CHAPTER III. TECHNIQUES FOR SOME PARTIAL DIFFERENTIAL EQUATIONS

Section 3.1 Classification of Second Order, Linear Equations

In this section, we will be interested in a rather general second order, linear operator. In two variables, we consider the operator

$$L(u) = \sum_{p=1}^{2}\sum_{p=1}^{2} A_{pq} \frac{\partial^2 u}{\partial x_p \partial x_q} + \sum_{p=1}^{2} B_p \frac{\partial u}{\partial x_p} + Cu.$$

Written more simply:

$$L(u) = a\, u_{xx} + b\, u_{xy} + c\, u_{yy} + d\, u_x + e\, u_y + f.$$

There is an analogus formula for three variables. We suppose that u is smooth enough so that

$$\partial^2 u/\partial x_1\, \partial x_2 = \partial^2 u/\partial x_2\, \partial x_1;$$

that is, we can interchange the order of differentiation. The matrix A, the vector B, and the number C do not depend on u; we take the matrix A to be symmetric.

We shall continue to be interested in an equation which has the form L(u) = f. In this section, u is a function on R^2 or R^3 and the equations are to hold in an open, connected region D of the plane. We will also assume that the boundary of the region is piecewise smooth, and denote this boundary by ∂D. Just as in ordinary differential equations, so in partial differential equations, some boundary conditions will be needed to solve the equations. We will take the boundary conditions to be linear and have the general form

$$B(u) = a\, u + b\, \partial u/\partial \eta,$$

where $\partial u/\partial \eta$ is the derivative taken in the direction of the outward normal to the region.

The techniques of studying partial differential operators and the properties of these operators change depending on the "type" of operator. These operators have been classified into three principal types. The classifications are made according to the nature of the coefficients in the equation which defines the operator. The operator is called an <u>elliptic</u> operator if the eigenvalues of A are nonzero and have the same algebraic sign. The operator is <u>hyperbolic</u> if the eigenvalues have opposite signs and is <u>parabolic</u> if not all the eigenvalues are nonzero.

EXAMPLES

(a) We consider an example which arises in a physical situation and is called the <u>one dimensional heat equation</u>. Here, u(t,x) represents the heat on a line at time t and position x. One should be given an initial distribution of temperature which is denoted u(0,x), and some boundary conditions which arise in the context of the problem. For example, it might be assumed that the ends are held at some fixed temperature for all time. In this case, boundary conditions for a line of length L would be $u(t,0) = \alpha$ and $u(t,L) = \beta$. Or, one might assume that the ends are insulated. A mathematical statement of this is that the rate of flow of heat over the ends is zero:

$$\frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t,L) = 0.$$

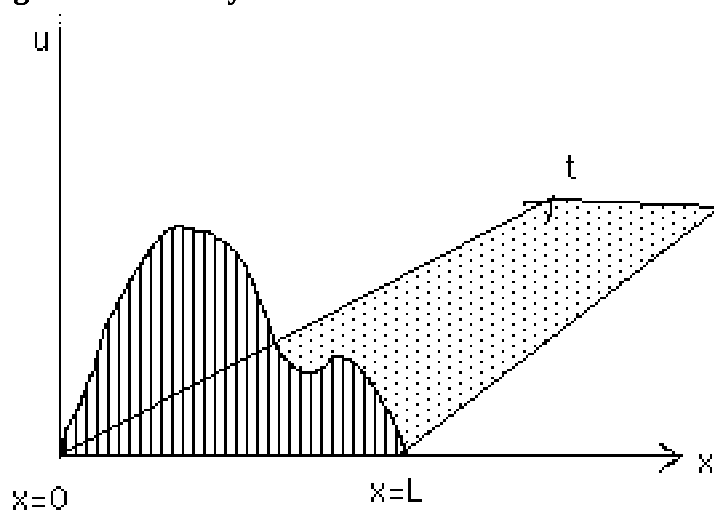
The manner in which u changes in time is derived from the physical principle which states that the heat flux at any point is proportional to the temperature gradient at that point and leads to the equation

$$\frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x).$$

Geometrically, one may think of the problem as one of defining the graph of u whose domain is the infinite strip bounded in the first quadrant by the parallel lines x = 0 and x = L The function u is known along the x axis between x = 0 and x = L, To define u on the infinite strip, move in the t direction according to the equation

$$\frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x),$$

while maintaining the boundary conditions.



