SECTION 2.6 THE SECOND ALTERNATIVE

We now discuss the problems where the Second Alternative holds. The supposition is that there is a nontrivial solution for $L(y) = 0$, $B_1(y) = B_2(y) = 0$. The Fredholm Theorems assure us that, if f is continuous, then there is a solution for $L(y) = f$, with $B_1(y) = B_2(y) = 0$ provided

$$< f, w > = \int_0^1 f(x)\, w(x)\, dx = 0$$

for all solutions w of the equation $L^*(w) = 0$, $B_1^*(w) = B_2^*(w) = 0$. As before, we will construct Green's functions G such that, in case f satisfies the above requirement, then

$$y(x) = \int_0^1 G(x,t)\, f(t)\, dt$$

provides a solution for $L(y) = f$.

In this second alternative, there may be many solutions for the equation $L(y) = f$. Consequently, we expect there may be many Green's functions. In the technique developed below, $G(,t)$ is always in

$$M= \{y: B_0(y) = B_1(y) = 0 \}.$$

This is not necessarily true for Green's functions constructed by other methods: see for example the construction found by Don Jones while a graduate research assistant at GEORGIA TECH and given in an appendix.

We again divide the problems into three cases according to the nature of the boundary conditions. We shall illustrate methods of construction.

The first case to consider is where the boundary conditions arise as <u>initial conditions</u>. This case is not pertinent for the initial value problem has a unique solution. Thus, case one is always in the first alternative.

EXAMPLE:(Second Alternative,<u>unmixed, two point boundary conditions</u>)

Suppose that $L(y) = y'' + y' - 2y$, $B_1(y) = y(0) - y'(0)$, and $B_2(y) = y(1) - y'(1)$. It is the purpose of this example to show that there is no function G such that $L(G(,t))(x) = \delta(x,t)$. Note that $L^*(z) = z'' - z' -2z$ and $M^* = \{y: 2z(1) = z'(1), 2z(0) = z'(0)\}$. Nontrivial functions in the nullspace of $\{L^*, B_1^*, B_2^*\}$ are multiples of e^{2t}. Hence, we are in the second alternative. The Fredholm Alternative theorem suggests that there will be no function G such that, if t is in (0,1), then the distribution equation $L(G(,t)) (x) = \delta(x,t)$ holds unless

$$\int_0^1 \delta(x,t)\, e^{2t}\, dt = 0.$$

Of course, the value of this integral is not zero.

For this situation, we must modify the construction of the Green's function.

CONSTRUCTION OF G IN THE SECOND ALTERNATIVE, n^{th} ORDER

Step (1) Find the nullspace of { L*, M*}
Step (2) Find an orthonormal basis for this nullspace. Call this basis $v_1, v_2, \ldots v_m$, $m \leq n$.
Step (3) Construct u_p such that $L(u_p) = v_p$, $p = 1, 2, \ldots n$.
Step (4) Construct G such that $L(G(\ ,t))(x) = \delta(x,t) - \sum_{p=1}^{m} v_p(x)v_p(t)$.

THEOREM If $0 < t < 1$, then there is $G(\ ,t)$ such that

$$L(G(\ ,t))(x) = \delta(x,t) - \sum_{p=1}^{n} v_p(x)v_p(t).$$

INDICATION OF PROOF. By the Fredholm Alternative Theorems, there will be such a function G provided

$$0 = \left< \delta(x,t) - \sum_{p=1}^{n} v_p(x)v_p(t) , w(t) \right> \int_0^1 [\delta(x,t) - \sum_{p=1}^{n} v_p(x)v_p(t)] \, w(t) \, dt$$

for all w in the nullspace of L*. This can be verified by writing w in terms of this orthonormal basis , $w(x) = \sum_{p=1}^{n} \alpha_p v_p(x)$, and evaluating the dot product.

HOW TO CONSTRUCT G SUCH THAT

$$L(G(\ ,t))(x) = \delta(x,t) - \sum_{p=1}^{n} v_p(x)v_p(t).$$

First, find linearly independent solutions $\{y_p\}_{p=1}^{n}$ of the homogeneous equation $L(y) = 0$. Then, find solutions $\{u_p\}_{p=1}^{m}$, $m < n$, for the equations

$L(u_p)(x) = v_p(x)$. It is not required that these solutions should satisfy any special boundary conditions.

The problem of finding G is now a problem of finding constants $\{C_p\}_{p=1}^{n}$ and $\{D_p\}_{p=1}^{n}$ such that

$$G(x,t) = \begin{cases} \sum_{p=1}^{n} C_p\, y_p(x) - \sum_{p=1}^{m} v_p(t)\, u_p(x) & \text{if } x < t \\ \sum_{p=1}^{n} D_p\, y_p(x) - \sum_{p=1}^{m} v_p(t)\, u_p(x) & \text{if } t < x \end{cases}.$$

where $\{C_p\}_{p=1}^{n}$ and $\{D_p\}_{p=1}^{n}$ are determined by these 2n equations:

(a) $B_p(G(\ ,t)) = 0$, $p=1,2,\ldots n$,

(b) $0 = \partial^p G(x,t)/\partial x^p|_{x=t+} - \partial^p G(x,t)/\partial x^p|_{x=t-}$, $0 \le p \le n-2$,

(c) $1/a_n(t) = \partial^{n-1} G(x,t)/\partial x^{n-1}|_{x=t+} - \partial^{n-1} G(x,t)/\partial x^{n-1}|_{x=t-}$.

