Section 3.2: A Standard Form for Second Order Linear Equations

The ideas of the previous section suggested a connection with quadratic forms in analytic geometry. If presented with a quadratic equation in two variables, one could likely decide if the equation represented a parabola, hyperbola, or ellipse in the plane. However, if asked to draw a graph of this conic section in the plane, one would start recalling that there are several forms that are easy to draw:

$ax^2 + by^2 = c^2$, and the special case $x^2 + y^2 = c^2$,

$ax^2 - by^2 = c^2$, and the special case $x^2 - y^2 = c^2$,

or $y - ax^2 = 0$ and $x - by^2 = 0$.

These quadratic equations represent the familiar conic sections: ellipses, hyperbolas and parabolas, respectively. If a quadratic equation is given that is not in these special forms, then one must recall procedures to transform the equations algebraically into these standard forms. Performing these algebraic procedures corresponds to a geometric idea of translations and rotations.

For example, the equation

$$x^2 - 3y^2 - 8x + 30y = 60$$

represents a hyperbola. To draw the graph of the hyperbola, one algebraically factors the equation or, geometrically, translates the axes:

$$(x - 4)^2 - 3(y - 5)^2 = 1.$$

Now, the response that this is a hyperbola with center {4,5} is expected. More detailed information about the direction of the major and minor axes could be made, but these are not notions that we will wish to carry over to the process of getting second order partial differential equations into standard forms.

There is another idea more appropriate. Rather than keeping the hyperbola in the Euclidean plane where it now has the equation

$$x^2 - 3y^2 = 1$$

in the translated form, think of this hyperbola in the cartesian plane, and do not insist that the x axis and the y axis have the same scale. In this particular case, keep the x axis the same size and expand the y axis so that every unit is the old unit multiplying by $\sqrt{3}$. Algebraically, one commonly writes that there are new coordinates {x',y'} related to the old coordinates by

$$x = x' , \qquad \sqrt{3}\, y = y'.$$

The algebraic effect is that the equation is transformed into an equation in {x',y'} coordinates:

$$x'^2 - y'^2 = 1$$

Pay attention to the fact that it is now a mistake to carry over too much of the geometric language for the form. For example if the original quadratic equation had been

$$x^2 + 3y^2 - 8x + 30y = 60$$

and we had translated axes to produce

$$x^2 + 3y^2 = 60,$$

and then rescaled the axes to get

$$x^2 + y^2 = 60$$

we have not changed an ellipse into a circle for a circle is a geometric object whose very definition involves the notion of distance. The process of changing the scale on the X axis and the Y axis certainly destroys the entire notion of distance being the same in all directions.

Rather, the rescaling is an idea that is algebraically simplifying.

Before we pursue the idea of rescaling and translating in second order partial differential equations in order to come up with standard forms, we need to recall that there is also the troublesome need to rotate the axis in order to get some quadratic forms into the standard one. For example, if the equation is

$$xy = 2,$$

we quickly recognize this as a quadratic equation. Even more, we could draw the graph. If pushed, we would identify the resulting geometric figure as a hyperbola. We ask for more here since these geometric ideas are more readily transformed into ideas about partial differential equations if they are converted into algebraic ideas. The question, then, is how do we achieve the algebraic representation of the hyperbola in standard form?

One recalls from analytic geometry, or recognizes from simply looking at the picture of the graph of the equation, that this hyperbola that has been rotated out of standard form. To see it in standard form, we must rotate the axes. One forgets the details of how this rotation is performed, but should know a reference to find the scheme.

Here is the rotation needed to remove the xy term in the equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

The new coordinates {x', y'} are given by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

where α is $\pi/4$ if a = c and is $\frac{1}{2}\arctan\left(\frac{b}{a-c}\right)$ if $a \neq c$. What is the same,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

Thus, substitute $x = x' \cos(\alpha) - y' \sin(\alpha)$ and $y = x' \sin(\alpha) + y' \cos(\alpha)$ into the equation, where α is as indicated above and the cross term, bxy, will disappear.

Given a general quadratic, there are three things that need to be done to get it into standard form: get rid of the xy terms, factor all the x terms and the y terms separately, and rescale the axes so that the coefficients of the x^2 term and the y^2 terms are the same. Geometrically this corresponds, as we have recalled, to a rotation, translation, and expansion, respectively. From the geometric point of view, it does not matter which is done first: the rotation and then the translation, or vice versa. Algebraically, it is easier to remove the xy terms first, for then the factoring is easier.