We could also have a source term. Physically, this could be thought of as a heater (or refrigerator) adding or removing heat at some rate along the strip. Such an equation could be written as

$$\frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) + F(t,x).$$

Boundary and initial conditions would be as before. In order to rewrite this equation in the context of this course, we should conceive of the equation as L[u] = f , with appropriate boundary conditions. The operator L is

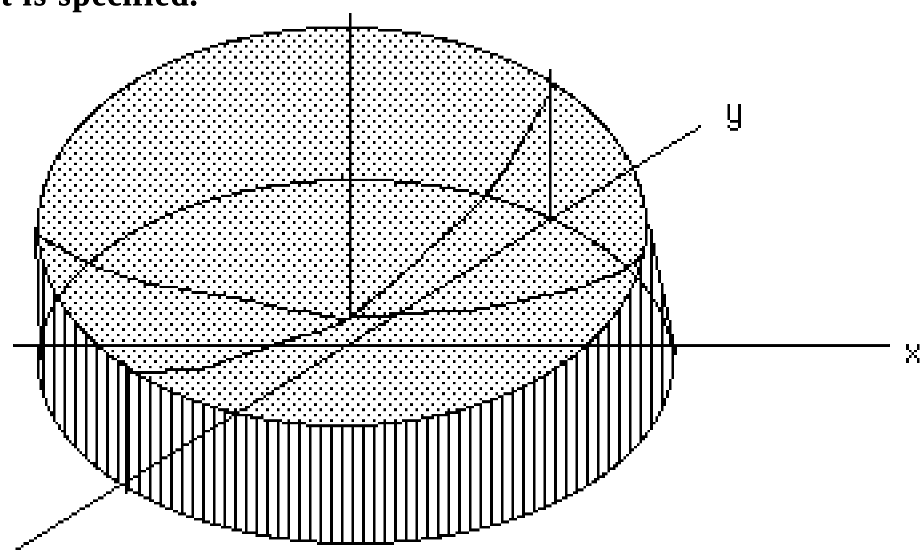
$$L[u] \quad = \quad \frac{\partial u}{\partial t}(t,x) - \frac{\partial^2 u}{\partial x^2}(t,x).$$

This is a parabolic operator; in fact, the matrix A of the above definition is given by $A = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$

(b) Another common operator is the <u>Laplacian operator</u>:

$$L[u] = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} .$$

A physical situation in which it arises is in the problem of finding the shape of a drum under force. Suppose that the bottom of the drum sits on the unit disc in the xy-plane and that the sides of the drum lie above the unit circle. We do not suppose that the sides are at a uniform height, but that the height is specified.



That is, we know u(x,y) for {x,y} on the boundary ot the drum. We also suppose that there is a force pulling down, or pushing up, on the drum at each point and that this force is not changing in time. An example of such a force might be the pull of gravity. The question is, what is the shape of the drum? As we shall see, the appropriate equations take the form: Find u if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y) \text{ for } x^2 + y^2 < 1$$

with $\quad u(x,y)$ **specified for** $x^2 + y^2 = 1$.

This is an elliptic problem for the Laplacian operator is an elliptic operator.

(c) An example of a hyperbolic problem is the <u>one dimensional wave equation</u>. One can think of this equation as describing the motion of a taunt string after an initial perturbation and subject to some outside force. Appropriate boundary conditions are given. To think of this as being a plucked string with the observer watching the up and down motion in time is not a bad perspective, and certainly gives intutive understanding. Here is

another perspective, however, which will be more useful in the context of finding the Greens function to solve this one dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + f(t,x) \quad \text{for } 0 < x < L,$$

$$u(t,x) = u(t,L) = 0 \quad \text{for } t > 0,$$

$$u(0,x) = g(x)$$

$$\text{and} \quad \frac{\partial u}{\partial t}(0,x) = h(x) \text{ for } 0 < x < L.$$

As in example (a), the problem is to describe u in the infinite strip within the first quadrant of the xt-plane bounded by the x axis and the lines x = 0 and x = L. Both u and its first derivative in the t direction are known along the x axis. Along the other boundaries, u is zero. What must be the shape of the graph above the infinite strip?

To classify this as a hyperbolic problem, think of the operator L as

$$L[u] = \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2}$$

and re-write it in the appropriate form for classification.

There are standard procedures for changing more general partial differential equations to the familiar standard forms.