CONTINUATION OF THE PREVIOUS EXAMPLE

Recall that $L(y) = y'' + y' - 2y$, $B_1(y) = y(0) - y'(0)$, and $B_2(y) = y(1) - y'(1)$. Linearly independent solutions for $L(y) = 0$ are e^{-2x} and e^x. A normalized basis for the one-dimensional nullspace of $\{L^*, B_1^*, B_2^*\}$ is αe^{2x} where α is the positive number given by

$$\alpha^2 = 1 \Big/ \int_0^1 (e^{2x})^2\, dx = 4/(e^4-1).$$

A solution u for the equation $y'' + y' - 2y = \alpha e^{2x}$ is $u(x) = \alpha e^{2x}/4$.

Now, G is given by:

$$G(x,t) = \begin{cases} Ae^{-2x} + Be^{x} - \alpha^2 e^{2(x+t)}/4 & \text{if } x < t \\ Ce^{-2x} + De^{x} - \alpha^2 e^{2(x+t)}/4 & \text{if } t < x. \end{cases}$$

The four constants - A,B,C, and D - can be solved by these four equations:

(1) $0 = B_1(G(\ ,t)) = G(0,t) - \partial G(x,t)/\partial x|_{x=0} = 3A + \alpha^2 e^t/4$

(2) $0 = B2(G(\ ,t)) = G(1,t) - \partial G(x,t)/\partial x|_{x=1} = 3Ce^{-2} + a^2 e^{2(1+t)}/4$,

(3) $0 = G(t^+,t) - G(t^-,t) = (C-A)e^{-2t} + (D-B)e^t$, and

(4) $1 = \partial G(x,t)/\partial x|_{x=t+} - \partial G(x,t)/\partial x|_{x=t-} = -2(C-A)e^{-2t} + (D-B)e^t$.

Upon solving this system of four equations and four unknowns, an infinity of solutions will be found determined by these three equations:

$A = -\alpha^2 e^{2t}/12$, $C = Ae^4$, $D-B = e^{-t}/3$.

EXERCISE: FOR EACH OF THE FOLLOWING, GIVE L^*, B_1^*, B_2^*, and G.

(a) $L(y) = y'' + y' - 2y$, $B_1(y) = y(0) - y'(0)$, $B_2(y) = y(1) - y'(1)$.
(b) $L(y) = 4y'' - y$, $B_1(y) = y(0) - 2y'(0)$, $B_2(y) = y(1) - 2y'(1)$.
(c) $L(y) = y'' - 2y' - 3y$, $B_1(y) = 3y(0) - y'(0)$, $B_2(y) = 3y(1) - y'(1)$.

EXAMPLE(Second Alternative, mixed, two point boundary conditions.)
Suppose that $L(y) = y''$, $B_1(y) = y(0) + y(1)$, $B_2(y) = y'(0) - y'(1)$. Then $L^*(z) = z''$, $B_1^*(z) = z(0) - z(1)$, $B_2^*(z) = z'(0) + z'(1)$. All solutions of $\{L, B_1, B_2\}$ are multiples of $2x - 1$. A nontrivial solution of $[L^*, B_1^*, B_2^*\}$ is the constant function 1. Also, the function $V(x) = 1$ forms a basis for the null space of $0 = L^*(z)$ in M^*. The function $u(x) = x^2/2$ satisfies $L(u) = 1$. Thus

$$G(x,t) = \begin{cases} A + Bx - x^2/2 & \text{if } x < t \\ C + Dx - x^2/2 & \text{if } t < x. \end{cases}$$

We have four unknowns; we have the following four equations:
(1) $0 = G(0,t) + G(1,t) = A + C + D - 1/2$
(2) $0 = dG(x,t)/dx|_{x=0} - dG(x,t)/dx|_{x=1} = B - (D-1)$
(3) $0 = G(t^+,t) - G(t^-,t) = C - A + (D-B)t$
(4) $1 = dG(x,t)/dx|_{x=t+} - dG(x,t)/dx|_{x=t-} = D - B$.
As expected, there is an infinity of solutions to these equations which may be found by choosing D and then

$$B = D - 1$$
$$2C = -(t+D) + 1/2$$
$$A = C + t.$$

EXERCISE: I Verify that each of these problems is second alternative and find L^*, B_1^*, B_2^*, and G.
(1) $L(y) = y''$, $B_1(y) = y(0) - y(1)$, $B_2(y) = y'(0) - y'(1)$,
(2) $L(y) = y'' + 9\pi^2 y$, $B_1(y) = y(0) - y(1)$, $B_2(y) = y'(0) + y'(1)$,
(3) $L(y) = y'' + y' - 2y$, $B_1(y) = e\,y(0) - y(1)$, $B_2(y) = e\,y'(0) - y'(1)$.
II. Construct L^*, B^* and G for each of the following L's and with periodic boundary conditions $y(0) = y(1)$, $y'(0) = y'(1)$:
(a) $L(y) = y''$,
(b) $L(y) = y'' + \pi^2 y$
(c) $L(y) = 2y'' + y' - y$,

NONHOMOGENEOUS BOUNDARY CONDITIONS

To solve the equations $L(y) = f$, $B_1(y) = \alpha$, $B_2(y) = \beta$, first construct G for the problem $L(y) = f$, $B_1(y) = 0$, $B_2(y) = 0$. Then construct functions z_1 and z_2 such that $B_1(z_1) = 0$, $B_2(z_1) \neq 0$, and $B_1(z_2) \neq 0$, $B_2(z_2) = 0$. The solution for the original problem is

$$y(x) = \int_0^1 G(x,t)f(t)\,dt + \beta \frac{z_1(x)}{B_2(z_1)} + \alpha \frac{z_2(x)}{B_1(z_2)} \text{ 。}$$