The purpose of the previous paragraphs recalling how to change algebraic equations representing two dimensional conic sections into standard form was to suggest that the same ideas carry over almost

unchanged for the second degree partial differential equations. The techniques will change these equations into the standard forms for elliptic, hyperbolic, or parabolic partial differential equations. The purpose for doing this is that the techniques for solving equations are different in the three classes, if this is possible at all. Even more, there are important resemblances among the solutions of one class and striking differences between the solutions of one class and those of another class. The remainder of these notes will be primarily concerned with finding solutions to elliptic equations, but will discuss the standard form given by the Laplacian

$$ {}^2u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2} $$

If one has an elliptic equation that is not in the standard form of the Laplacian, the purpose of the remainder of this section is to present methods to change it into this form. The techniques are similar to those used in the analytic geometry. Having the standard form, one might then solve the equation involving the Laplacian. Finally, the solution should be transformed back into the original coordinate system.

We will illustrate the procedure for transformation of a second order equation into standard form. Consider the equation

$$ 4\frac{\partial^2 u}{\partial x^2} - 24\frac{\partial^2 u}{\partial x \partial y} + 11\frac{\partial^2 u}{\partial x^2} - 12\frac{\partial u}{\partial x} - 9\frac{\partial u}{\partial y} - 5u = 0. $$

We would like to transform the equation into the form

$$ {}^2 u + cu = 0. $$

In the original equation, if we think of the equation as

$$ a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + f\,u = 0. $$

Then, a = 4, b = -12, c = 11, so that $b^2 -$ a c = 100 and the equation is hyperbolic. The transformations will be made in three steps which will correspond to a rotation, a translation, and a rescaling in the earlier discussion.

The first step is the introduction of new coordinates (ξ,η) by rotation of axes so that in the transformed equation the mixed second partial derivative does not appear. Let

$$ \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} $$

or,

$$ \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}. $$

Using the chain rule, $\quad \frac{\partial}{\partial x} = \cos(\alpha)\frac{\partial}{\partial \xi} - \sin(\alpha)\frac{\partial}{\partial \eta}$

and $$\frac{\partial}{\partial y} = \sin(\alpha)\frac{\partial}{\partial \xi} + \cos(\alpha)\frac{\partial}{\partial \eta}.$$

It follows that

$$\frac{\partial^2}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial}{\partial x} = \left(\cos(\alpha)\frac{\partial}{\partial \xi} - \sin(\alpha)\frac{\partial}{\partial \eta}\right)\left(\cos(\alpha)\frac{\partial}{\partial \xi} - \sin(\alpha)\frac{\partial}{\partial \eta}\right)$$

so that

$$\frac{\partial^2}{\partial x^2} = \cos^2(\alpha)\frac{\partial^2}{\partial \xi^2} - 2\sin(\alpha)\cos(\alpha)\frac{\partial^2}{\partial \xi\partial \eta} + \sin^2(\alpha)\frac{\partial^2}{\partial \eta^2}.$$

In a similar manner,

$$\frac{\partial^2}{\partial x\,\partial y} =$$

$$\sin(\alpha)\cos(\alpha)\frac{\partial^2}{\partial \xi^2} + (\cos^2(\alpha) - \sin^2(\alpha))\frac{\partial^2}{\partial \xi\,\partial \eta} - \sin(\alpha)\cos(\alpha)\frac{\partial^2}{\partial \eta^2},$$

and

$$\frac{\partial^2}{\partial y} = \sin^2(\alpha)\frac{\partial^2}{\partial \xi^2} + 2\sin(\alpha)\cos(\alpha)\frac{\partial^2}{\partial \xi\,\partial \eta} + \cos^2(\alpha)\frac{\partial^2}{\partial \eta^2}.$$

The original equation described u as a function of x and y. We now define v as a function of ξ and η by $v(\xi,\eta) = u(x,y)$. The variables ξ and η are related to x and y as described by the rotation above. Of course, we have not specified α yet. This comes next.

