial x\ u_0(x) + C_0(x,t)\ u'_0(x) + \partial C_1(x,t)/\partial x\ u_1(x) + C_1(x,t)\ u'_1(x)$$
$$= C_0(x,t)\ u'_0(x) + C_1(x,t)\ u'_1(x).$$

This last equality follows from (b). To find

$$\partial G(x,t)/\partial x\,|_{x=t+} - \partial G(x,t)/\partial x\,|_{x=t-}$$
$$= [C_0(t^+,t)-C_0(t^-,t)]\ u'_0(t) + [C_1(t^+,t)-C_1(t^-,t)]\ u'_1(t),$$

we must evaluate

$$[C_0(t^+,t) - C_0(t^-,t)] = \int_{t^-}^{t^+} \partial C_0(x,t)/\partial x\ dx$$

$$= \int_{t^-}^{t^+} \frac{-u_1(x)\ \delta(x,t)}{a_2(x)w(x)}\,dx = \frac{-u_1(t)}{a_2(t)\ w(t)}$$

In a similar manner,

$[C_1(t^+,t) - C_1(t^-,t)] = \quad u_0(t) / a_2(t)\, w(t).$

Hence,

$$\partial G(x,t)/\partial x \,|_{x=t+} - \partial G(x,t)/\partial x \,|_{x=t-} =$$

$$\frac{-u_1(t)\, u_0'(t) + u_0(t)\, u_1'(t)}{a_2(t)\, w(t)} = \frac{1}{a_2(t)}$$

Hence, the inverse of the differential operator L on the set M is obtained by finding the function G(,t) in M which satisfies

$$L(G(\ ,t))(x) = \delta(x,t).$$

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Exercise 2.5

I. Solve each of the following problems, L(G(,t))(x) = δ(x,t) where L and M are specified.

(A) $L(y) = y'' + 3y' + 2y$, $M = \{y:\ y''$ exists and $y(0) = 0 = y'(0)\}$.

(B) $L(y) = y'' + 3y' + 2y$, $M = \{y:\ y''$ exists and $y(0) = 0 = y(1)\}$.

(C) $L(y) = y'' + 3y' + 2y$, $M = \{y:\ y''$ exists and $y(0) = y(1), y'(0) = y'(1)\}$.

II. Solve these three problems:

(A) $y'' + 3y' + 2y = x^2$, $y(0) = 0 = y'(0)\}$.

(B) $y'' + 3y' + 2y = 1$, $y(0) = 0 = y(1)\}$.

(C) $y'' + 3y' + 2y = \sin(\pi x)$, $y(0) = y(1), y'(0) = y'(1)\}$.