The equation satisfied by v is

$$[4c^2 - 24sc + 11s^2]\frac{\partial^2 v}{\partial \xi^2} + [14sc - 24(c^2 - s^2)]\frac{\partial^2 v}{\partial \xi\partial \eta}$$

$$+ [4s^2 + 24sc + 11c^2]\frac{\partial^2 v}{\partial \eta^2} + [-12c - 9s]\frac{\partial v}{\partial \xi}$$

$$+ [12s - 9c]\frac{\partial v}{\partial \eta} - 5v,$$

where we have used the abbreviations $s = \sin(\alpha)$ and $c = \cos(\alpha)$. The coefficient of the mixed partials will vanish if α is chosen so that

$$14\sin(\alpha)\cos(\alpha) - 24(\cos^2(\alpha) - \sin^2(\alpha)) = 0,$$

that is,

$$\tan(2\alpha) = 24/7.$$

This means $\sin(\alpha) = 3/5$ and $\cos(\alpha) = 4/5$.

After substitution of these values, the equation satisfied by v becomes

$$\frac{\partial^2 v}{\partial \xi^2} - 4\frac{\partial^2 v}{\partial \eta^2} + 3\frac{\partial v}{\partial \xi} + v = 0.$$

This special example, together with the foregoing discussions of analytic geometry makes the following statement believable: Every second order partial differential equation with constant coefficients can be transformed into one in which mixed partials are absent.

We are now ready for the second step: to remove the first order term. For economy of notation, let us assume that the given equation is already in the form

$$\frac{\partial^2 u}{\partial x^2} - 4\frac{\partial^2 u}{\partial y^2} + 3\frac{\partial u}{\partial x} + u = 0.$$

Define v by

$$v(x,y) = e^{-\beta x}\, u(x,y) \quad \text{or} \quad u(x,y) = e^{\beta x}\, v(x,y),$$

where β will be chosen so that the transformed equation will have the first order derivative removed. Differentiating u and substituting into the equation we get that

$$\frac{\partial^2 v}{\partial x^2} - 4\frac{\partial^2 v}{\partial y^2} + (2\beta + 3)\frac{\partial v}{\partial x} + (\beta^2 + 3\beta + 1)v = 0.$$

If we choose $\beta = -3/2$, we have

$$\frac{\partial^2 v}{\partial x^2} - 4\frac{\partial^2 v}{\partial y^2} - \frac{5}{4}v = 0.$$

Notice that this transformation to achieve an equation lacking the first derivative with respect to x is generally possible when the coefficient on the second derivative with respect to x is not zero, and is otherwise impossible. The same statements hold for derivatives with respect to y.

The final step is rescaling. We choose variables ξ and η by $\xi = \mu x$ and $\eta = \nu y$, where μ and ν are chosen so that in the transformed equation the coefficients of $\frac{\partial^2 v}{\partial \xi^2}$, $\frac{\partial^2 v}{\partial \eta^2}$, and v are equal in absolute value. We have

$$\frac{\partial^2 u}{\partial x^2} = \mu^2\frac{\partial^2 v}{\partial \xi^2} \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = \nu^2\frac{\partial^2 v}{\partial \eta^2}.$$

Our equation becomes

$$\mu^2\frac{\partial^2 v}{\partial \xi^2} + 4\nu^2\frac{\partial^2 v}{\partial \eta^2} - \frac{5}{4}v = 0.$$

The condition that

$$\mu^2 = 4\nu^2 = \frac{5}{4}$$

will be satisfied if $\mu = \frac{\sqrt{5}}{2}$ and $\nu = \frac{\sqrt{5}}{4}$. Then, we obtain the standard form

$$^2v - v = \frac{\partial^2 v}{\partial \xi^2} + \frac{\partial^2 v}{\partial \eta^2} - v = 0.$$

EXERCISES:

I. Transform the following equations into standard form:

(a) $$3\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial y^2} - u = 0.$$

(b) $$4\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + 4\frac{\partial^2 u}{\partial y^2} + u = 0.$$

(c) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 3\frac{\partial u}{\partial x} - 4\frac{\partial u}{\partial y} + 25u = 0.$$

(d) $$\frac{\partial^2 u}{\partial x^2} - 3\frac{\partial^2 u}{\partial y^2} + 2\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} + u = 0.$$

(e) $$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} + 3u = 0.$$

II. Show that the equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} + \gamma u = f(x,y)$$

where γ is any constant, can be transformed into

$$\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial y} = g(x,y).$$

III. Show that by rotation of the axis by 45° the equations

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 u}{\partial x \partial y} = 0.$$

can be transformed into one another. Find the general solution for both equations.